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On solving a singular Volterra integral equation

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Abstract. In the paper, a general solution of the singular Volterra integral equation of the second kind is found. The feature of the considered integral equation lies in the fact that the integral of its kernel does not tend to zero as the upper limit approaches the lower one, thus making the Picard method inapplicable. It is shown that the corresponding homogeneous integral equation has a non-zero solution, which is found in an explicit form. Such integral equations arise, for example, in boundary value problems of heat conduction in degenerate regions, where the boundaries change with time [10]-[2]. Additionally, these equations appear in mathematical modeling of thermophysical processes in the electric arc of high-current switching devices [8]-[7].

1. Introduction

The growing interest in the theory of Volterra integral equations is driven by two main factors. Firstly, there is an expanding range of applications for these equations. Secondly, it is increasingly recognized that Volterra equations are not merely a special case of Fredholm integral equations, but rather form a distinct class with their own unique challenges. Classical literature typically addresses solutions to second-kind Volterra integral equations primarily in situations where these equations are regular. For the Volterra integral equation

$$\mu(t) - \lambda \int_0^t K(t,\tau)\mu(\tau)d\tau = g(t). \tag{1}$$

It is assumed that the kernel $K(t,\tau)$ of the equation is sufficiently well-behaved or may have only weak singularities. In these cases, as is well known, equation (1) has a unique solution for any value of λ, which can be found using the method of successive approximations, and the corresponding homogeneous equation has only the trivial solution.

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However, there are often cases where important practical problems lead to singular (irregular) Volterra equations. These include, for example, Volterra integral equations with an infinite integration interval or equations whose kernels have singularities of sufficiently high order, sometimes even of non-integrable types. In such cases, the inhomogeneous equation may have more than one solution, and the corresponding homogeneous equation may have a non-zero solution [18]-[17].

Let us consider a few examples of singular (irregular) Volterra integral equations. For instance, it is clear that the homogeneous equation [12]:

$$\mu(t) - \lambda \int_0^t \frac{1}{\tau} \cdot \mu(\tau) d\tau = 0, \quad t > 0$$

has a continuous solution

$$\mu(t) = C \cdot t^n, \forall \lambda = n.$$

or a less trivial equation

$$\mu(t) - \lambda \int_0^t J_0(t-\tau) \cdot \frac{\mu(\tau)}{\tau} d\tau = 0, \quad t > 0$$

$\forall \lambda = n$, has a solution $\mu(t) = C \cdot J_n(t)$, where $J_\nu(t)$ is the Bessel function of order ν.

The solution of the homogeneous Picard-Goursat integral equation with a variable lower limit [13]

$$\mu(t) - \lambda \int_t^\infty (t-\tau)^n \cdot \mu(\tau) d\tau = 0, \lambda > 0$$

has the form:

$$\mu(t) = C \cdot e^{-\gamma t}, \quad \gamma = (-\lambda n!)^{\frac{1}{n+1}}$$

This solution is unique for $n = 0, 1, 2, 3$. For $n \geq 4$, the general solution of the homogeneous Picard-Goursat equation is:

$$\mu(t) = C \cdot e^{-\gamma t} + \sum_{k=1}^{\left[\frac{n}{4}\right]} e^{-\alpha_k t} \left[C_k^{(1)} \cos(\beta_k t) + C_k^{(2)} \sin(\beta_k t) \right],$$

where $[a]$ is the integer part of the number a, and the coefficients α_k, β_k are determined by the expressions:

$$\alpha_k = (-\lambda n!)^{\frac{1}{n+1}} \cdot \cos\left(\frac{2\pi k}{n+1}\right) \cdot \beta_k = (-\lambda n!)^{\frac{1}{n+1}} \cdot \sin\left(\frac{2\pi k}{n+1}\right).$$

An integral equation arising when solving problems in the theory of heat conduction [16]:

$$\mu(t) - \lambda \int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} \cdot \mu(\tau) d\tau = 0, \quad \lambda > 0$$

has eigenfunctions of the form

$$\mu(t) = C \cdot t^\alpha, \alpha \geq 0$$

for all

$$\lambda = \frac{1}{B(0,5+\alpha, 0,5)},$$

where $B(p,q)$ is the beta function.

Indeed, in such cases, the non-zero solutions are called eigenfunctions of the given kernel $K(t,\tau)$, and the corresponding values of the parameter λ are called characteristic values. It is important to note that even in the case of Volterra-type equations, the resulting eigenfunctions can be useful, for example, in finding the solution to an inhomogeneous equation. Therefore, the peculiarity of solving singular second-kind Volterra equations, which fundamentally differ from standard Volterra equations, lies in the fact that they are, in a certain sense, intermediate between solving Volterra and Fredholm equations.

2. Problem statement

Find the solution to the following singular second-kind Volterra integral equation with respect to the unknown function $\mu(t)$:

$$\mathbf{N}_\lambda\mu \equiv (I-\lambda\mathbf{N})\mu \equiv \mu(t)-\lambda\int_0^t N(t,\tau)\mu(\tau)d\tau = g(t), t>0 \tag{2}$$

where $\lambda>0$ and the kernel

$$N(t,\tau)=N_h(t,\tau)+N_r(t,\tau),$$
$$N_h(t,\tau)=\frac{t^\nu}{(t-\tau)\tau^\nu}\exp\left[-\frac{t^2+\tau^2}{4a^2(t-\tau)}\right]\cdot I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right),$$
$$N_r(t,\tau)=\frac{t^\nu\tau^{1-\nu}}{2a^2(t-\tau)}\exp\left[-\frac{t^2+\tau^2}{4a^2(t-\tau)}\right]\cdot I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)$$

where $0<\nu<1$, $I_\nu(z)$ is the Infeld function (Bessel function of the imaginary argument of the first kind) of order ν. The function $g(t)$ is a given function from the class

$$t^{1-\nu}\exp\left[\frac{t}{4a^2}\right]g(t)\in M(\mathbb{R}_+)=L_\infty(\mathbb{R}_+)\cap C(\mathbb{R}_+). \tag{3}$$

The function $\mu(t)$ also belongs to the class:

$$t^{1-\nu}\exp\left[\frac{t}{4a^2}\right]\mu(t)\in M(\mathbb{R}_+)=L_\infty(\mathbb{R}_+)\cap C(\mathbb{R}_+).$$

The feature of the integral equation (2) lies in the following property of its kernel:

$$\lim_{t\to 0}\lambda\int_0^t N(t,\tau)d\tau=\frac{\lambda}{\nu}.$$

For example, such equations arise in the study of thermal stresses that occur during rapid cooling or heating of a body, as well as in the modeling of thermophysical processes in the electric arc of high-current switching devices [8]-[7].

Considering the equality

$$\frac{t^2+\tau^2}{4a^2(t-\tau)}=\frac{t-\tau}{4a^2}+\frac{t\tau}{2a^2(t-\tau)}$$

equation (2) can be represented in the following form:

$$\mu(t)-\lambda\int_0^t\left\{\frac{t^\nu}{(t-\tau)\tau^\nu}\exp\left[-\frac{t-\tau}{4a^2}\right]\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]\cdot I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)+\right.$$
$$\left.+\frac{t^\nu\tau^{1-\nu}}{2a^2(t-\tau)}\exp\left[-\frac{t-\tau}{4a^2}\right]\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right\}\mu(\tau)d\tau=g(t),\quad t>0.$$

If we now introduce the following notations:

$$t^{1-\nu}\exp\left[\frac{t}{4a^2}\right]\mu(t)=\mu_1(t),\quad t^{1-\nu}\exp\left[\frac{t}{4a^2}\right]g(t)=g_1(t) \tag{4}$$

then the last integral equation is transformed into the following equation:

$$\mathbf{M}_\lambda\mu\equiv(I-\lambda\mathbf{M})\mu_1\equiv\mu_1(t)-\lambda\int_0^t M(t,\tau)\mu_1(\tau)d\tau=g_1(t) \tag{5}$$

whose kernel has the form:

$$
\begin{aligned}
M(t,\tau) &= M_h(t,\tau) + M_r(t,\tau),\\
M_h(t,\tau) &= \frac{t}{\tau(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]\cdot I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right),\\
M_r(t,\tau) &= \frac{t}{2a^2(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right).
\end{aligned}
$$

Remark 2.1. If a solution to the singular integral equation (5) is found, then, by virtue of relations (3), a solution to the original integral equation (2) will be obtained.

The feature of this integral equation follows from the following remark.

Remark 2.2. For the kernel of the integral equation (5), $\forall t > 0, \forall \nu \in (0,1)$, the following equality holds:

$$
\lim_{t\to 0}\lambda\int_0^t M(t,\tau)d\tau = \frac{\lambda}{\nu}, \tag{6}
$$

and

$$
\int_0^t M_h(t,\tau)d\tau = \frac{1}{\nu}, \quad \lim_{t\to 0}\int_0^t M_r(t,\tau)d\tau = 0. \tag{7}
$$

Indeed,

$$
\begin{aligned}
\int_0^t M_h(t,\tau)\,d\tau &= \int_0^t \frac{t}{\tau(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)d\tau =\\
&= \left\|\frac{t\tau}{2a^2(t-\tau)} = z\right\| = \int_0^\infty \frac{1}{z^2}\cdot z\cdot e^{-z}\cdot I_\nu(z)\,dz =\\
&= \int_0^\infty \frac{1}{z}e^{-z} I_\nu(z)\,dz = \left\|\begin{matrix}(2.15.4.3)\\ [14][\text{p. } 272]\end{matrix}\right\| = \frac{1}{\sqrt{\pi}}\cdot\Gamma\begin{pmatrix}\nu, \frac{1}{2}\\ 1+\nu\end{pmatrix} = \frac{1}{\nu}.
\end{aligned}
$$

Let us show the validity of the second equality.

$$
\begin{aligned}
\int_0^t M_r(t,\tau)\,d\tau &= \int_0^t \frac{t}{2a^2(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)d\tau =\\
&= \left\|\frac{t\tau}{2a^2(t-\tau)} = z\right\| = \int_0^\infty \frac{t}{t+2a^2 z}\cdot e^{-z}\cdot I_\nu(z)\,dz \le \frac{t}{2a^2}\int_0^\infty \frac{1}{z}e^{-z} I_\nu(z)\,dz =\\
&= \frac{t}{2a^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\Gamma\begin{pmatrix}\nu, \frac{1}{2}\\ 1+\nu\end{pmatrix} = \frac{t}{2a^2\nu} \xrightarrow{t\to 0} 0.
\end{aligned}
$$

Remark 2.3. From relation (6), it follows that for $\lambda < \nu$, the integral equation (2) has a unique solution, which can be found using the method of successive approximations. That is, we will further investigate the integral equation (2) for $\lambda \ge \nu$, which we will call the singular Volterra integral equation.

3. Solution of the characteristic singular integral equation

First, let us solve the following integral equation:

$$
\mathbf{M}_{h\lambda}\mu_1 \equiv (I - \lambda\mathbf{M}_h)\,\mu_1 \equiv \mu_1(t) - \lambda\int_0^t M_h(t,\tau)\mu_1(\tau)d\tau = g_1(t) \tag{8}
$$

which we will call the characteristic equation for equation (5). The solution of the complete integral equation (5) will be found using the method of Carleman-Vekua equivalent regularization, that is, by solving the characteristic equation.

If in equation (8) we introduce new variables x and y instead of the variables t and τ:

$$t = \frac{1}{y}, \quad \tau = \frac{1}{x}; \quad \mu_1(t) = \mu_1\left(\frac{1}{y}\right) = \mu_2(y), \quad g_1(t) = g_1\left(\frac{1}{y}\right) = g_2(y)$$

then equation (8) will be reduced to the following integral equation with respect to the unknown function $\mu_2(y)$:

$$\mu_2(y) - \lambda \int_y^\infty M_{h_-}(y-x)\mu_2(x)dx = g_2(y) \tag{9}$$

here

$$M_{h_-}(y-x) = \frac{1}{x-y} \cdot \exp\left(-\frac{1}{2a^2(x-y)}\right) \cdot I_\nu\left(\frac{1}{2a^2(x-y)}\right).$$

The method of successive approximations is not applicable to equation (9) due to the property (7). This distinguishes it from the classical second-kind Volterra integral equations, for which a unique solution exists [12]-[13].

4. Solution of the corresponding homogeneous equation

Let us find the solution of the homogeneous singular integral equation:

$$\mu_2(y) - \lambda \int_y^\infty M_{h_-}(y-x)\mu_2(x)dx = 0. \tag{10}$$

This integral equation has a solution of the form

$$\mu_2(y) = e^{-\gamma y}$$

where γ is the root of the following transcendental equation [13, p. 507] with respect to the parameter γ:

$$\lambda \int_0^\infty M_{h_-}(-z) \cdot e^{-\gamma z} dz = 1. \tag{11}$$

The left-hand side of this equation is the image of the function $M_h(-z)$ obtained using the Laplace transform with the parameter $(-\gamma)$. For example, a simple root $(-\gamma_0)$ corresponds to the eigenfunction

$$\mu_2^{(0)}(y) = C \cdot e^{-\gamma_0 y}.$$

Indeed, applying the Laplace transform [4, p. 158], [13, pp. 498-499] to equation (10), we obtain

$$\widehat{\mu_2}(p) \cdot \left[1 - \lambda \widehat{M_{h_-}}(-p)\right] = 0, \quad \operatorname{Re} p < 0.$$

First, we find the image of the function

$$\widehat{M_{h_-}}(-p) = 2K_\nu\left(\frac{\sqrt{-p}}{a}\right) I_\nu\left(\frac{\sqrt{-p}}{a}\right)$$

where formula (29.169) [4, p. 350] is used. Then equation (11) takes the form:

$$1 - 2\lambda \cdot K_\nu\left(\frac{\sqrt{-p}}{a}\right) \cdot I_\nu\left(\frac{\sqrt{-p}}{a}\right) = 0, \quad \operatorname{Re} p < 0,$$

where $K_\nu(\sqrt{-p}/a)$ is the Macdonald function (Bessel function of the second kind with imaginary argument). This equation has a unique root for each value $\lambda \geq \nu$, which we denote by $-p_\lambda$. This means that equation (10) has a non-zero solution for each value $\lambda \geq \nu$:

$$\mu_2^{(\lambda)}(y) = C \cdot e^{p_\lambda y}, \quad p_\lambda < 0, \quad C = \text{ const.}$$

And the equation

$$\mathbf{M}_{h\lambda}\mu_1 \equiv (I - \lambda \mathbf{M}_h)\,\mu_1 \equiv \mu_1(t) - \lambda \int_0^t M_h(t,\tau)\mu_1(\tau)d\tau = 0$$

has the following non-zero solution:

$$\mu_1^{(\lambda)}(t) = C \cdot e^{\frac{p_\lambda}{t}}, \quad p_\lambda < 0, \quad C = \text{ const.}$$

This means that the homogeneous integral equation

$$\mathbf{N}_{h\lambda}\mu \equiv (I - \lambda \mathbf{N}_h)\,\mu \equiv \mu(t) - \lambda \int_0^t N_h(t,\tau)\mu(\tau)d\tau = 0, t > 0$$

has the following non-zero solution as well:

$$\mu^{(\lambda)}(t) = C \cdot \frac{1}{t^{1-\nu}} \cdot e^{\frac{p_\lambda}{t} - \frac{t}{4a^2}} \in M(\mathbb{R}_+)\,, \quad p_\lambda < 0 \quad C = \text{ const.}$$

Let us formulate the obtained result in the form of the following lemma.

Lemma 4.1. *The values $\lambda \in (0,\nu)$ are regular numbers of the operator $\mathbf{N}_{h\lambda}$, and the values $\lambda \in (\nu,+\infty)$ are characteristic numbers of this operator, with the corresponding eigenfunctions having the form:*

$$\mu^{(\lambda)}(t) = C \cdot \frac{1}{t^{1-\nu}} \cdot e^{\frac{p_\lambda}{t} - \frac{t}{4a^2}}.$$

5. Solution of the inhomogeneous characteristic integral equation. Construction of the resolvent

To find a particular solution of the inhomogeneous equation, we apply the method of model solutions [13, p. 561]. Then the solution of equation (9) will have the form:

$$\mu_2(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\widehat{g_2}(p)}{1-\lambda\widehat{M_{h_-}}(-p)} e^{py}dp = g_2(y) + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{R_-}(-p)\widehat{g_2}(p)e^{py}dp,$$

here

$$\widehat{g_2}(p) = \int_0^\infty g_2(y)e^{-py}dy, \quad \widehat{R_-}(-p) = \frac{\lambda\widehat{M_{h_-}}(-p)}{1-\lambda\widehat{M_{h_-}}(-p)}, \quad \operatorname{Re} p < 0.$$

If $(\widehat{R}_-)(-p) \risingdotseq R_-(y)$, then the particular solution of equation (9) has the form:

$$\mu_2(y) = g_2(y) + \int_y^\infty R_-(y-x)g_2(x)dx.$$

Let us write the image of the resolvent in the following form:

$$\widehat{R_-}\left(\frac{\sqrt{-p}}{a}\right) = \frac{2\lambda K_\nu\left(\frac{\sqrt{-p}}{a}\right)\cdot I_\nu\left(\frac{\sqrt{-p}}{a}\right)}{1-2\lambda K_\nu\left(\frac{\sqrt{-p}}{a}\right)\cdot I_\nu\left(\frac{\sqrt{-p}}{a}\right)}, \quad \operatorname{Re} p < 0$$

and find the original of the resolvent [11, p. 519]:

$$R_{-}(y) = A(\lambda, p_\lambda) \cdot e^{p_\lambda y}, \quad p_\lambda < 0$$

here

$$A(\lambda, p_\lambda) = \frac{a \cdot (-p_\lambda)}{a(\lambda - \beta) + 2\lambda \cdot \sqrt{-p_\lambda} I_\nu \left(\frac{\sqrt{-p_\lambda}}{a} \right) \cdot K_{\nu-1} \left(\frac{\sqrt{-p_\lambda}}{a} \right)} > 0.$$

Returning to the original variables, we obtain the solution of the inhomogeneous integral equation (5) in the following form:

$$\mu_1(t) = g_1(t) + \int_0^t R_{-}(t, \tau) g_1(\tau) d\tau + C \cdot \mu_1^{(\lambda)}(t) \tag{12}$$

here $C \cdot \mu_1^{(\lambda)}(t)$ — the general solution of homogeneous characteristic integral equation,

$$R_{-}(t, \tau) = A(\lambda, p_\lambda) \cdot e^{p_\lambda \cdot \frac{t-\tau}{t\tau}} \cdot \frac{1}{\tau^2}.$$

Hereafter, we will consider that

$$\begin{gathered} A(\lambda, p_\lambda) \cdot e^{p_\lambda \cdot \left(\frac{t-\tau_1}{t\tau_1} \right)} \leq C(\lambda, p_\lambda) \cdot \frac{(t\tau_1)^\alpha}{(t-\tau_1)^\alpha}, \\ 0 < \alpha < 1 - \beta, \quad C(\lambda, p_\lambda) = \text{ const.} \end{gathered} \tag{13}$$

6. Solution of the singular integral equation (5). Carleman-Vekua regularization method

To solve the integral equation (5), we represent it in the form:

$$\mu_1(t) - \lambda \int_0^t M_h(t, \tau) \cdot \mu_1(\tau) d\tau = g_1(t) + \lambda \int_0^t M_r(t, \tau) \cdot \mu_1(\tau) d\tau.$$

Using formula (12) we obtain:

$$\begin{gathered} \mu_1(t) - \lambda \int_0^t M_r(t, \tau) \cdot \mu_1(\tau) d\tau - \lambda \int_0^t R_{-}(t, \tau) \left\{ \int_0^\tau M_r(\tau, \tau_1) \cdot \mu_1(\tau_1) d\tau_1 \right\} d\tau = \\ = g_1(t) + \int_0^t R_{-}(t, \tau) g_1(\tau) d\tau + C \cdot \mu_1^{(\lambda)}(t). \end{gathered}$$

We change the order of integration in the repeated integral and then interchange the variables τ and τ_1. Then we obtain the following integral equation:

$$\mathbf{K}_\lambda \mu_1 \equiv (I - \lambda \mathbf{K}) \mu_1 \equiv \mu_1(t) - \lambda \int^t K(t, \tau) \cdot \mu_1(\tau) d\tau = F_g(t) + C \cdot \mu_1^{(\lambda)}(t) \tag{14}$$

here

$$K(t, \tau) = M_r(t, \tau) + \int_\tau^t R_{-}(t, \tau_1) \cdot M_r(\tau_1, \tau) d\tau_1,$$

$$F_g(t) = g_1(t) + \int_0^t R_{-}(t, \tau) g_1(\tau) d\tau.$$

We will show that the integral equation (14) can be solved by the method of successive approximations.

Theorem 6.1. *The kernel $K(t,\tau)$ of the integral equation* (14) *has a weak singularity, since the following estimate holds for it:*

$$K(t,\tau) \leq B(\lambda, p_\lambda) \cdot \frac{t^\alpha}{\tau^{1-\alpha}(t-\tau)^\alpha}$$

here $B(\lambda, p_\lambda) =$ const, $0 < \alpha < 1-\nu$.

Proof. It has been shown previously that

$$M_r(t,\tau) \leq B_1(\nu) \cdot t, \quad B_1(\nu) = \text{ const.}$$

Now let's estimate the second term of the kernel:

$$Q(t,\tau) = \int_\tau^t R_-(t,\tau_1) \cdot M_r(\tau_1,\tau)\, d\tau_1 =$$

$$= \int_\tau^t A(\lambda, p_\lambda) \cdot e^{p_\lambda \cdot \frac{t-\tau_1}{t\tau_1}} \cdot \frac{1}{\tau_1^2} \cdot \frac{\tau_1}{2a^2(\tau_1-\tau)} \exp\left[-\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right] I_\nu\left(\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right) d\tau_1.$$

Using inequality (13), we have:

$$\begin{aligned}
Q(t,\tau) &\leq \frac{C(\lambda,p_\lambda)}{2a^2} \int_\tau^t \frac{(t\tau_1)^\alpha}{(t-\tau_1)^\alpha} \cdot \frac{1}{\tau_1} \cdot \frac{1}{\tau_1-\tau} \exp\left[-\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right] I_\nu\left(\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right) d\tau_1 = \\
&= C(\lambda,p_\lambda) \cdot \frac{t^\alpha}{2a^2} \int_\tau^t \frac{{\tau_1}^{\alpha-1}}{(t-\tau_1)^\alpha} \cdot \frac{1}{\tau_1-\tau} \exp\left[-\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right] I_\nu\left(\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right) d\tau_1 = \\
&= \left\| \frac{\tau_1\tau}{2a^2(\tau_1-\tau)} = z \right\| = \\
&= \frac{C(\lambda,p_\lambda)\, t^\alpha}{2a^2} \int_{\frac{t\tau}{2a^2(t-\tau)}}^\infty \left(\frac{2a^2\tau z}{2a^2z-\tau}\right)^{\alpha-1} \left(\frac{2a^2z-\tau}{2a^2(t-\tau)z-t\tau}\right)^\alpha \frac{\left(2a^2z-\tau\right)\cdot 2a^2\tau^2}{\tau^2\left(2a^2z-\tau\right)^2} \cdot e^{-z} \cdot I_\nu(z)dz = \\
&= \frac{C(\lambda,p_\lambda)\, t^\alpha}{2a^2\tau} \int_{\frac{t\tau}{2a^2(t-\tau)}}^\infty \left(\frac{2a^2\tau z}{2a^2(t-\tau)z-t\tau}\right)^\alpha \cdot \frac{1}{z} \cdot e^{-z} \cdot I_\nu(z)dz = \\
&= \frac{C(\lambda,p_\lambda)\, t^\alpha}{2a^2\tau^{1-\alpha}(t-\tau)^\alpha} \int_{\frac{t\tau}{2a^2(t-\tau)}}^\infty \left(\frac{1}{z-\frac{t\tau}{2a^2(t-\tau)}}\right)^\alpha \cdot \frac{1}{z^{1-\alpha}} \cdot e^{-z} \cdot I_\nu(z)dz.
\end{aligned}$$

By making the substitution

$$z - \frac{t\tau}{2a^2(t-\tau)} = \eta$$

and using the inequality

$$e^{-z} \cdot I_\nu(z) \leq D \cdot \frac{z^\nu}{(1+z)^{\frac{1}{2}+\nu}},$$

we obtain:

$$\begin{aligned}
Q(t,\tau) &\leq \frac{C(\lambda,p_\lambda)\cdot Dt^\alpha}{2a^2\tau^{1-\alpha}(t-\tau)^\alpha} \int_0^\infty \left(\frac{1}{\eta}\right)^\alpha \cdot \frac{1}{\left(\eta+\frac{t\tau}{2a^2(t-\tau)}\right)^{1-\alpha-\nu}} \cdot \frac{1}{\left(1+\eta+\frac{t\tau}{2a^2(t-\tau)}\right)^{\frac{1}{2}+\nu}} d\eta = \\
&\leq \frac{C(\lambda,p_\lambda)\cdot Dt^\alpha}{2a^2\tau^{1-\alpha}(t-\tau)^\alpha} \cdot \int_0^\infty \frac{(\eta)^{\nu-1}}{(1+\eta)^{\frac{1}{2}+\nu}} d\eta = \frac{C(\lambda,p_\lambda)\cdot D}{2a^2} \cdot B\left(\nu,\frac{1}{2}\right) \cdot \frac{t^\alpha}{\tau^{1-\alpha}(t-\tau)^\alpha}.
\end{aligned}$$

The theorem is proved. □

Thus, by virtue of 6.1, equation (14) has a unique solution, which can be found by the method of successive approximations. The homogeneous equation corresponding to (5):

$$\mathbf{M}_\lambda \mu \equiv (I - \lambda \mathbf{M})\mu_1 \equiv \mu_1(t) - \lambda \int_0^t M(t,\tau)\mu_1(\tau)d\tau = 0, \quad t > 0 \tag{15}$$

will be equivalent to the inhomogeneous equation

$$\mathbf{K}_\lambda \mu_1 \equiv (I - \lambda \mathbf{K})\mu_1 \equiv \mu_1(t) - \lambda \int_0^t K(t,\tau) \cdot \mu_1(\tau)d\tau = C \cdot \mu_1^{(\lambda)}(t),$$

which for $\forall \lambda \geq \nu$, $\forall C = \text{const}$ has a unique solution, which is an eigenfunction of the homogeneous equation (15).

Thus, the following theorem holds.

Theorem 6.2. *In the space $M(R_+)$ equation* (5) *is solvable for any function $g_1(t) \in M(R_+)$, and its general solution has the form*

$$\mu_1(t) = [\mathbf{K}_\lambda]^{-1}\left[F_g(t)\right] + C \cdot [\mathbf{K}_\lambda]^{-1}\left[\mu_1^{(\lambda)}(t)\right], \quad C = \text{ const.}$$

The solution of the original singular integral equation (2) will be found using Remark 2.1.

7. Conclusion

Recently, the field of application of singular second-kind Volterra integral equations has significantly expanded. These equations have kernels with discontinuities or strong singularities that cannot be transformed into non-singular ones. The Picard method is not applicable to them, and the homogeneous equations can have non-zero solutions.

These equations find applications in various scientific fields, such as boundary value problems of heat conduction for regions that change over time and degenerate into a point, as well as in describing heat transfer in a moving medium where the velocity depends on the coordinates. They are relevant for new technologies in metallurgy, crystal production, laser technologies, and other fields.

Solving such equations typically presents difficulties when using traditional methods. Therefore, the study of methods for solving singular Volterra integral equations is important. Finding the eigenfunctions of these equations will allow for a more accurate determination of the classes of well-posedness of the corresponding boundary value problems.

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