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Certain aspects of rough deferred statistical cluster points in normed linear spaces

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Abstract. For any real valued sequence $x = (x_k)$, the number γ is called deferred statistical cluster point of x, provided that for every $\varpi > 0$, the set $\left\{p(n) < k \leq q(n) : \left|x_k - \gamma\right| < \varpi\right\}$ have a non-zero deferred density, where $p = (p(n))$ and $q = (q(n))$ are the sequence of non-negative integers satisfying $p(n) < q(n)$ and $\lim\limits_{n\to\infty} q(n) = \infty$. In the present article, we introduce and investigate the concept of rough deferred statistical cluster points in a normed linear space. Several fundamental properties are established, and significant inclusion relations are derived, enhancing the theoretical framework of statistical convergence.

1. Introduction and preliminaries

Fast [10] and Steinhaus [24] introduced statistical convergence independently as an extension of usual convergence in the context of real sequences. The core concept driving statistical convergence is rooted in the notion of natural density. Further exploration of this field and additional applications of statistical convergence can be found in the works of Fridy [11, 12], Šalát [21], Mohiuddine et al. [17], and Tripathy [25, 26] and many others [13, 14, 18, 20, 22].

Agnew [1] generalized Cesàro mean to deferred Cesàro mean to obtain a more useful method having stronger features. Deferred Cesàro mean, defined with the help of the sequences $(p(n))_{n\in\mathbb{N}}$ and $(q(n))_{n\in\mathbb{N}}$, seems like a new form of Cesàro mean, but it is a more effective summation method in terms of its features. For example, although the Cesàro mean follows the Silverman Toeplitz theorem, the deferred Cesàro mean, which seems to be a generalization of it, has the following property that the Cesàro mean does not provide in addition to this theorem

$$\text{“for each } k \in \mathbb{N}, a_{n,k} = 0 \text{ for almost all } n \in \mathbb{N}\text{”.}$$

The provision of this property by the deferred Cesàro mean makes it more efficient in lower triangular methods in that it converts bounded sequences to convergent sequences. The reason why the deferred Cesàro mean is applied to many problems compared to other methods is because of its properties.

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Let $p = (p(n))$ and $q = (q(n))$ are the sequence of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{1}$$

Suppose $K \subseteq \mathbb{N}$, and assume that $K_{p,q}(n)$ indicates the set $\{k : p(n) < k \leq q(n), k \in K\}$. Then, the deferred density of K is defined as

$$\mathfrak{D}_{p,q}(K) := \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|K_{p,q}(n)\right|, \tag{2}$$

provided that the limit exists. Here the vertical bars in (2) indicate the number of elements of the set $K_{p,q}(n)$.

As $\mathfrak{D}_{p,q}(K)$ may not exist for all $K \subset \mathbb{N}$, , it is practical to employ the upper deferred asymptotic density of K, which is defined as follows:

$$\mathfrak{D}^*_{p,q}(K) = \limsup_{n\to\infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), k \in K\}|}{q(n) - p(n)}.$$

It is evident that:
i) $\mathfrak{D}_{p,q}(K) = \mathfrak{D}^*_{p,q}(K)$ provided that $\mathfrak{D}_{p,q}(K)$ exists,
ii) $\mathfrak{D}^*_{p,q}(K) > 0$ iff $\mathfrak{D}_{p,q}(K) \neq 0$,
iii) $\mathfrak{D}^*_{p,q}(K) \leq \mathfrak{D}^*_{p,q}(M)$, provided that $K \subset M$.

A sequence of real numbers, denoted as $x = (x_k)$ is considered to be deferred statistically convergent [16] to $x_0 \in \mathbb{R}$ provided that for any $\varpi > 0$,

$$\mathfrak{D}_{p,q}\left(\{k \in \mathbb{N} : |x_k - x_0| \geq \varpi\}\right) = 0.$$

The element γ is referred to as a deferred statistical cluster point [27] of x, provided that for every $\varpi > 0$,

$$\mathfrak{D}_{p,q}\left(\left\{p(n) < k \leq q(n) : \left|x_k - \gamma\right| < \varpi\right\}\right) \neq 0.$$

Several investigations in this direction can be accessed from the works of Et et al. [6, 7] and Şengül et al. [23], and many others [2, 5, 8, 9, 15, 27].

In a different context, the exploration of rough convergence in finite-dimensional normed spaces was initially undertaken by Phu [19]. Phu primarily demonstrated that the set $LIM^r x$ possesses the properties of being bounded, closed, and convex, unveiling the noteworthy characteristics of this intriguing concept. Notably, the concept of rough convergence arises organically in numerical analysis, bearing interesting applications in that domain. Aytar [3, 4] further advanced this field by combining the concepts of rough convergence and statistical convergence, introducing rough statistical convergence and rough statistical cluster points.

2. Main results

Throughout the paper, $p = (p(n)), p' = (p'(n)), q = (q(n))$ and $q' = (q'(n))$ are the sequence of non-negative integers satisfying

$$p(n) < q(n), p'(n) < q'(n), \lim_{n\to\infty} q(n) = \infty \text{ and } \lim_{n\to\infty} q'(n) = \infty.$$

Also E, E', F and F' denotes the range set of the sequences p, p', q and q', respectively.

2.1. Characteristics of rough deferred statistical cluster points

Definition 2.1. Assume that $(X, \|\cdot\|)$ represents a normed linear space. For $r \geq 0$, the vector $\mu \in X$ is termed as the r-deferred statistical cluster point of $x = (x_k)$, if

$$\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0$$

holds for every $\varpi > 0$. The set encompassing all r-deferred statistical cluster points of x is denoted as $\Gamma^r_x(p,q)$.

Theorem 2.2. *Let $x = (x_k)$ be any sequence in the normed linear space $(X, \|\cdot\|)$. Then, the set $\Gamma_x^r(p,q)$ is closed for any $r \geq 0$ as well as for any choices of p, q.*

Proof. =Assume that $\Gamma_x^r(p,q) \neq \emptyset$ and take $y = (y_k) \subseteq \Gamma_x^r$ so that $\lim_{k\to\infty} y_k = y_*$. Let us demonstrate that for every $\varpi > 0$

$$\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - y_*\right\| < r + \varpi\right\}\right) \neq 0.$$

Fix $\varpi > 0$. Since $\lim_{k\to\infty} y_k = y_*$, there exists an $k_0 = k_0(\varpi) \in \mathbb{N}$ such that $\left\|y_k - y_*\right\| < \frac{\varpi}{2}$ for all $k > k_0$. Fix j_0 such that $j_0 > k_0$. Then we have $\left\|y_{j_0} - y_*\right\| < \frac{\varpi}{2}$. Let j be any point of the set $\left\{k \in \mathbb{N} : \left\|x_k - y_{j_0}\right\| < r + \frac{\varpi}{2}\right\}$. Since $\left\|x_j - y_{j_0}\right\| < r + \frac{\varpi}{2}$, we get

$$\begin{aligned}\left\|x_j - y_*\right\| &\leq \left\|x_j - y_{j_0}\right\| + \left\|y_{j_0} - y_*\right\| \\ &< r + \frac{\varpi}{2} + \frac{\varpi}{2} = r + \varpi,\end{aligned}$$

which shows that $j \in \left\{k \in \mathbb{N} : \left\|x_k - y_*\right\| < r + \varpi\right\}$. Hence we have

$$\left\{k \in \mathbb{N} : \left\|x_k - y_{j_0}\right\| < r + \frac{\varpi}{2}\right\} \subseteq \left\{k \in \mathbb{N} : \left\|x_k - y_*\right\| < r + \varpi\right\}.$$

As the deferred density of the set on the left-hand side in the above inclusion relation is non-zero, it follows that the deferred density of the set on the right-hand side is also non-zero. Therefore we have $y_* \in \Gamma_x^r(p,q)$. □

Theorem 2.3. *For any sequence $x = (x_k)$ with $r > 0$, $x_* \in \Gamma_x^r(p,q)$ iff there exists a sequence $y = (y_k)$ so that $x_* \in \Gamma_y(p,q)$ and $\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - y_k\right\| > r\right\}\right) = 0$.*

Proof. Fix r and ϖ. Assume that $x_* \in \Gamma_x^r(p,q)$. Hence we have $\mathfrak{D}_{p,q}(K) \neq 0$, where

$$K := \{k \in \mathbb{N} : \|x_k - x_*\| < r + \varpi\}.$$

Define

$$y_k := \begin{cases} x_*, & \|x_k - x_*\| \leq r \text{ and } k \in K \\ x_k + r\frac{x_* - x_k}{\|x_k - x_*\|}, & \|x_k - x_*\| > r \text{ and } k \in K \\ z_k, & k \notin K, \end{cases} \tag{3}$$

where the sequence $z = (z_k)$ is arbitrary. It is clear that

$$\left\|y_k - x_*\right\| = \begin{cases} 0, & \text{if } \|x_k - x_*\| \leq r \\ \|x_k - x_*\| - r, & \text{otherwise} \end{cases} \tag{4}$$

and

$$\left\|x_k - y_k\right\| \leq r$$

for every $k \in K$. Now, let us demonstrate the validity of the inclusion:

$$K \subseteq \left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}. \tag{5}$$

If $k_0 \in K$, then we have $\left\|x_{k_0} - x_*\right\| < r + \varpi$. Therefore, the two following scenarios may occur:
(i) If $\left\|x_{k_0} - x_*\right\| \leq r$, then from (4), we get $\left\|y_{k_0} - x_*\right\| = 0$, i.e., $k_0 \in \left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}$.
(ii) If $\left\|x_{k_0} - x_*\right\| > r$, then from (4), we get

$$\left\|y_{k_0} - x_*\right\| = \left\|x_{k_0} - x_*\right\| - r < r + \varpi - r = \varpi, \text{ i.e., } k_0 \in \left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}.$$

Since $\mathfrak{D}_{p,q}(K) \neq 0$, , according to the inclusion relation (5), we can conclude that

$$\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}\right) \neq 0.$$

Conversely, suppose that $x_* \in \Gamma_y(p,q)$ and fix $\varpi > 0$. Then, we have

$$\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}\right) \neq 0.$$

Take $j \in \left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\}$. We can write

$$\begin{aligned}\left\|x_j - x_*\right\| &\leq \left\|x_j - y_j\right\| + \left\|y_j - x_*\right\| \\ &< r + \varpi.\end{aligned}$$

Therefore we get $j \in \{k \in \mathbb{N} : \|x_k - x_*\| < r + \varpi\}$, which shows that the inclusion

$$\left\{k \in \mathbb{N} : \left\|y_k - x_*\right\| < \varpi\right\} \subseteq \{k \in \mathbb{N} : \|x_k - x_*\| < r + \varpi\}$$

holds. From this inclusion, we have $\mathfrak{D}_{p,q}\left(\{k \in \mathbb{N} : \|x_k - x_*\| < r + \varpi\}\right) \neq 0$. □

Theorem 2.4.

$$\Gamma_x^r(p,q) = \bigcup_{c \in \Gamma_x(p,q)} \bar{B}_r(c),$$

where $\bar{B}_r(c) := \{y \in X : \|y - c\| \leq r\}$.

Proof. Assume that $\mu \in \bigcup_{c \in \Gamma_x(p,q)} \bar{B}_r(c)$. This implies the existence of a vector $c \in \Gamma_x(p,q)$ so that $\mu \in \bar{B}_r(c)$, i.e., $\|c - \mu\| \leq r$. Fix $\varpi > 0$. Since $c \in \Gamma_x(p,q)$, there exists a set $K = K(\varpi) := \{k \in \mathbb{N} : \|x_k - c\| < \varpi\}$ with $\mathfrak{D}_{p,q}(K) \neq 0$. We have

$$\begin{aligned}\left\|x_k - \mu\right\| &\leq \|x_k - c\| + \|c - \mu\| \\ &< \varpi + r\end{aligned}$$

for every $k \in K$. Thus, we conclude that $\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < \varpi + r\right\}\right) \neq 0$, thereby completing the first part of the proof.

To establish the converse inclusion, consider $\mu \in \Gamma_x^r(p,q)$. Then, we have

$$\mathfrak{D}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < \varpi + r\right\}\right) \neq 0 \tag{6}$$

for every $\varpi > 0$. Let us show that $\mu \in \bigcup_{c \in \Gamma_x(p,q)} \bar{B}_r(c)$. Assume that this condition is not met. Then, we get $\mu \notin \bar{B}_r(c)$, i.e., $\|\mu - c\| > r$ for every $c \in \Gamma_x(p,q)$. Since the set $\Gamma_x(p,q)$ is closed, there exists a vector $\tilde{c} \in \Gamma_x(p,q)$ such that $\|\mu - \tilde{c}\| = \min\{\|\mu - c\| : c \in \Gamma_x(p,q)\}$. We can write $t := \|\mu - \widetilde{c}\| > r$, because $\|\mu - c\| > r$ for all $c \in \Gamma_x(p,q)$. Define $\widetilde{\varpi} := \frac{t-r}{3}$. Then, we get

$$X \backslash B_{\tilde{\varpi}}\left(\Gamma_x(p,q)\right) \supseteq \{y \in X : \|\mu - y\| < \widetilde{\varpi} + r\}, \tag{7}$$

where $B_{\tilde{\varpi}}\left(\Gamma_x(p,q)\right) = \{y \in X : \min\{\|y - c\| : c \in \Gamma_x(p,q)\} < \tilde{\varpi}\}$. According to the definition of $\Gamma_x(p,q)$, it follows that the set $\{k : x_k \notin B_{\tilde{\varpi}}\left(\Gamma_x(p,q)\right)\}$ has deferred density zero. Following the inclusion (7), we can conclude that

$$\{k : x_k \notin B_{\widetilde{\varpi}}\left(\Gamma_x(p,q)\right)\} \supseteq \left\{k : \left\|x_k - \mu\right\| < \widetilde{\varpi} + r\right\}. \tag{8}$$

Hence, based on the inclusion (8), it follows that the set $\left\{k : \left\|x_k - \mu\right\| < \widetilde{\varpi} + r\right\}$ has deferred density zero, which contradicts (6). □

2.2. Some inclusion results for $\Gamma_x^r(p,q)$

Theorem 2.5. *Let $r \geq 0$. Assume that $\lim_{n\to\infty} \frac{q'(n)-p(n)}{q(n)-p(n)} = d \neq 0$. Then, the following results are true:*
(i) If $F' \setminus F$ is finite then $\Gamma_x^r(p,q') \subseteq \Gamma_x^r(p,q)$.
(ii) If $F' \triangle F$ is finite then $\Gamma_x^r(p,q') = \Gamma_x^r(p,q)$.

Proof. (i) Since the set $F' \setminus F$ is finite, then there exists a positive natural number N such that

$$\{q'(n) : n \geqslant N\} \subset \{q(n) : n \in \mathbb{N}\}.$$

For $n \geqslant N$, let $j(n)$ be a strictly increasing sequence such that $q'(n) := q(j(n))$. Let $\mu \in \Gamma_x^r(p,q')$.
Then, for any $\varpi > 0$,

$$\begin{aligned}
&\mathfrak{D}_{p,q'}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0\\
\Longrightarrow\ &\mathfrak{D}^*_{p,q'}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) > 0\\
\Longrightarrow\ &\limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \leq q'(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q'(n) - p(n)} > 0\\
\Longrightarrow\ &\limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \leq q(j(n)) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(j(n)) - p(n)} > 0. \qquad (9)
\end{aligned}$$

Now as the following inequality

$$\frac{\left|\left\{p(n) < k \leq q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} \geq \frac{q(j(n)) - p(n)}{q(n) - p(n)} \cdot \frac{\left|\left\{p(n) < k \leq q(j(n)) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(j(n)) - p(n)}$$

holds, so by (9),

$$\mathfrak{D}^*_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) = \limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \leq q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} > 0.$$

In other words, $\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0$. Hence, $\mu \in \Gamma_x^r(p,q)$ and this completes the proof.
(ii) The proof of this part is straightforward, hence omitted. □

Theorem 2.6. *Let $r \geq 0$. Assume that $\lim_{n\to\infty} \frac{q(n)-p(n)}{q(n)-p'(n)} = d \neq 0$. Then, the following results are true:*
(i) If $E' \setminus E$ is finite then $\Gamma_x^r(p,q) \subseteq \Gamma_x^r(p',q)$.
(ii) If $E' \triangle E$ is finite then $\Gamma_x^r(p,q) = \Gamma_x^r(p',q)$.

Proof. Since the set $E' \setminus E$ is finite, then there exists a positive natural number N such that

$$\{p'(n) : n \geqslant N\} \subset \{p(n) : n \in \mathbb{N}\}.$$

For $n \geqslant N$, let $j(n)$ be a strictly increasing sequence such that $p'(n) := p(j(n))$. Let $\mu \in \Gamma_x^r(p,q)$. Then, for any $\varpi > 0$,

$$\begin{aligned}
&\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0\\
\Longrightarrow\ &\mathfrak{D}^*_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) > 0\\
\Longrightarrow\ &\limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \leq q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} > 0. \qquad (10)
\end{aligned}$$

Now as the following inequality

$$\frac{\left|\left\{p(j(n)) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(j(n))} \ge \frac{q(n) - p(n)}{q(n) - p(j(n))} \cdot \frac{\left|\left\{p(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)}$$

holds, so by (10),

$$\mathfrak{D}^*_{p',q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) = \limsup_{n\to\infty} \frac{\left|\left\{p'(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p'(n)} > 0.$$

In other words, $\mathfrak{D}_{p',q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0$. Hence, $\mu \in \Gamma^r_x(p', q)$ and this completes the proof.

(ii) The proof of this part is straightforward, hence omitted. □

Theorem 2.7. *Let $r \ge 0$. Assume that $p(n) \le p'(n) < q'(n) \le q(n)$ and $\lim_{n\to\infty} \frac{q'(n)-p'(n)}{q(n)-p(n)} = d \neq 0$ both holds simultaneously. Then, $\Gamma^r_x(p', q') \subseteq \Gamma^r_x(p, q)$.*

Proof. Let $\mu \in \Gamma^r_x(p', q')$. Then, for any $\varpi > 0$,

$$\begin{aligned}
&\mathfrak{D}_{p',q'}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0\\
\Longrightarrow\ &\mathfrak{D}^*_{p',q'}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) > 0\\
\Longrightarrow\ &\limsup_{n\to\infty} \frac{\left|\left\{p'(n) < k \le q'(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q'(n) - p'(n)} > 0. \qquad (11)
\end{aligned}$$

Now as the following inequality

$$\frac{\left|\left\{p(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} \ge \frac{q'(n) - p'(n)}{q(n) - p(n)} \cdot \frac{\left|\left\{p'(n) < k \le q'(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q'(n) - p'(n)}$$

holds, so by (11),

$$\mathfrak{D}^*_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) = \limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} > 0.$$

In other words, $\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0$. Hence, $\mu \in \Gamma^r_x(p, q)$ and this completes the proof. □

Theorem 2.8. *Let $r \ge 0$. Assume that $q(n) \le n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{q(n)-p(n)}{n} = d \neq 0$. Then, $\Gamma^r_x \supseteq \Gamma^r_x(p, q)$.*

Proof. Let $\mu \in \Gamma^r_x(p, q)$. Then, for any $\varpi > 0$,

$$\begin{aligned}
&\mathfrak{D}_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0\\
\Longrightarrow\ &\mathfrak{D}^*_{p,q}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) > 0\\
\Longrightarrow\ &\limsup_{n\to\infty} \frac{\left|\left\{p(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)} > 0. \qquad (12)
\end{aligned}$$

Now as the following inequality

$$\frac{\left|\left\{k \le n : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{n} \ge \frac{q(n) - p(n)}{n} \cdot \frac{\left|\left\{p(n) < k \le q(n) : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{q(n) - p(n)}$$

holds, so by (12),

$$\mathfrak{D}^*\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) = \limsup_{n\to\infty} \frac{\left|\left\{k \le n : \left\|x_k - \mu\right\| < r + \varpi\right\}\right|}{n} > 0.$$

In other words, $\mathfrak{D}\left(\left\{k \in \mathbb{N} : \left\|x_k - \mu\right\| < r + \varpi\right\}\right) \neq 0$. Hence, $\mu \in \Gamma_x^r$ and this completes the proof. □

3. Conclusion

In this paper, we introduced and explored the concept of rough deferred statistical cluster points within normed linear spaces. Theorem 2.2 proves the closedness property of the set $\Gamma_x^r(p, q)$ for any $r \geq 0$. Theorem 2.3 establishes the necessary and sufficient condition for a number x_* to be a $r-$deferred statistical cluster point of a sequence $x = (x_k)$. Theorem 2.5, Thorem 2.6, Theorem 2.7, and Theorem 2.8 reveals significant inclusion relations for variation in the sequences p and q. Future work could explore the generalization of these concepts to broader mathematical structures or their implications in practical computational contexts.

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