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# Bott-Duffin core inverse

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**Abstract.** The paper focuses on a new generalized inverse, Bott-Duffin core inverse, which is a generalization of the Bott-Duffin inverse. Several properties, characterizations and representations of Bott-Duffin core inverse are presented. We discuss the constrained matrix approximation problem in the Frobenius norm by using the Bott-Duffin core inverse.

## 1. Introduction

In this paper, $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices. If $L$ is a subspace of $\mathbb{C}^n$, we use the notation $L \leqslant \mathbb{C}^n$. Bott and Duffin, in their famous paper [3], introduced the "constrained inverse" of a square matrix as an important tool in the electrical network theory. This inverse is called in their honor the Bott-Duffin inverse in [2]. Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and let $P_L$ be the orthogonal projection on $L$. If $AP_L + P_{L^\perp}$ is nonsingular, then the Bott-Duffin inverse of $A$ with respect to $L$, denoted by $A^{(-1)}_{(L)}$, is defined by $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1}$.

In [4], Chen defined the generalized Bott-Duffin inverse of $A$ (denoted by $A^{(\dagger)}_{(L)}$) . It is particularly worth noting that the form of definition, $A^{(\dagger)}_{(L)} = P_L(AP_L + P_{L^\perp})^{\dagger}$, is a natural extension of $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1}$. It is interesting to consider that if $AP_L + P_{L^\perp}$ is core invertible, can a new generalized inverse be formed?

Let $\mathbb{C}^{CM}_n$ be the set of $n \times n$ matrices of index one, that is,

$$\mathbb{C}^{CM}_n = \left\{ A \in \mathbb{C}^{n\times n} \middle| \operatorname{rank}(A^2) = \operatorname{rank}(A) \right\}.$$

Let us recall that $A \in \mathbb{C}^{n\times n}$ has the core inverse if and only if $A \in \mathbb{C}^{CM}_n$. For the convenience of describing the article, we first provide the following definition:

**Definition 1.1.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$. If $(AP_L + P_{L^\perp}) \in \mathbb{C}^{CM}_n$, then*

$$A^{(\#)}_{(L)} = P_L(AP_L + P_{L^\perp})^{\#}, \tag{1}$$

*is called the Bott-Duffin core inverse of $A$ with respect to $L$.*

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Our main contributions can be summarized as below:

(1) We give some properties of Bott-Duffin core inverse, especially showing Bott-Duffin core inverse is an outer inverse with prescribed range and null space.

(2) Some characterizations of Bott-Duffin core inverse are provided by using range space, projections, matrix equations and EP-property.

(3) Though a appropriate matrix decomposition, we conclude the explicit representations of Bott-Duffin core inverse. Moreover, we give the limit expression for Bott-Duffin core inverse.

(4) We study the constrained matrix approximation problem in the Frobenius norm by using the Bott-Duffin core inverse. Moreover, we give the unique solution to two classes of matrix equation, and provide a Cramer's rule for the unique solution.

This paper is organized as follows. In Section 2, we introduce some necessary notations, definitions and lemmas. In Section 3, we give some properties of Bott-Duffin core inverse. In Section 4, we present several characterizations of the Bott-Duffin core inverse in terms of range space, projections, matrix equations and EP-property. In Section 5, some representations of the Bott-Duffin core inverse are provided. The applications of Bott-Duffin core inverse in solving two classes of matrix equation are given in Section 6.

## 2. Notations and Preliminaries

The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, $A^*$, $A^T$ and $\operatorname{rank}(A)$ represent the range space, null space, conjugate transpose, transpose and rank of $A \in \mathbb{C}^{m\times n}$, respectively. We denote the identity matrix in $\mathbb{C}^{n\times n}$ by $I_n$. The symbol $O$ stands for the null matrix. $L^\perp$ means the orthogonal complement subspace of $L$. The dimension of $L$ is denoted by $\dim(L)$. $P_{L,M}$ stands for the oblique projection onto $L$ along $M$, where $L, M \leqslant \mathbb{C}^n$ and $L \oplus M = \mathbb{C}^n$.

Additionally, the Moore–Penrose inverse $A^\dagger \in \mathbb{C}^{n\times m}$ of $A \in \mathbb{C}^{m\times n}$ is the unique matrix verifying the following matrix equations (see [2, 9, 11])

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies $XAX = X$ is called an outer inverse of $A$ and is denoted by $A^{(2)}$. Let $L \leqslant \mathbb{C}^n$, $\dim L = l \leqslant \operatorname{rank}(A)$ and $S \leqslant \mathbb{C}^m$, $\dim S = m - l$. There exists a unique outer inverse $X$ of $A$ such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$. In this case, the matrix $X$ is called the outer inverse with prescribed range and null space and is denoted by $A^{(2)}_{T,S}$" (see [2, 11]).

The group inverse of $A \in \mathbb{C}^{CM}_n$ is the unique matrix $A^\# \in \mathbb{C}^{n\times n}$ verifying the following matrix equations (see [2, 8, 11])

$$AA^\# A = A,\ A^\# AA^\# = A^\#,\ AA^\# = A^\# A. \tag{2}$$

For a given matrix $A \in \mathbb{C}^{CM}_n$, the core inverse of $A$ is defined to be the unique matrix $A^{\text{\textcircled{\#}}} \in \mathbb{C}^{n\times n}$ satisfying (see [1])

$$AA^{\text{\textcircled{\#}}} = AA^\dagger,\ \mathcal{R}(A^{\text{\textcircled{\#}}}) \subset \mathcal{R}(A). \tag{3}$$

Moreover, Wang and Liu [12] prove that the core inverse of $A \in \mathbb{C}^{CM}_n$ is the unique matrix satisfying

$$AA^{\text{\textcircled{\#}}}A = A,\ A(A^{\text{\textcircled{\#}}})^2 = A^{\text{\textcircled{\#}}},\ (AA^{\text{\textcircled{\#}}})^* = AA^{\text{\textcircled{\#}}}. \tag{4}$$

Henceforth, the symbol $\mathbb{C}^{EP}_n$ will stand for the set of $n \times n$ EP matrices, i.e.

$$\mathbb{C}^{EP}_n = \{A \mid A \in \mathbb{C}^{n\times n}, AA^\dagger = A^\dagger A\} = \{A \mid A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\}.$$

**Lemma 2.1.** [10] *Let $A \in \mathbb{C}_n^{CM}$. Then:*

*(a)* $A^{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$;

*(b)* $A^{\textcircled{\#}}A = A^{\#}A = AA^{\#} = P_{\mathcal{R}(A),\mathcal{N}(A)}$.

**Lemma 2.2.** *Let $A \in \mathbb{C}_n^{CM}$ and let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix. Then,*

*(i)* $(UAU^*)^{\#} = UA^{\#}U^*$;

*(ii)* $(UAU^*)^{\textcircled{\#}} = UA^{\textcircled{\#}}U^*$.

*Proof.* (*i*). It can be verified directly by (2).

(*ii*). From [1, Theorem 1], we have

$$A^{\textcircled{\#}} = A^{\#}AA^{\dagger}. \tag{5}$$

Then, $(UAU^*)^{\textcircled{\#}} = (UAU^*)^{\#}UAU^*(UAU^*)^{\dagger} = UA^{\#}U^*UAU^*UA^{\dagger}U^* = UA^{\textcircled{\#}}U^*$. □

**Lemma 2.3.** [4, Lemma 1] *For any $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$, we have*

$$\mathcal{R}(AP_L + P_{L^\perp}) = \mathcal{R}(P_LAP_L + P_{L^\perp}) = AL + L^\perp = P_LAL \oplus L^\perp,$$

*and*

$$\mathcal{N}(P_LA + P_{L^\perp}) = \mathcal{N}(P_LAP_L + P_{L^\perp}) = (A^*L)^\perp \cap L = \mathcal{N}(P_LAP_L) \cap L.$$

**Lemma 2.4.** [11, Theorem 1.3.2] *Let $P_1$ be the projection on $R_1$ along $N_1$, $P_2$ the projection on $R_2$ along $N_2$, then $P = P_1 + P_2$ is a projection if and only if*

$$P_1P_2 = P_2P_1 = O.$$

*In this case, $P$ is a projection on $R = R_1 \oplus R_2 = \mathcal{R}(P_1) \oplus \mathcal{R}(P_2)$ along $N = N_1 \cap N_2 = \mathcal{N}(P_1) \cap \mathcal{N}(P_2)$.*

**Lemma 2.5.** [6, Theorem 1] *Let $T = \begin{bmatrix} A & O \\ B & D \end{bmatrix}$ be a partitioned matrix of $\mathbb{C}^{m\times m}$, where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{(m-n)\times n}$ and $D \in \mathbb{C}^{(m-n)\times(m-n)}$. Suppose $A^{\textcircled{\#}}$ and $D^{\textcircled{\#}}$ exist. Set $E_D = I_{m-n} - DD^{\dagger}$ and $F_A = I_n - A^{\dagger}A$. Then, $T^{\textcircled{\#}}$ exists if and only if $E_DBF_A = O$. In this case*

$$T^{\textcircled{\#}} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

*where*

$$\begin{aligned}
T_{11} &= A^{\textcircled{\#}}\Big(I + (E_DBA^{\dagger})^*E_DBA^{\dagger}\Big)^{-1}, \\
T_{12} &= A^{\textcircled{\#}}\Big(E_DBA^{\dagger}\Big)^*\Big(I + E_DBA^{\dagger}\Big(E_DBA^{\dagger}\Big)^*\Big)^{-1}, \\
T_{21} &= \Big(\Big(I - DD^{\#}\Big)BA^{\dagger} - D^{\textcircled{\#}}B\Big)A^{\textcircled{\#}}\Big(I + \Big(E_DBA^{\dagger}\Big)^*E_DBA^{\dagger}\Big)^{-1}, \\
T_{22} &= D^{\textcircled{\#}} + \Big(\Big(I - DD^{\#}\Big)BA^{\dagger} - D^{\textcircled{\#}}B\Big)A^{\textcircled{\#}}\Big(E_DBA^{\dagger}\Big)^*\Big(I + E_DBA^{\dagger}\Big(E_DBA^{\dagger}\Big)^*\Big)^{-1}.
\end{aligned}$$

**Lemma 2.6.** [6, Theorem 2] *Let $S = \begin{bmatrix} A & C \\ O & D \end{bmatrix}$ be a partitioned matrix of $\mathbb{C}^{m\times m}$, where $A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{n\times(m-n)}$ and $D \in \mathbb{C}^{(m-n)\times(m-n)}$. Suppose $A^{\textcircled{\#}}$ and $D^{\textcircled{\#}}$ exist. Set $E_A = I_n - AA^{\dagger}$ and $F_D = I_{m-n} - D^{\dagger}D$. Then, $S^{\textcircled{\#}}$ exists if and only if $E_ACF_D = O$. In this case*

$$S^{\textcircled{\#}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

*where*

$$\begin{aligned}
S_{11} &= A^{\textcircled{\#}} + \left(\left(I - AA^{\#}\right)CD^{\dagger} - A^{\textcircled{\#}}C\right)D^{\textcircled{\#}}\left(I + \left(E_ACD^{\dagger}\right)^{*}E_ACD^{\dagger}\right)^{-1}\left(E_ACD^{\dagger}\right)^{*},\\
S_{12} &= \left(\left(I - AA^{\#}\right)CD^{\dagger} - A^{\textcircled{\#}}C\right)D^{\textcircled{\#}}\left(I + \left(E_ACD^{\dagger}\right)^{*}E_ACD^{\dagger}\right)^{-1},\\
S_{21} &= D^{\textcircled{\#}}\left(I + (E_ACD^{\dagger})^{*}E_ACD^{\dagger}\right)^{-1}\left(E_ACD^{\dagger}\right)^{*},\\
S_{22} &= D^{\textcircled{\#}}\left(I + \left(E_ACD^{\dagger}\right)^{*}E_ACD^{\dagger}\right)^{-1}.
\end{aligned}$$

## 3. The properties of Bott-Duffin core inverse

Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$. In order to discuss some properties of the Bott-Duffin core inverse, we will consider an appropriate matrix decomposition of $A$ with respect to $L$. Since there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$P_L = U\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^*, \tag{6}$$

where $l = \dim(L)$. On the basis of (6), the decomposition of $P_L$, a matrix $A$ can be written as

$$A = U\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}U^*, \tag{7}$$

where $A_L \in \mathbb{C}^{l\times l}$, $B_L \in \mathbb{C}^{l\times(n-l)}$, $C_L \in \mathbb{C}^{(n-l)\times l}$, $D_L \in \mathbb{C}^{(n-l)\times(n-l)}$.

Using this decomposition, we give the necessary and sufficient condition for the existence of $A_{(L)}^{(\textcircled{\#})}$ as well as the representation of $A_{(L)}^{(\textcircled{\#})}$.

**Theorem 3.1.** *Let $P_L$ and $A$ be given by* (6) *and* (7)*, respectively. Then $A_{(L)}^{(\textcircled{\#})}$ exists if and only if $A_L \in \mathbb{C}_l^{CM}$. In this case,*

$$A_{(L)}^{(\textcircled{\#})} = U\begin{bmatrix} A_L{}^{\textcircled{\#}} & O \\ O & O \end{bmatrix}U^*. \tag{8}$$

*Proof.* From (6) and (7), we have

$$AP_L + P_{L^\perp} = U\begin{bmatrix} A_L & O \\ C_L & I_{n-l} \end{bmatrix}U^*. \tag{9}$$

In [1], Baksalary pointed out that $(AP_L + P_{L^\perp})^{\textcircled{\#}}$ exists if and only if $(AP_L + P_{L^\perp}) \in \mathbb{C}_n^{CM}$. Using (9), we can verify $(AP_L + P_{L^\perp}) \in \mathbb{C}_n^{CM}$ if and only if $A_L \in \mathbb{C}_n^{CM}$. From Lemma 2.2, (9) and Lemma 2.5, we have

$$(AP_L + P_{L^\perp})^{\textcircled{\#}} = U\begin{bmatrix} A_L{}^{\textcircled{\#}} & O \\ -C_LA_L{}^{\textcircled{\#}} & I_{n-l} \end{bmatrix}U^*.$$

By using (1), we can get (8). □

The basic properties of $A_{(L)}^{(\textcircled{\#})}$ are given in the following theorem.

**Theorem 3.2.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$. If $(AP_L + P_{L^\perp}) \in \mathbb{C}_n^{CM}$, then the following statements hold:*

*(i)* $A_{(L)}^{(\textcircled{\#})} = P_LA_{(L)}^{(\textcircled{\#})} = A_{(L)}^{(\textcircled{\#})}P_L = P_LA_{(L)}^{(\textcircled{\#})}P_L$;

*(ii)* $\mathcal{R}(A_{(L)}^{(\#)}) = T$ *and* $\mathcal{N}(A_{(L)}^{(\#)}) = T^{\perp}$;

*(iii)* $A_{(L)}^{(\#)} A A_{(L)}^{(\#)} = A_{(L)}^{(\#)}$;

*(iv)* $AA_{(L)}^{(\#)} = P_{AT,T^{\perp}}$ *and* $A_{(L)}^{(\#)} A = P_{T,(A^*T)^{\perp}}$;

*(v)* $P_L A A_{(L)}^{(\#)} = P_T$ *and* $A_{(L)}^{(\#)} A P_L = P_{T,S}$;

*(vi)* $A_{(L)}^{(\#)} = P_T A_{(L)}^{(\#)} = A_{(L)}^{(\#)} P_T = P_{T,S} A_{(L)}^{(\#)}$;

*(vii)* $P_T(A - AA_{(L)}^{(\#)} A) = (A - AA_{(L)}^{(\#)} A) P_{T,S} = O$;

*(viii)* $A_{(L)}^{(\#)} = A_{T,T^{\perp}}^{(2)} = (AP_L)_{T,T^{\perp}}^{(2)} = (P_L A)_{T,T^{\perp}}^{(2)}$;

*(ix)* $A_{(L)}^{(\#)} = (P_L A P_L)^{\#}$.

*Proof.* (*i*). From (1), multiplying $A_{(L)}^{(\#)} = P_L(AP_L + P_{L^{\perp}})^{\#}$ by $P_L$ from the left, we have $P_L A_{(L)}^{(\#)} = A_{(L)}^{(\#)}$. By Lemma 2.3, we get $L^{\perp} \subset \mathcal{R}(AP_L + P_{L^{\perp}})$, then it follows that $P_{\mathcal{R}(AP_L+P_{L^{\perp}}),\mathcal{N}(AP_L+P_{L^{\perp}})} P_{L^{\perp}} = P_{L^{\perp}}$. Note the fact that

$$\begin{aligned} P_L(AP_L + P_{L^{\perp}})^{\#} P_{L^{\perp}} &= P_L(AP_L + P_{L^{\perp}})^{\#}(AP_L + P_{L^{\perp}})P_{L^{\perp}} \\ &= P_L(AP_L + P_{L^{\perp}})(AP_L + P_{L^{\perp}})^{\#} P_{L^{\perp}} \\ &= P_L P_{\mathcal{R}(AP_L+P_{L^{\perp}}),\mathcal{N}(AP_L+P_{L^{\perp}})} P_{L^{\perp}} \\ &= P_L P_{L^{\perp}} \\ &= O. \end{aligned}$$

Then, by (1),

$$\begin{aligned} A_{(L)}^{(\#)} P_L &= P_L(AP_L + P_{L^{\perp}})^{\#} P_L + P_L(AP_L + P_{L^{\perp}})^{\#} P_{L^{\perp}} \\ &= P_L(AP_L + P_{L^{\perp}})^{\#}(P_L + P_{L^{\perp}}) \\ &= A_{(L)}^{(\#)}. \end{aligned}$$

Consequently,

$$P_L A_{(L)}^{(\#)} P_L = P_L A_{(L)}^{(\#)} = A_{(L)}^{(\#)}.$$

(*ii*). It follows from (1) and Lemma 2.1 (a) that $\mathcal{R}(A_{(L)}^{(\#)}) = \mathcal{R}(P_L(AP_L + P_{L^{\perp}})^{\#}) = P_L\mathcal{R}((AP_L + P_{L^{\perp}})^{\#}) = P_L\mathcal{R}(AP_L + P_{L^{\perp}}) = \mathcal{R}(P_L A P_L) = T$. By (1) and (4), we have

$$\begin{aligned} \mathcal{N}(A_{(L)}^{(\#)}) = \mathcal{N}(P_L(AP_L + P_{L^{\perp}})^{\#}) \subset \mathcal{N}(P_L A P_L(AP_L + P_{L^{\perp}})^{\#}) &= \mathcal{N}(P_L(AP_L + P_{L^{\perp}})(AP_L + P_{L^{\perp}})^{\#}) \\ &= [\mathcal{R}((AP_L + P_{L^{\perp}})(AP_L + P_{L^{\perp}})^{\#} P_L)]^{\perp}. \end{aligned}$$

From (3), Lemma 2.3 and Lemma 2.4, we get $(AP_L + P_{L^{\perp}})(AP_L + P_{L^{\perp}})^{\#} = P_{\mathcal{R}(AP_L+P_{L^{\perp}})} = P_{T\oplus L^{\perp}} = P_T + P_{L^{\perp}}$. Since $P_T P_L = P_T$, it follows that $\mathcal{N}(A_{(L)}^{(\#)}) \subset [\mathcal{R}((AP_L + P_{L^{\perp}})(AP_L + P_{L^{\perp}})^{\#} P_L)]^{\perp} = [\mathcal{R}((P_T + P_{L^{\perp}})P_L)]^{\perp} = T^{\perp}$. Since $\dim(\mathcal{N}(A_{(L)}^{(\#)})) = n - \dim(\mathcal{R}(A_{(L)}^{(\#)})) = n - \dim(T) = \dim(T^{\perp})$, we can obtain $\mathcal{N}(A_{(L)}^{(\#)}) = T^{\perp}$.

(*iii*). From the proof of (*i*), we have $P_L(AP_L + P_{L^\perp})^{\textcircled{\#}} P_{L^\perp} = O$. Thus,

$$\begin{aligned} A_{(L)}^{(\textcircled{\#})} A A_{(L)}^{(\textcircled{\#})} &= P_L\,(AP_L + P_{L^\perp})^{\textcircled{\#}}\, AP_L\,(AP_L + P_{L^\perp})^{\textcircled{\#}} \\ &= P_L\,(AP_L + P_{L^\perp})^{\textcircled{\#}}\,(AP_L + P_{L^\perp})\,(AP_L + P_{L^\perp})^{\textcircled{\#}} \\ &= P_L\,(AP_L + P_{L^\perp})^{\textcircled{\#}} \\ &= A_{(L)}^{(\textcircled{\#})}. \end{aligned}$$

(*iv*). From (*ii*) and (*iii*), note the facts that $\mathcal{R}(AA_{(L)}^{(\textcircled{\#})}) = A\mathcal{R}(A_{(L)}^{(\textcircled{\#})}) = AT$, $\mathcal{N}(AA_{(L)}^{(\textcircled{\#})}) = \mathcal{N}(A_{(L)}^{(\textcircled{\#})}) = T^\perp$ and $AA_{(L)}^{(\textcircled{\#})}AA_{(L)}^{(\textcircled{\#})} = AA_{(L)}^{(\textcircled{\#})}$, we have $AA_{(L)}^{(\textcircled{\#})} = P_{AT,T^\perp}$. The proof of $A_{(L)}^{(\textcircled{\#})}A = P_{T,(A^*T)^\perp}$ is similar.

(*v*). Since $P_LP_T = P_T$, premultiplying the equation $(AP_L + P_{L^\perp})(AP_L + P_{L^\perp})^{\textcircled{\#}} = P_T + P_{L^\perp}$ with $P_L$ gives

$$P_LAP_L(AP_L + P_{L^\perp})^{\textcircled{\#}} = P_T.$$

Thus, $P_LAA_{(L)}^{(\textcircled{\#})} = P_T$. By (*i*), (*ii*), (1), Lemma 2.1 (b) and Lemma 2.3,

$$\begin{aligned} A_{(L)}^{(\textcircled{\#})}AP_LA_{(L)}^{(\textcircled{\#})}AP_L &= P_L(AP_L + P_{L^\perp})^{\textcircled{\#}}(AP_L + P_{L^\perp})P_L(AP_L + P_{L^\perp})^{\textcircled{\#}}AP_L \\ &= P_L(AP_L + P_{L^\perp})^{\#}(AP_L + P_{L^\perp})A_{(L)}^{(\textcircled{\#})}AP_L \\ &= P_LP_{\mathcal{R}(AP_L+P_{L^\perp}),\mathcal{N}(AP_L+P_{L^\perp})}A_{(L)}^{(\textcircled{\#})}AP_L \\ &= P_LP_{(P_LAL\oplus L^\perp),\mathcal{N}(AP_L+P_{L^\perp})}A_{(L)}^{(\textcircled{\#})}AP_L \\ &= P_LA_{(L)}^{(\textcircled{\#})}AP_L \\ &= A_{(L)}^{(\textcircled{\#})}AP_L. \end{aligned}$$

We can also derive from the above equation that $A_{(L)}^{(\textcircled{\#})}AA_{(L)}^{(\textcircled{\#})} = A_{(L)}^{(\textcircled{\#})}$. Then $\mathcal{R}(A_{(L)}^{(\textcircled{\#})}) = \mathcal{R}(A_{(L)}^{(\textcircled{\#})}AP_LA_{(L)}^{(\textcircled{\#})}) \subset \mathcal{R}(A_{(L)}^{(\textcircled{\#})}AP_L)$. It is clear that $\mathcal{R}(A_{(L)}^{(\textcircled{\#})}AP_L) \subset \mathcal{R}(A_{(L)}^{(\textcircled{\#})})$. By (*ii*), $\mathcal{R}(A_{(L)}^{(\textcircled{\#})}AP_L) = \mathcal{R}(A_{(L)}^{(\textcircled{\#})}) = T$. Since $S \subset \mathcal{N}(A_{(L)}^{(\textcircled{\#})}P_LAP_L) = \mathcal{N}(A_{(L)}^{(\textcircled{\#})}AP_L)$ and $\text{rank}(A_{(L)}^{(\textcircled{\#})}AP_L) = \text{rank}(P_LAP_L)$, it follows that $\mathcal{N}(A_{(L)}^{(\textcircled{\#})}AP_L) = S$.

(*vi*). From (*i*) and (*v*), (*vi*) can be directly derived.

(*vii*). By (*v*), (*vii*) can be directly derived.

(*viii*). By (*ii*) and (*iii*), we have $A_{(L)}^{(\textcircled{\#})} = A_{T,T^\perp}^{(2)}$. From (*i*), we have $A_{(L)}^{(\textcircled{\#})}AP_LA_{(L)}^{(\textcircled{\#})} = A_{(L)}^{(\textcircled{\#})}$. Thus, $A_{(L)}^{(\textcircled{\#})} = (AP_L)_{T,T^\perp}^{(2)}$. The proof of $A_{(L)}^{(\textcircled{\#})} = (P_LA)_{T,T^\perp}^{(2)}$ is similar.

(*ix*). From (*i*) and (*v*), we get $P_LAP_LA_{(L)}^{(\textcircled{\#})}P_LAP_L = P_TAP_L$. Since $P_T = P_TP_L$ and $T = \mathcal{R}(P_LAP_L)$, $P_TAP_L = P_TP_LAP_L = P_LAP_L$. By (*i*), (*v*) and (*vi*), $P_LAP_L(A_{(L)}^{(\textcircled{\#})})^2 = P_TA_{(L)}^{(\textcircled{\#})} = A_{(L)}^{(\textcircled{\#})}$. In terms of (*i*) and (*v*), we can obtain $(P_LAP_LA_{(L)}^{(\textcircled{\#})})^* = P_LAP_LA_{(L)}^{(\textcircled{\#})}$. It follows from (4) that $A_{(L)}^{(\textcircled{\#})} = (P_LAP_L)^{\textcircled{\#}}$. □

## 4. Some characterizations of the Bott-Duffin core inverse

In this section, we provide several characterizations of the BD-inverse core inverse of $A \in \mathbb{C}^{n\times n}$ (in the case when it exists) mainly in terms of range space, projections, matrix equations and EP-property. In the following theorem, using Theorem 3.2 (*ii*), we present some characterizations of Bott-Duffin core inverse.

**Theorem 4.1.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$ be such that $A_{(L)}^{(\textcircled{\#})}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

*(a)* $X = A_{(L)}^{(\textcircled{\#})}$;

*(b)* $\mathcal{R}(X) = T$ *and* $AX = P_{AT,T^{\perp}}$*;*

*(c)* $\mathcal{R}(X) = T$ *and* $P_L AX = P_T$*;*

*(d)* $\mathcal{R}(X) = T$*,* $XAX = X$ *and* $XP_T = X$*;*

*(e)* $\mathcal{R}(X) = T$*,* $XA = P_{T,(A^*T)^{\perp}}$ *and* $XP_T = X$*.*

*Proof.* $(a) \Rightarrow (b)$. This follows directly by Theorem 3.2 (*ii*) and (*iv*).

$(b) \Rightarrow (c)$. From Theorem 3.2 (*iv*) and (*v*), we have $P_L AX = P_L P_{AT,T^{\perp}} = P_T$.

$(c) \Rightarrow (d)$. Since $\mathcal{R}(X) = T$ and $P_L AX = P_T$, we have $\mathrm{rank}(X) = \dim(T)$ and $\mathcal{N}(X) = T^{\perp}$, which implies $XP_T = X$. From $L^{\perp} \subset T^{\perp}$, we get $XP_L = X$. Thus $XAX = XP_L AX = XP_T = X$.

$(d) \Rightarrow (e)$. From $XAX = X$, it is clear that $XAXA = XA$. Since $\mathcal{R}(X) = T$ and $XAX = X$, it follows that $\mathcal{R}(XA) = \mathcal{R}(X) = T$. By $\mathcal{R}(X) = T$ and $XP_T = X$, we get $\mathrm{rank}(X) = \dim(T)$ and $T^{\perp} \subset \mathcal{N}(X)$, which implies $\mathcal{N}(X) = T^{\perp}$. It follows from $\mathcal{N}(X) = T^{\perp}$ that $\mathcal{N}(XA) = [\mathcal{R}(XA)^*]^{\perp} = (A^*\mathcal{N}(X)^{\perp})^{\perp} = (A^*T)^{\perp}$. Thus, $XA = P_{T,(A^*T)^{\perp}}$.

$(e) \Rightarrow (a)$. Since $\mathcal{R}(X) = T$ and $XA = P_{T,(A^*T)^{\perp}}$, it follows that $XAX = X$. From $\mathrm{rank}(X) = \dim(T)$ and $XP_T = X$, we have $\mathcal{N}(X) = T^{\perp}$. Thus by Theorem 3.2 (*viii*), we get $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$. □

**Remark 4.2.** *Let* $A \in \mathbb{C}^{n\times n}$*,* $L \leqslant \mathbb{C}^n$*,* $T = \mathcal{R}(P_L AP_L)$ *and* $S = \mathcal{N}(P_L AP_L)$ *be such that* $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ *exists and let* $X \in \mathbb{C}^{n\times n}$*. The following statements are equivalent:*

*(a)* $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$*;*

*(b)* $\mathcal{R}(X^*) = T$ *and* $XA = P_{T,(A^*T)^{\perp}}$*;*

*(c)* $\mathcal{R}(X^*) = T$ *and* $XAP_L = P_{T,S}$*;*

*(d)* $\mathcal{R}(X^*) = T$*,* $XAX = X$ *and* $P_T X = X$*;*

*(e)* $\mathcal{R}(X^*) = T$*,* $AX = P_{AT,T^{\perp}}$ *and* $P_T X = X$*.*

*Proof.* It is similar to the proof of the Theorem 4.1. We only provide the proof of $(e) \Rightarrow (a)$.

$(e) \Rightarrow (a)$. It is well known that $\mathcal{R}(X^*) = T$ if and only if $\mathcal{N}(X) = T^{\perp}$. Since $\mathcal{N}(X) = T^{\perp}$ and $AX = P_{AT,T^{\perp}}$, it follows that $XAX = X$. From $\mathrm{rank}(X) = \dim(T)$ and $P_T X = X$, we have $\mathcal{R}(X) = T$. Therefore, by Theorem 3.2 (*viii*), we get $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$. □

By Theorem 3.2, we know that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ is an outer inverse of $A$. Using this property, some characterizations of $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ are given in the following theorem.

**Theorem 4.3.** *Let* $A \in \mathbb{C}^{n\times n}$*,* $L \leqslant \mathbb{C}^n$*,* $T = \mathcal{R}(P_L AP_L)$ *and* $S = \mathcal{N}(P_L AP_L)$ *be such that* $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ *exists and let* $X \in \mathbb{C}^{n\times n}$*. The following statements are equivalent:*

*(a)* $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$*;*

*(b)* $XAX = X$*,* $XP_T = X$ *and* $XA = P_{T,(A^*T)^{\perp}}$*;*

*(c)* $XAX = X$*,* $XAP_L = P_{T,S}$ *and* $AX = P_{AT,T^{\perp}}$*;*

*(d)* $XAX = X$*,* $P_T X = X$ *and* $AX = P_{AT,T^{\perp}}$*;*

*(e)* $XAX = X$*,* $P_L AX = P_T$ *and* $XA = P_{T,(A^*T)^{\perp}}$*;*

*(f)* $XAX = X$*,* $P_T XP_T = X$ *and* $\mathrm{rank}(X) = \dim(T)$*.*

*Proof.* ($a$) $\Rightarrow$ ($b$). This follows directly by Theorem 3.2 ($iii$), ($vi$) and ($iv$).

($b$) $\Rightarrow$ ($c$). From Theorem 3.2 ($iv$) and ($v$) , we have $XAP_L = P_{T,(A^*T)^\perp}P_L = P_{T,S}$. It follows from $XAX = X$ that $AXAX = AX$, $\mathcal{R}(XA) = \mathcal{R}(X)$ and $\mathcal{N}(AX) = \mathcal{N}(X)$. In terms of $XA = P_{T,(A^*T)^\perp}$, we have $\mathcal{R}(X) = T$, which implies $\mathcal{R}(AX) = AT$. From $XP_T = X$ and $\mathcal{R}(X) = T$, we get $\mathcal{N}(X) = T^\perp$. Thus, $AX = P_{AT,T^\perp}$.

($c$) $\Rightarrow$ ($d$). From $XAX = X$ and $AX = P_{AT,T^\perp}$, we have $\mathcal{N}(X) = \mathcal{N}(AX) = T^\perp$, which implies $\mathrm{rank}(X) = \dim(T)$. Since $XAX = X$ and $XAP_L = P_{T,S}$, it follows that $T \subset \mathcal{R}(XA) = \mathcal{R}(X)$. Therefore, $\mathcal{R}(X) = T$ holds , which means $P_TX = X$.

($d$) $\Rightarrow$ ($e$). Similar to ($b$) $\Rightarrow$ ($c$).

($e$) $\Rightarrow$ ($f$). From $XAX = X$ and $XA = P_{T,(A^*T)^\perp}$, we have $\mathcal{R}(X) = T$, which means $P_TX = X$ and $\mathrm{rank}(X) = \dim(T)$. It follows from $P_LAX = P_T$ and $\mathrm{rank}(X) = \dim(T)$ that $\mathcal{N}(X) = L^\perp$ implies $XP_T = X$. Thus, $P_TXP_T = X$.

($f$) $\Rightarrow$ ($a$). In terms of $P_TXP_T = X$ and $\mathrm{rank}(X) = \dim(T)$, it clear that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = T^\perp$. By Theorem 3.2 ($viii$), we can obtain $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$. $\square$

From Theorem 3.2 ($iv$), we have

$$X = A^{(\#\!\!\!\!\bigcirc)}_{(L)} \Rightarrow AX = P_{AT,T^\perp}\ ,\ XA = P_{T,(A^*T)^\perp}. \tag{10}$$

It is interesting to remark that the reverse of (10) is invalid as will be illustrated in the following example.

**Example 4.4.** *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, L = \mathcal{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X = \begin{bmatrix} \frac{8}{19} & \frac{11}{19} & -\frac{9}{19} & 0 \\ \frac{1}{19} & -\frac{1}{19} & \frac{6}{19} & 0 \\ \frac{21}{19} & \frac{36}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

*Then, we have*

$$P_{AT,T^\perp} = \begin{bmatrix} \frac{10}{19} & \frac{9}{19} & \frac{3}{19} & 0 \\ \frac{9}{19} & \frac{10}{19} & -\frac{3}{19} & 0 \\ \frac{3}{19} & -\frac{3}{19} & \frac{18}{19} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, P_{T,(A^*T)^\perp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{(\#\!\!\!\!\bigcirc)}_{(L)} = \begin{bmatrix} \frac{8}{19} & \frac{11}{19} & -\frac{9}{19} & 0 \\ \frac{1}{19} & -\frac{1}{19} & \frac{6}{19} & 0 \\ \frac{21}{19} & \frac{36}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

*We can directly verify $AX = P_{AT,T^\perp}$ and $XA = P_{T,(A^*T)^\perp}$, but $X \neq A^{(\#\!\!\!\!\bigcirc)}_{(L)}$.*

In the following theorem, we add other conditions in $AX = P_{AT,T^\perp}$ and $XA = P_{T,(A^*T)^\perp}$ to characterize the Bott-Duffin core inverse.

**Theorem 4.5.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ be such that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

*(a)* $X = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$;

*(b)* $AX = P_{AT,T^\perp}$, $XA = P_{T,(A^*T)^\perp}$ *and* $XAX = X$;

*(c)* $AX = P_{AT,T^\perp}$, $XA = P_{T,(A^*T)^\perp}$ *and* $\mathrm{rank}(X) = \dim(T)$;

*(d)* $AX = P_{AT,T^\perp}$, $XA = P_{T,(A^*T)^\perp}$ *and* $XP_T = X$;

*(e)* $AX = P_{AT,T^\perp}$, $XA = P_{T,(A^*T)^\perp}$ *and* $P_TX = X$.

*Proof.* ($a$) ⇒ ($b$). This follows directly by Theorem 3.2 ($iii$) and ($iv$).

($b$) ⇒ ($c$). From $XA = P_{T,(A^*T)^\perp}$ and $XAX = X$, we have $\mathcal{R}(X) = \mathcal{R}(XA) = T$. Thus, $\text{rank}(X) = \dim(T)$.

($c$) ⇒ ($d$). From $AX = P_{AT,T^\perp}$ and $\text{rank}(X) = \dim(T)$, we have $\mathcal{N}(X) = T^\perp$, which implies that $XP_T = X$.

($d$) ⇒ ($e$). Since $AX = P_{AT,T^\perp}$ and $XP_T = X$, it follows that $\mathcal{N}(X) = L^\perp$, which means $\text{rank}(X) = \dim(T)$. In terms of $XA = P_{T,(A^*T)^\perp}$, we have $T = \mathcal{R}(XA) \subset \mathcal{R}(X)$. Thus, $\mathcal{R}(X) = T$, it can derive $P_TX = X$.

($e$) ⇒ ($a$). It follows from $XA = P_{T,(A^*T)^\perp}$ and $P_TX = X$ that $\mathcal{R}(X) = T$, $\text{rank}(X) = \dim(T)$ and $XAX = X$. From $AX = P_{AT,T^\perp}$, we have $\mathcal{N}(X) = L^\perp$. By Theorem 3.2 ($viii$), we can obtain $X = A^{(\#)}_{(L)}$. □

Motivated by Theorem 4.5, we consider characterizing Bott-Duffin core inverse just using two conditions which are one of $AX = P_{AT,T^\perp}$ and $XA = P_{T,(A^*T)^\perp}$ and another matrix equation.

**Theorem 4.6.** *Let* $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ *and* $T = \mathcal{R}(P_LAP_L)$ *be such that* $A^{(\#)}_{(L)}$ *exists and let* $X \in \mathbb{C}^{n\times n}$*. The following statements are equivalent:*

*(a)* $X = A^{(\#)}_{(L)}$;

*(b)* $AX = P_{AT,T^\perp}$ *and* $P_TXP_T = X$;

*(c)* $AX = P_{AT,T^\perp}$ *and* $P_TAX^2P_T = X$;

*(d)* $XA = P_{T,(A^*T)^\perp}$ *and* $P_TXP_T = X$;

*(e)* $XA = P_{T,(A^*T)^\perp}$ *and* $P_TX^2AP_T = X$.

*Proof.* ($a$) ⇒ ($b$). This follows directly by Theorem 3.2 ($iv$) and ($vi$).

($b$) ⇒ ($c$). It is clear that $P_TP_{AT,T^\perp} = P_T$, then $P_TAX^2P_T = P_TXP_T = X$.

($c$) ⇒ ($a$). From $AX = P_{AT,T^\perp}$ and $P_TXP_T = X$, we have $XAX = X$ and $\mathcal{R}(X) \subset T$ which mean $\mathcal{N}(X) = \mathcal{N}(AX) = L^\perp$ and $\mathcal{R}(X) = T$. By Theorem 3.2 ($viii$), $X = A^{(\#)}_{(L)}$.

The rest of the proof follows similarly. □

Using the Theorem 3.2 ($ii$), we can conclude that $A^{(\#)}_{(L)} \in \mathbb{C}^{EP}_n$. In the following theorem, we discuss other characterizations of the Bott-Duffin core inverse.

**Theorem 4.7.** *Let* $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ *and* $S = \mathcal{N}(P_LAP_L)$ *be such that* $A^{(\#)}_{(L)}$ *exists and let* $X \in \mathbb{C}^{n\times n}$*. The following statements are equivalent:*

*(a)* $X = A^{(\#)}_{(L)}$;

*(b)* $X \in \mathbb{C}^{EP}_n$, $XA = P_{T,(A^*T)^\perp}$ *and* $P_TX = X$;

*(c)* $X \in \mathbb{C}^{EP}_n$, $XAP_L = P_{T,S}$ *and* $P_TX = X$;

*(d)* $X \in \mathbb{C}^{EP}_n$, $AX = P_{AT,T^\perp}$ *and* $XP_T = X$;

*(e)* $X \in \mathbb{C}^{EP}_n$, $P_LAX = P_T$ *and* $XP_T = X$.

*Proof.* ($a$) ⇒ ($b$). This follows directly by Theorem 3.2 ($ii$), ($iv$) and ($vi$).

($b$) ⇒ ($c$). From $T \subset L$, we have $P_LX = P_LP_TX = P_TX = X$, then $XAP_LXAP_L = XAP_L$. Since $XA = P_{T,(A^*T)^\perp}$, it follows that $\mathcal{R}(XA) \supset \mathcal{R}(XAP_L) \supset \mathcal{R}(XAP_LXA) = \mathcal{R}(XA)$. Therefore, $\mathcal{R}(XAP_L) = T$. Note the fact that $\mathcal{N}(XAP_L) = [\mathcal{R}(P_L(XA)^*)]^\perp = (P_L\mathcal{N}(XA)^\perp)^\perp = (P_LA^*T)^\perp = \mathcal{N}(P_TAP_L)$, it follows form $P_LP_T = P_T$ and $A^{(\#)}_{(L)}P_L = A^{(\#)}_{(L)}$ that $S \subset \mathcal{N}(P_TAP_L) \subset \mathcal{N}(A^{(\#)}_{(L)}AP_L) = \mathcal{N}(P_{T,S}) = S$, which means $\mathcal{N}(P_TAP_L) = S$. Thus, $XAP_L = P_{T,S}$.

$(c) \Rightarrow (d)$. Since $P_TX = X$ and $T \subset L$, multiplying $XAP_L$ by $X$ from the right, we get $XAX = X$. Then $\mathcal{N}(AX) = \mathcal{N}(X)$ and $AX$ is idempotent. From $XAP_L = P_{T,S}$ and $P_TX = X$, we get $\mathcal{R}(X) = T$. Hence $\mathcal{R}(AX) = AT$. Since $X \in \mathbb{C}_n^{EP}$ and $\mathcal{R}(X) = T$, we have $\mathcal{N}(X) = T^{\perp}$. Thus, $XP_T = X$ and $AX = P_{AT,T^{\perp}}$.

$(d) \Rightarrow (e)$. From $AX = P_{AT,T^{\perp}}$ and Theorem 3.2 $(v)$, it is clear that $P_LAX = P_T$.

$(e) \Rightarrow (a)$. Since $XP_T = X$ and $P_LAX = P_T$, it follows that $\mathcal{N}(X) = T^{\perp}$. From $X \in \mathbb{C}_n^{EP}$ and $\mathcal{N}(X) = T^{\perp}$, we have $\mathcal{R}(X) = T$. From $XP_T = X$ and $L^{\perp} \subset T^{\perp}$, multiplying $P_LAX$ by $X$ from the left, we get $XAX = X$. Thus $X = A^{(2)}_{T,T^{\perp}} = A^{(\#)}_{(L)}$. □

## 5. Different representations of the Bott-Duffin core inverse

In this section, we give some representations of the Bott-Duffin core inverse.

**Theorem 5.1.** *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$. Let $a, b \in \mathbb{C}$ be such that $ab \neq 0$. If $A^{(\#)}_{(L)}$ exists, then*

$$\begin{aligned} A^{(\#)}_{(L)} &= aP_L(aAP_L + bP_{L^{\perp}})^{\#} \\ &= aP_L(aP_LAP_L + bP_{L^{\perp}})^{\#} \\ &= a(aP_LAP_L + bP_{L^{\perp}})^{\#}P_L \\ &= a(aP_LAP_L + bP_{L^{\perp}})^{\#} - \frac{a}{b}P_{L^{\perp}}. \end{aligned}$$

*Proof.* Let $P_L$ and $A$ be given by (6) and (7), respectively. We have

$$aAP_L + bP_{L^{\perp}} = U\begin{bmatrix} aA_L & O \\ aC_L & bI_{n-l} \end{bmatrix}U^*. \tag{11}$$

Using Lemma 2.2, (11) and Lemma 2.5, it follows that

$$(aAP_L + bP_{L^{\perp}})^{\#} = U\begin{bmatrix} \frac{1}{a}A_L^{\#} & O \\ -\frac{a}{b}C_LA_L^{\#} & \frac{1}{b}I_{n-l} \end{bmatrix}U^*. \tag{12}$$

From (6), (8) and (12),

$$\begin{aligned} aP_L(aAP_L + bP_{L^{\perp}})^{\#} &= U\begin{bmatrix} aI_n & O \\ O & O \end{bmatrix}\begin{bmatrix} \frac{1}{a}A_L^{\#} & O \\ -\frac{a}{b}C_LA_L^{\#} & \frac{1}{b}I_{n-l} \end{bmatrix}U^* \\ &= U\begin{bmatrix} A_L^{\#} & O \\ O & O \end{bmatrix}U^* \\ &= A^{(\#)}_{(L)}. \end{aligned}$$

The rest of proof follows similar. □

**Theorem 5.2.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$. Let $a, b, c, d \in \mathbb{C}$ be such that $a + b \neq 0$ and $cd \neq 0$. If $A^{(\#)}_{(L)}$ exists, then*

$$\begin{aligned} A^{(\#)}_{(L)} &= (a + b)P_T(aAP_{T,S} + dP_{L^{\perp}} + bP_TAP_{T,S})^{\#} \\ &= (a + b)(aP_TA + dP_{L^{\perp}} + bP_TAP_{T,S})^{\#}P_T \\ &= c(cP_TAP_{T,S} + dP_{L^{\perp}})^{\#} - \frac{c}{d}P_{L^{\perp}}. \end{aligned}$$

*Proof.* Let $P_L$ and $A$ be given by (6) and (7), respectively. We have

$$P_T = U\begin{bmatrix} A_L A_L^{\#} & O \\ O & O \end{bmatrix} U^* \tag{13}$$

and

$$P_{T,S} = U\begin{bmatrix} A_L^{\#} A_L & O \\ O & O \end{bmatrix} U^*. \tag{14}$$

By Lemma 2.5, it follows that

$$\begin{aligned} & (a+b)P_T(aAP_{T,S} + dP_{L^\perp} + bP_TAP_{T,S})^{\#} \\ = \;& U\begin{bmatrix} (a+b)A_LA_L^{\#} & O \\ O & O \end{bmatrix}\begin{bmatrix} (a+b)A_L & O \\ aC_LA_L^{\#}A_L & dI_{n-l} \end{bmatrix}^{\#} U^* \\ = \;& U\begin{bmatrix} (a+b)A_LA_L^{\#} & O \\ O & O \end{bmatrix}\begin{bmatrix} \frac{1}{(a+b)}A_L^{\#} & O \\ -\frac{a}{d(a+b)}C_LA_L^{\#} & \frac{1}{d}I_{n-l} \end{bmatrix} U^* \\ = \;& U\begin{bmatrix} A_L^{\#} & O \\ O & O \end{bmatrix} U^* = A_{(L)}^{(\#)}. \end{aligned}$$

Similar, from Lemma 2.6, we have

$$\begin{aligned} & (a+b)(aP_TA + dP_{L^\perp} + bP_TAP_{T,S})^{\#}P_T \\ = \;& (a+b)\left(U\begin{bmatrix} (a+b)A_L & aA_LA_L^{\#}B_L \\ O & dI_{n-l} \end{bmatrix}^{\#}\begin{bmatrix} A_LA_L^{\#} & O \\ O & O \end{bmatrix} U^*\right) \\ = \;& (a+b)\left(U\begin{bmatrix} \frac{1}{(a+b)}A_L^{\#} & -\frac{a}{d(a+b)}A_L^{\#}B_L \\ O & \frac{1}{d}I_{n-l} \end{bmatrix}\begin{bmatrix} A_LA_L^{\#} & O \\ O & O \end{bmatrix} U^*\right) \\ = \;& U\begin{bmatrix} A_L^{\#} & O \\ O & O \end{bmatrix} U^* = A_{(L)}^{(\#)}. \end{aligned}$$

The rest of the proof follows similarly. □

**Remark 5.3.** *Under the hypotheses of Theorem 5.2 and additional assumption $a = 0$, we have the following equation:*

$$\begin{aligned} A_{(L)}^{(\#)} &= bP_T(dP_{L^\perp} + bP_TAP_{T,S})^{\#} \\ &= b(dP_{L^\perp} + bP_TAP_{T,S})^{\#}P_T, \end{aligned}$$

*while $b = 0$, we have the following equation:*

$$\begin{aligned} A_{(L)}^{(\#)} &= aP_T(aAP_{T,S} + dP_{L^\perp})^{\#} \\ &= a(aP_TA + dP_{L^\perp})^{\#}P_T. \end{aligned}$$

In the next theorem, we present representations for the Bott-Duffin core inverse, using the projections $P = P_{T^\perp,AT}$ and $Q = P_{(A^*T)^\perp,T}$.

**Theorem 5.4.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$ be such that $A_{(L)}^{(\#)}$ exists. For any $a, b, c, d \in \mathbb{C}$ such that $cd \neq 0$ and $a + b \neq 0$, the following statements hold:*

(*a*) $A_{(L)}^{(\#)} = P_{T,S}(aAP_{T,S} + bP_TAP_{T,S} + cPP_{L^\perp})^{\#}(a+b)(I_n - P)$;

(*b*) $A^{(\textcircled{\#})}_{(L)} = (a+b)(I_n - Q)P_{T,S}(aAP_{T,S} + bP_TAP_{T,S} + cP_{L^\perp}Q)^{\textcircled{\#}}$;

(*c*) $A^{(\textcircled{\#})}_{(L)} = cP_{T,S}(cP_TAP_{T,S} + dP_{T,S}Q)^{\textcircled{\#}}$;

(*d*) $A^{(\textcircled{\#})}_{(L)} = c(cP_TAP_{T,S} + dP_{T,S}Q)^{\textcircled{\#}}P_T$,

*where* $P = P_{T^\perp,AT}$ *and* $Q = P_{(A^*T)^\perp,T}$.

*Proof.* (*a*). From (7), (8) and Theorem 3.2 (*iv*), we have

$$P = I_n - P_{AT,T^\perp} = U\begin{bmatrix} I_l - A_L A_L^{\textcircled{\#}} & O \\ -C_L A_L^{\textcircled{\#}} & I_{n-l} \end{bmatrix}U^*. \tag{15}$$

Using (6), (7), (13), (14), (15) and Lemma 2.5, we can obtain

$$\begin{aligned}(aAP_{T,S} + bP_TAP_{T,S} + cPP_{L^\perp})^{\textcircled{\#}} &= \left(U\begin{bmatrix} (a+b)A_L & O \\ aC_LA_L^{\textcircled{\#}}A_L & cI_{n-l} \end{bmatrix}U^*\right)^{\textcircled{\#}} \\ &= U\begin{bmatrix} \frac{1}{a+b}A_L^{\textcircled{\#}} & O \\ -\frac{a}{c(a+b)}C_LA_L^{\textcircled{\#}} & \frac{1}{c}I_{n-l} \end{bmatrix}U^*.\end{aligned}$$

Hence,

$$\begin{aligned}&P_{T,S}(aAP_{T,S} + bP_TAP_{T,S} + cPP_{L^\perp})^{\textcircled{\#}}(a+b)(I_n - P) \\ = {}& U\begin{bmatrix} A_L^{\textcircled{\#}}A_L & O \\ O & O \end{bmatrix}\begin{bmatrix} \frac{1}{a+b}A_L^{\textcircled{\#}} & O \\ -\frac{a}{c(a+b)}C_LA_L^{\textcircled{\#}} & \frac{1}{c}I_{n-l} \end{bmatrix}\begin{bmatrix} (a+b)A_LA_L^{\textcircled{\#}} & O \\ (a+b)C_LA_L^{\textcircled{\#}} & O \end{bmatrix}U^* \\ = {}& U\begin{bmatrix} A_L^{\textcircled{\#}} & O \\ O & O \end{bmatrix}U^* \\ = {}& A^{(\textcircled{\#})}_{(L)}.\end{aligned}$$

(*b*). From (7), (8) and Theorem 3.2 (*iv*), we have

$$Q = I_n - P_{T,(A^*T)^\perp} = U\begin{bmatrix} I_l - A_L^{\textcircled{\#}}A_L & -A_L^{\textcircled{\#}}B_L \\ O & I_{n-l} \end{bmatrix}U^*. \tag{16}$$

The rest of the proof follows similarly. □

**Example 5.5.** *Let*

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \text{ and } L = \mathcal{R}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

*From (15) and (16), we have*

$$P = I - P_{AT,T^\perp} = \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} & 0 \\ -\frac{4}{9} & \frac{4}{9} & -\frac{2}{9} & 0 \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} & 0 \\ \frac{2}{9} & -\frac{2}{9} & -\frac{8}{9} & 1 \end{bmatrix},\ Q = I - P_{T,(A^*T)^\perp} = \begin{bmatrix} \frac{4}{3} & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{9} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

*By the direct calculation,*

$$
\begin{aligned}
& P_{T,S}(aAP_{T,S} + bP_TAP_{T,S} + cPP_{L^\perp})^{\#\!\!\!\!\!\bigcirc}(a+b)(I_n - P) \\
= & \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{3(a+b)} & 0 & \frac{2}{3(a+b)} & 0 \\ \frac{-1}{9(a+b)} & \frac{1}{9(a+b)} & \frac{4}{9(a+b)} & 0 \\ \frac{4}{9(a+b)} & \frac{2}{9(a+b)} & \frac{-4}{9(a+b)} & 0 \\ \frac{2a}{9c(a+b)} & \frac{-2a}{9c(a+b)} & \frac{-8a}{9c(a+b)} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} \frac{5(a+b)}{9} & \frac{4(a+b)}{9} & \frac{-2(a+b)}{9} & 0 \\ \frac{4(a+b)}{9} & \frac{5(a+b)}{9} & \frac{2(a+b)}{9} & 0 \\ \frac{-2(a+b)}{9} & \frac{2(a+b)}{9} & \frac{8(a+b)}{9} & 0 \\ \frac{-2(a+b)}{9} & \frac{2(a+b)}{9} & \frac{8(a+b)}{9} & 0 \end{bmatrix} \\
= & \begin{bmatrix} -\frac{1}{3} & 0 & \frac{2}{3} & 0 \\ -\frac{1}{9} & \frac{1}{9} & \frac{4}{9} & 0 \\ \frac{4}{9} & \frac{2}{9} & -\frac{4}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
= & A_{(L)}^{(\#\!\!\!\!\!\bigcirc)}.
\end{aligned}
$$

In [14], Yuan and Zuo present several limit expressions for some generalized inverses. Motivated by this result, in the following theorem we give some similar expressions for Bott-Duffin core inverse.

**Theorem 5.6.** *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A_{(L)}^{(\#\!\!\!\!\!\bigcirc)}$ exists. Then*

(*a*) $A_{(L)}^{(\#\!\!\!\!\!\bigcirc)} = \lim_{\lambda\to 0} P_LAP_LA^*(\lambda I_n + (P_LA)^2P_LA^*)^{-1}P_L$;

(*b*) $A_{(L)}^{(\#\!\!\!\!\!\bigcirc)} = \lim_{\lambda\to 0} P_LA(\lambda I_n + P_LA^*(P_LA)^2)^{-1}P_LA^*P_L$;

(*c*) $A_{(L)}^{(\#\!\!\!\!\!\bigcirc)} = \lim_{\lambda\to 0} (\lambda I_n + P_LAP_LA^*P_LA)^{-1}P_LAP_LA^*P_L$.

*Proof.* Let $A_L$ be given in (7). From [14, Corollary 2.3], it follows that

$$A_L^{\#\!\!\!\!\!\bigcirc} = \lim_{\lambda\to 0} A_LA_L^*(\lambda I_n + A_L^2A_L^*)^{-1}. \tag{17}$$

Let $M = P_LAP_LA^*(\lambda I_n + (P_LA)^2P_LA^*)^{-1}P_L$. By (6) and (7), we have

$$
\begin{aligned}
M &= U\begin{bmatrix} A_LA_L^* & A_LC_L^* \\ O & O \end{bmatrix}\begin{bmatrix} \lambda I_l + A_L^2A_L^* & A_L^2C_L^* \\ O & \lambda I_{n-l} \end{bmatrix}^{-1}\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^* \\
&= U\begin{bmatrix} A_LA_L^* & A_LC_L^* \\ O & O \end{bmatrix}\begin{bmatrix} (\lambda I_l + A_L^2A_L^*)^{-1} & -(\lambda I_l + A_L^2A_L^*)^{-1}A_L^2C_L^* \\ O & \frac{1}{\lambda}I_{n-l} \end{bmatrix}\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^* \\
&= U\begin{bmatrix} A_LA_L^*(\lambda I_l + A_L^2A_L^*)^{-1} & O \\ O & O \end{bmatrix}U^*.
\end{aligned}
$$

Hence, from (8) and (17), we have

$$
\begin{aligned}
\lim_{\lambda\to 0} M &= \lim_{\lambda\to 0} U\begin{bmatrix} A_LA_L^*(\lambda I_l + A_L^2A_L^*)^{-1} & O \\ O & O \end{bmatrix}U^* \\
&= U\begin{bmatrix} A_L^{\#\!\!\!\!\!\bigcirc} & O \\ O & O \end{bmatrix}U^* \\
&= A_{(L)}^{(\#\!\!\!\!\!\bigcirc)}.
\end{aligned}
$$

Assertions (*b*) and (*c*) can be proved similarly. □

**Example 5.7.** *Let the matrix A and the subspace L be given as in the Example 5.5. By simple calculation, we have*

$$M = \begin{bmatrix} \frac{5\lambda^2+45\lambda}{\lambda^3+24\lambda^2-135\lambda} & \frac{5\lambda^2}{\lambda^3+24\lambda^2-135\lambda} & \frac{-90\lambda}{\lambda^3+24\lambda^2-135\lambda} & 0 \\ \frac{5\lambda^2+15\lambda}{\lambda^3+24\lambda^2-135\lambda} & \frac{6\lambda^2-15\lambda}{\lambda^3+24\lambda^2-135\lambda} & \frac{2\lambda^2-60\lambda}{\lambda^3+24\lambda^2-135\lambda} & 0 \\ \frac{-60\lambda}{\lambda^3+24\lambda^2-135\lambda} & \frac{2\lambda^2-30\lambda}{\lambda^3+24\lambda^2-135\lambda} & \frac{4\lambda^2+60\lambda}{\lambda^3+24\lambda^2-135\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Thus,*

$$\lim_{\lambda\to 0} M = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{2}{3} & 0 \\ -\frac{1}{9} & \frac{1}{9} & \frac{4}{9} & 0 \\ \frac{4}{9} & \frac{2}{9} & -\frac{4}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{(L)}^{(\#\!\!\!\bigcirc)}.$$

## 6. The Bott-Duffin core inverse and constrained matrix approximation problem

The Frobenius norm is a matrix form of an $m \times n$ matrix $A$ defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \left|a_{i,j}\right|^2},$$

where $a_{i,j}$ represents the elements in the $i$-th row and $j$-th column of matrix $A$. In the following theorem, we study the constrained matrix approximation problem in the Frobenius norm by using the Bott-Duffin core inverse. Consider the following equation:

$$P_L A x = b, \tag{18}$$

where $P_L A P_L \in \mathbb{C}_n^{CM}$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_L A P_L)$. When $b \notin \mathcal{R}(P_L A)$, (18) is unsolvalble, it has least-squares solutions. Therefore, we consider the least-squares solutions of (18) under the certain condition $x \in T$, i.e.,

$$\|P_L A x - b\|_F = \min \quad \text{subject to} \quad x \in T. \tag{19}$$

**Theorem 6.1.** *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A_{(L)}^{(\#\!\!\!\bigcirc)}$ exists. And let $b \in \mathbb{C}^n$. Then,*

$$x = A_{(L)}^{(\#\!\!\!\bigcirc)} b \tag{20}$$

*is the unique solution of (19).*

*Proof.* Since $x \in T$, it follows that there exists $y \in \mathbb{C}^n$ for which $x = P_L A P_L y$. Then, $x$ is the solution of (19) if and only if $y$ is the solution of

$$\left\|P_L A P_L A P_L y - b\right\|_F = \min.$$

Denote

$$U^* y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ and } U^* b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $y_1, b_1 \in \mathbb{C}^l$. From (6) and (7), we have

$$\|P_L A x - b\|_F^2 = \left\|\begin{bmatrix} {A_L}^2 y_1 - b_1 \\ -b_2 \end{bmatrix}\right\|_F^2 = \left\|{A_L}^2 y_1 - b_1\right\|_F^2 + \|b_2\|_F^2.$$

Since $A_{(L)}^{(\#)}$ exists, we have $A_L \in \mathbb{C}_l^{CM}$. According to [5, Corollary 6], in (7), matrix $A_L \in \mathbb{C}^{l\times l}$ of rank $r$ can be represented in the form

$$A_L = V\begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} V^*, \tag{21}$$

where $l = \dim(L)$, $V \in \mathbb{C}^{l\times l}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \dots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of $A_L$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(l-r)}$ satisfy

$$KK^* + LL^* = I_r$$

In [1, Lemma 2], Baksalary and Trenkler point out that if $A_L \in \mathbb{C}_l^{CM}$ be of the form (21). Then

$$A_L{}^{\#} = V\begin{bmatrix} (\Sigma K)^{-1} & O \\ O & O \end{bmatrix} V^*. \tag{22}$$

Denote

$$V^* y_1 = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix} \text{ and } V^* b_1 = \begin{bmatrix} b_1{}' \\ b_2{}' \end{bmatrix},$$

where $y_1{}', b_1{}' \in \mathbb{C}^r$. It follows from (21) that

$$\begin{aligned} \left\|A_L{}^2 y_1 - b_1\right\|_F^2 &= \left\|\begin{bmatrix} (\Sigma K)^2 y_1{}' + \Sigma K \Sigma L y_2{}' - b_1{}' \\ -b_2{}' \end{bmatrix}\right\|_F^2 \\ &= \left\|(\Sigma K)^2 y_1{}' + \Sigma K \Sigma L y_2{}' - b_1{}'\right\|_F^2 + \left\|b_2{}'\right\|_F^2. \end{aligned}$$

Since $\Sigma K$ is invertible, we have $\min_{y_1,y_2} \left\|(\Sigma K)^2 y_1{}' + \Sigma K\Sigma L y_2{}' - b_1{}'\right\|_F^2 = 0$, that is, $\|P_L A P_L x - b\|_F = \min = \sqrt{\|b_2\|_F^2 + \left\|b_2{}'\right\|_F^2}$, in which $y_2{}' \in \mathbb{C}^{l-r}$ is arbitrary, and $y_1{}' = -(\Sigma K)^{-1}\Sigma L y_2{}' + (\Sigma K)^{-2} b_1{}'$. It follows from (22) that

$$\begin{aligned} x &= P_L A P_L y = U\begin{bmatrix} A_L & O \\ O & O \end{bmatrix} U^* y = U\begin{bmatrix} A_L y_1 \\ O \end{bmatrix} = U\begin{bmatrix} V\begin{bmatrix} (\Sigma K)^{-1} b_1{}' \\ O \end{bmatrix} \\ O \end{bmatrix} \\ &= U\begin{bmatrix} A_L{}^{\#} b_1 \\ O \end{bmatrix} = A_{(L)}^{(\#)} b, \end{aligned}$$

that is, (20) is the unique solution of (19). □

When $M \in \mathbb{C}^{n\times n}$ is nonsingular, it is well known that the solution of $Mx = b$ is unique and $x = M^{-1}b$, where $b \in \mathbb{C}^n$. Let $x = (x_1, x_2, \dots, x_n)^T$. Then,

$$x_j = \frac{\det(M(i \to b))}{\det(M)}, \ i = 1, 2, \dots, n \tag{23}$$

is called Cramer's rule for solving $Mx = b$. In the following Theorem, we give the unique least-square solution of (19).

**Theorem 6.2.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $b \in \mathbb{C}^n$, $T = \mathcal{R}(P_L A P_L)$ and $\mathrm{rank}(P_L A P_L) = r$ be such that $A_{(L)}^{(\#)}$ exists, and let $F \in \mathbb{C}^{n\times(n-r)}$ with $\mathrm{rank}(F) = n - r$ and $\mathcal{R}(F) = T^\perp$. Then, (19) has the unique solution $x = (x_1, x_2, \dots, x_n)^T$ satisfying*

$$x_i = \frac{\det\left(\begin{bmatrix} P_L A P_L(i \to b) & F \\ F^*(i \to 0) & O \end{bmatrix}\right)}{\det\left(\begin{bmatrix} P_L A P_L & F \\ F^* & O \end{bmatrix}\right)}, \tag{24}$$

*where $i = 1, 2, \ldots, n$.*

*Proof.* From [13, Lemma 3.3], we have

$$G = \begin{bmatrix} P_L A P_L & F \\ F^* & O \end{bmatrix}$$

is invertible and

$$G^{-1} = \begin{bmatrix} A_{(L)}^{(\#)} & (I_n - A_{(L)}^{(\#)} P_L A P_L) F (F^* F)^{-1} \\ (F^* F)^{-1} F^* & O \end{bmatrix}. \tag{25}$$

Then we get the unique solution $\hat{x} = G^{-1}\hat{b}$ of $G\hat{x} = \hat{b}$, in which $\hat{x}^* = \begin{bmatrix} x^* & y^* \end{bmatrix}^*$ and $\hat{b}^* = \begin{bmatrix} b^* & O \end{bmatrix}^*$. In terms of (25), it follows that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_{(L)}^{(\#)} & (I_n - A_{(L)}^{(\#)} P_L A P_L) F (F^* F)^{-1} \\ (F^* F)^{-1} F^* & O \end{bmatrix} \begin{bmatrix} b \\ O \end{bmatrix} = \begin{bmatrix} A_{(L)}^{(\#)} b \\ (F^* F)^{-1} F^* b \end{bmatrix}.$$

Applying (23), we can obtain (24). □

**Example 6.3.** *Let the matrix $A$ and the subspace $L$ be as in Example 5.5, and let*

$$b = \begin{bmatrix} 2 & 1 & 3 & 1 \end{bmatrix}^T, \; F = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

*It is clear that $b \notin T$, then (18) is unsolvable. Therefore, by using Theorem 6.1 and Theorem 6.2, we consider the least-squares solutions of (18). We can check* $\operatorname{rank}(F) = 2$ *and $\mathcal{R}(F) = T^\perp$. Let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ is the unique solution of (19). By $A_{(L)}^{(\#)}$ in Example 5.5, applying (20) or (24), we can derive the components of $x$ directly, i.e.*

$$x_1 = \frac{4}{3}, \; x_2 = \frac{11}{9}, \; x_3 = -\frac{2}{9}, \; x_4 = 0$$

In [3], let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ and $L \leqslant \mathbb{C}^n$, the constrained linear equation

$$Ax + y = b, \; x \in L, \; y \in L^\perp \tag{26}$$

arise in electrical network theory. When $AP_L + P_{L^\perp}$ is nonsingular, the constrained linear equation (26) has a unique solution

$$x = A_{(L)}^{(-1)} b, \; y = (I_n - A A_{(L)}^{(-1)}) b,$$

for any $b \in \mathbb{C}^n$. In the following theorem, we discuss the solution of (26) when $AP_L + P_{L^\perp} \in \mathbb{C}_n^{CM}$.

**Theorem 6.4.** *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $b \in \mathcal{R}(AP_L + P_{L^\perp})$ be such that $A_{(L)}^{(\#)}$ exists. The constrained linear equation (26) has a unique solution*

$$x = A_{(L)}^{(\#)} b, \; y = (I_n - A A_{(L)}^{(\#)}) b.$$

*Proof.* Let $z = x + y$, we have $P_L z = P_L(x + y) = P_L x + P_L y = x$ and $P_{L^\perp} z = P_{L^\perp}(x + y) = P_{L^\perp} x + P_{L^\perp} y = y$. Thus,

$$\begin{aligned} Ax + y = b \quad &\Leftrightarrow \quad AP_L z + P_{L^\perp} z = b \\ &\Leftrightarrow \quad (AP_L + P_{L^\perp}) z = b. \end{aligned} \tag{27}$$

From Theorem 3.1, $A_{(L)}^{(\#)}$ exists if and only if $AP_L + P_{L^\perp} \in \mathbb{C}_n^{CM}$. If $b \in \mathcal{R}(AP_L + P_{L^\perp})$, then the core-inverse solution of (27) is unique (see [7]), i.e. $z = (AP_L + P_{L^\perp})^{\#} b$. Thus, $x = P_L z = A_{(L)}^{(\#)} b$ and $y = (I_n - A A_{(L)}^{(\#)}) b$. □

**Example 6.5.** *Let the matrix A and the subspace L be as in Example 5.5, and let*

$$b^T = \begin{bmatrix} 5 & 8 & 6 & 5 \end{bmatrix}^T.$$

*It is easy to check* $b \in \mathcal{R}(AP_L + P_{L^\perp})$. *By Theorem 6.4 and* $A^{(\#)}_{(L)}$ *in Example 5.5, we can obtain the unique solution of equation (26):*

$$x = A^{(\#)}_{(L)} b = \begin{bmatrix} \frac{7}{3} & 3 & \frac{4}{3} & 0 \end{bmatrix}^T \text{ and } y = (I_4 - AA^{(\#)}_{(L)}) = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}^T.$$

## 7. Conclusion

The paper introduces a new generalized inverse, Bott-Duffin core inverse, which is a generalization of the Bott-Duffin inverse. We study its properties, characterizations and representations. Moreover, we discuss the application of Bott-Duffin core inverse, which is about constrained matrix approximation problem. On a basis of the current research background, there are many topics on the Bott-Dufiin core inverse which can be discussed. Some ideas are given as follows:

(1) It is possible to discuss the algebraic perturbation theory of Bott-Duffin core inverse and the expression of the algebraic perturbation of Bott-Duffin core inverse.

(2) Consider the relationships between the Bott-Duffin core inverse and other generalized inverses.

(3) The integral representation, continuity, and iterative calculation of the Bott-Duffin core inverse all can be discussed.

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