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Characterization of non-linear mixed bi-skew Jordan triple higher derivations on prime $*$-algebras

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Abstract. Let $\mathcal{A}$ be a prime $*$-algebra. For any $A, B \in \mathcal{A}$, define a new product $A \bullet B = AB^* + BA^*$. Let δ be a non-linear map satisfying $\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$ for all $A, B, C \in \mathcal{A}$. In this article, we show that δ is an additive $*$-derivation. Furthermore, we also discuss above result for higher derivable maps on $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be an associative $*$-algebra over the field of complex numbers $\mathbb{C}$. Recall that an algebra $\mathcal{A}$ is said to be prime if for any $A, B \in \mathcal{A}$, $A\mathcal{A}B$ equates to (0) then either $A = 0$ or $B = 0$. A linear map $\delta : \mathcal{A} \to \mathcal{A}$ is said to be a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Further, if $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$, then δ is called a $*$-derivation. If the linearity of δ is replaced by the additivity in the above definition, then δ is called an additive $*$-derivation. The products $A \circ B = AB + BA$ and $[A, B] = AB - BA$ are called Jordan and Lie product of $A, B \in \mathcal{A}$. These Jordan and Lie product with involution " $*$ ", defined as $A \star B = AB + BA^*$, $[A, B]_\star = AB - BA^*$, $A \bullet B = AB^* + BA^*$ and $[A, B]_\bullet = AB^* - BA^*$ are respectively termed as $*$-Jordan, $*$-Lie, bi-skew Jordan and bi-skew Lie product of $A, B \in \mathcal{A}$. A map (may not be linear) $\delta : \mathcal{A} \to \mathcal{A}$ is called a non-linear Jordan (resp. non-linear bi-skew Jordan) derivation if it satisfies $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$ (resp. $\delta(A \bullet B) = \delta(A) \bullet B + A \bullet \delta(B)$) for all $A, B \in \mathcal{A}$. Accordingly, a non-linear Jordan (or non-linear bi-skew Jordan) triple derivation is a map δ from $\mathcal{A}$ into itself which satisfies $\delta(A \circ B \circ C) = \delta(A) \circ B \circ C + A \circ \delta(B) \circ C + A \circ B \circ \delta(C)$ (or $\delta(A \bullet B \bullet C) = \delta(A) \bullet B \bullet C + A \bullet \delta(B) \bullet C + A \bullet B \bullet \delta(C)$) for all $A, B, C \in \mathcal{A}$. Many mathematicians characterized the maps concerning these products on different rings and algebras (see [3], [4], [6], [7], [8], [9], [13], [15], [17], [18], [19], [27], [30], [34] and the references therein). In [15], Khan and Alhazmi determined the structure of multiplicative bi-skew Jordan triple derivations on prime $*$-algebras. In fact, they proved that every multiplicative bi-skew Jordan triple derivation on a prime $*$-algebra, is an additive $*$-derivation. In recent years, several scholars paid more attention to mixed Jordan (Lie) products with involution " $*$ "

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and characterized the structures of maps concerning these products (see for example [1], [2], [5], [10], [16], [22], [23], [25], [26] [31], [32], [35]). For instance, Zhou et al. [35] proved that a non-linear mixed Lie triple derivation on a prime $*$-algebra, is an additive $*$-derivation. In [1], we have obtained the structure of non-linear mixed Jordan bi-skew Lie triple derivations on $*$-algebras. Yang and Zhang [32] characterized non-linear mixed Lie triple product preserving maps on factor von Neumann algebras. A natural question arises related to mixed bi-skew Jordan triple product (i.e., $A \circ B \bullet C$) that what would be the structure of a map δ on a prime $*$-algebra $\mathcal{A}$ satisfying $\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$ for all $A, B, C \in \mathcal{A}$. To settle this question, in Section 2, we prove that such a map is an additive $*$-derivation on $\mathcal{A}$. To further extend the result obtained in Section 2, we shift our focus to the mixed bi-skew Jordan triple higher derivations on prime $*$-algebras. The notion of higher derivations has been considered by many researchers on different rings and algebras (see [11], [12], [14], [20], [21], [24], [28], [29], [33]). Let us recall some terminologies related to (Jordan) higher derivations on an algebra $\mathcal{A}$. Let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a collection of linear maps $\delta_n : \mathcal{A} \to \mathcal{A}$ where $n \in \mathbb{N}$ (the set of all non-negative integers) such that $\delta_0 = id_{\mathcal{A}}$ (the identity map on $\mathcal{A}$). Then Δ is called

- a higher derivation if for each $n \in \mathbb{N}$ and for all $A, B \in \mathcal{A}$

$$\delta_n(AB) = \sum_{p+q=n} \delta_p(A)\delta_q(B);$$

- a Jordan higher derivation if for each $n \in \mathbb{N}$ and for all $A, B \in \mathcal{A}$

$$\delta_n(A \circ B) = \sum_{p+q=n} \delta_p(A) \circ \delta_q(B);$$

- a Jordan triple higher derivation if for each $n \in \mathbb{N}$ and for all $A, B, C \in \mathcal{A}$

$$\delta_n(A \circ B \circ C) = \sum_{p+q+r=n} \delta_p(A) \circ \delta_q(B) \circ \delta_r(C).$$

If the assumption of linearity is dropped (or replaced by additivity) in the above definitions, then Δ is called a non-linear higher, a non-linear Jordan higher and a non-linear Jordan triple higher (or an additive higher, an additive Jordan higher and an additive Jordan triple higher) derivation on $\mathcal{A}$, respectively. Analogously, we can define a (non-linear or an additive) bi-skew Jordan higher and a (non-linear or an additive) bi-skew Jordan triple higher derivation through the replacement of Jordan (triple) product by bi-skew Jordan (triple) product. Considering Jordan and bi-skew Jordan product i.e., $A \circ B = AB + BA$ and $A \bullet B = AB^* + BA^*$, we define non-linear mixed bi-skew Jordan triple higher derivation as follows: let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ (where $\mathbb{N}$ is the set of all non-negative integers) be a collection of maps $\delta_n : \mathcal{A} \to \mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$ (the identity map on $\mathcal{A}$), satisfying

$$\delta_n(A \circ B \bullet C) = \sum_{p+q+r=n} \delta_p(A) \circ \delta_q(B) \bullet \delta_r(C)$$

for all $A, B, C \in \mathcal{A}$ and for each $n \in \mathbb{N}$. Then Δ is called a non-linear mixed bi-skew Jordan triple higher derivation on $\mathcal{A}$.

Motivated by the work done on higher derivations ([11], [12], [14], [20], [21], [24], [28], [29], [33]), in Section 3, we prove that every non-linear mixed bi-skew Jordan triple higher derivations on a prime $*$-algebra $\mathcal{A}$, is an additive $*$-higher derivation on $\mathcal{A}$.

2. Non-linear mixed bi-skew Jordan triple derivations on prime $*$-algebras

In this section, we determine the structure of non-linear mixed bi-skew Jordan triple derivation on a prime $*$-algebra $\mathcal{A}$. In fact, we prove the following:

Theorem 2.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unity I and a nontrivial projection. Then a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C) \tag{1}$$

for all $A, B, C \in \mathcal{A}$, is an additive $*$-derivation.

Before proving Theorem 2.1, we give an example of a map δ on a prime $*$-algebra $\mathcal{A}$ that satisfies Equation (1).

Example 2.2. *Let $\mathcal{A} = M_2(\mathbb{C})$, the algebra of all 2×2 matrices over $\mathbb{C}$ (the field of complex numbers) and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the unity of $\mathcal{A}$. A map $* : \mathcal{A} \to \mathcal{A}$, defined by $*(A) = A^\theta$, where A^θ denotes the transpose conjugate of the matrix A, is an involution. Hence $\mathcal{A}$ is a prime $*$-algebra with unity I. Define a map $\delta : \mathcal{A} \to \mathcal{A}$ such that $\delta\left(\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}\right) = \begin{bmatrix} 0 & iz_2 \\ -iz_3 & 0 \end{bmatrix}$. Observe that δ is a $*$-derivation on $\mathcal{A}$. Therefore, it also satisfies*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathcal{A}$. Moreover, $\mathcal{A}$ contains a nontrivial projection $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and δ is also nontrivial.

Proof of Theorem 2.1: Let $\mathcal{A}$ be a prime $*$-algebra and $\mathbb{C}$ be the field of complex numbers. Take a projection $P_1 \in \mathcal{A}$ and let $P_2 = I - P_1$. We write $\mathcal{A}_{jk} = P_j\mathcal{A}P_k$ for $j, k = 1, 2$. Then, by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Note that, any element $A \in \mathcal{A}$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{jk} \in \mathcal{A}_{jk}$ $(j, k \in \{1, 2\})$. Let $\mathcal{H} = \{A \in \mathcal{A} \mid A^* = A\}$ and $\mathcal{K} = \{A \in \mathcal{A} \mid A^* = -A\}$, $\mathcal{H}_{12} = \{P_1HP_2 + P_2HP_1 \mid H \in \mathcal{H}\}$ and $\mathcal{H}_{ii} = P_i\mathcal{H}P_i$ $(i = 1, 2)$. Thus, for every $H \in \mathcal{H}$, $H = H_{11} + H_{12} + H_{22}$ for every $H_{12} \in \mathcal{H}_{12}$ and $H_{ii} \in \mathcal{H}_{ii}$ $(i = 1, 2)$.

In view of the above facts, the proof of Theorem 2.1 is given in a series of the following claims:

Claim 2.3. $\delta(0) = 0$.

$$\delta(0) = \delta(0 \circ 0 \bullet 0) = \delta(0) \circ 0 \bullet 0 + 0 \circ \delta(0) \bullet 0 + 0 \circ 0 \bullet \delta(0) = 0.$$

Claim 2.4.

(*i*) $\delta\left(\frac{1}{2}I\right) = 0$;

(*ii*) $\delta\left(-\frac{1}{2}I\right) = 0$;

(*iii*) $\delta\left(\frac{1}{2}iI\right) = 0$.

(*i*) Let $A = B = C = \frac{1}{2}I$ in Equation (1). Then, we can write

$$\begin{aligned}\delta\left(\frac{1}{2}I\right) &= \delta\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet \frac{1}{2}I\right)\\ &= \delta\left(\frac{1}{2}I\right) \circ \frac{1}{2}I \bullet \frac{1}{2}I + \frac{1}{2}I \circ \delta\left(\frac{1}{2}I\right) \bullet \frac{1}{2}I + \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta\left(\frac{1}{2}I\right)\\ &= \frac{3}{2}\left(\delta\left(\frac{1}{2}I\right) + \delta\left(\frac{1}{2}I\right)^*\right).\end{aligned} \tag{2}$$

From Equation (2), it is evident that $\delta\left(\frac{1}{2}I\right)$ is self-adjoint and hence again from Equation (2), we get $\delta\left(\frac{1}{2}I\right) = 0$.

(*ii*) If we put $A = B = \frac{1}{2}I$ and $C = -\frac{1}{2}I$ in Equation (1), we get

$$\delta\left(-\frac{1}{2}I\right) = \delta\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet -\frac{1}{2}I\right) = \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta\left(-\frac{1}{2}I\right) = \frac{1}{2}\left(\delta\left(-\frac{1}{2}I\right) + \delta\left(-\frac{1}{2}I\right)^*\right). \tag{3}$$

From Equation (3), we observe that $\delta\left(-\frac{1}{2}I\right)$ is self-adjoint, i.e.,

$$\delta\left(-\frac{1}{2}I\right)^* = \delta\left(-\frac{1}{2}I\right). \tag{4}$$

Also

$$\begin{aligned} 0 = \delta\left(\frac{1}{2}I\right) &= \delta\left(\frac{1}{2}I \circ -\frac{1}{2}I \bullet -\frac{1}{2}I\right) = \frac{1}{2}I \circ \delta\left(-\frac{1}{2}I\right) \bullet -\frac{1}{2}I + \frac{1}{2}I \circ -\frac{1}{2}I \bullet \delta\left(-\frac{1}{2}I\right) \\ &= -\left(\delta(-\frac{1}{2}I) + \delta(-\frac{1}{2}I)^*\right). \end{aligned}$$

Thus, we get

$$\delta\left(-\frac{1}{2}I\right)^* = -\delta\left(-\frac{1}{2}I\right). \tag{5}$$

From Equations (4) and (5), we obtain $\delta\left(-\frac{1}{2}I\right) = 0$.

(*iii*) Let $A = B = \frac{1}{2}iI$ and $C = \frac{1}{2}I$ in Equation (1). Then, we have

$$\begin{aligned} 0 = \delta\left(-\frac{1}{2}I\right) &= \delta\left(\frac{1}{2}iI \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right) = \delta\left(\frac{1}{2}iI\right) \circ \frac{1}{2}iI \bullet \frac{1}{2}I + \frac{1}{2}iI \circ \delta\left(\frac{1}{2}iI\right) \bullet \frac{1}{2}I \\ &= i\left(\delta\left(\frac{1}{2}iI\right) - \delta\left(\frac{1}{2}iI\right)^*\right). \end{aligned}$$

This gives

$$\delta\left(\frac{1}{2}iI\right)^* = \delta\left(\frac{1}{2}iI\right). \tag{6}$$

On the other hand, we have

$$0 = \delta(0) = \delta\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet \frac{1}{2}iI\right) = \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta\left(\frac{1}{2}iI\right) = \frac{1}{2}\left(\delta\left(\frac{1}{2}iI\right)^* + \delta\left(\frac{1}{2}iI\right)\right).$$

This implies that

$$\delta\left(\frac{1}{2}iI\right)^* = -\delta\left(\frac{1}{2}iI\right). \tag{7}$$

From Equations (6) and (7), we obtain $\delta\left(\frac{1}{2}iI\right) = 0$.

Claim 2.5. *For any $H \in \mathcal{H}$, $\delta(H)^* = \delta(H)$.*

Observe that $H = H \circ \frac{1}{2}I \bullet \frac{1}{2}I$. It follows from Claim 2.4 (*i*) that

$$\delta(H) = \delta\left(H \circ \frac{1}{2}I \bullet \frac{1}{2}I\right) = \delta(H) \circ \frac{1}{2}I \bullet \frac{1}{2}I = \frac{1}{2}\left(\delta(H) + \delta(H)^*\right).$$

Thus, $\delta(H)^* = \delta(H)$.

Claim 2.6. *For any $H \in \mathcal{H}$, we have*

(*i*) $\delta(-iH) = -i\delta(H)$;

(*ii*) $\delta(iH) = i\delta(H)$.

Observe that, $-iH \circ \frac{1}{2}I \bullet \frac{1}{2}I = H \circ \frac{1}{2}iI \bullet \frac{1}{2}I = 0$. It follows from Claim 2.4 that

$$\delta\left(-iH \circ \frac{1}{2}I \bullet \frac{1}{2}I\right) = \delta\left(H \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right).$$

This implies that

$$\delta(-iH) \circ \frac{1}{2}I \bullet \frac{1}{2}I = \delta(H) \circ \frac{1}{2}iI \bullet \frac{1}{2}I.$$

From Claim 2.5, we obtain

$$\delta(-iH) + \delta(-iH)^* = 0. \tag{8}$$

Next, since $H = -iH \circ \frac{1}{2}iI \bullet \frac{1}{2}I = \frac{1}{2}I \circ H \bullet \frac{1}{2}I$, then

$$\delta\left(-iH \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right) = \delta\left(\frac{1}{2}I \circ H \bullet \frac{1}{2}I\right).$$

Hence

$$\delta(-iH) \circ \frac{1}{2}iI \bullet \frac{1}{2}I = \frac{1}{2}I \circ \delta(H) \bullet \frac{1}{2}I.$$

This gives

$$\delta(-iH) - \delta(-iH)^* = -2i\delta(H). \tag{9}$$

Addition of Equations (8) and (9) leads to

$$\delta(-iH) = -i\delta(H).$$

Similarly, we can prove that $\delta(iH) = i\delta(H)$.

Claim 2.7. *For any $H_{ii} \in \mathcal{H}_{ii}, (i = 1, 2)$ and $H_{12} \in \mathcal{H}_{12}$, we have*

$$\delta(H_{ii} + H_{12}) = \delta(H_{ii}) + \delta(H_{12}).$$

Let $M = \delta(H_{11} + H_{12}) - \delta(H_{11}) - \delta(H_{12})$. Then, by Claim 2.5, we have $M^* = M$. Proving the claim, we have to show that $M = 0$. We prove the Claim for $i = 1$. we can write

$$\begin{aligned}
&\delta(P_2 \circ (H_{11} + H_{12}) \bullet P_2)\\
&= \delta(P_2 \circ H_{11} \bullet P_2) + \delta(P_2 \circ H_{12} \bullet P_2)\\
&= \delta(P_2) \circ (H_{11} + H_{12}) \bullet P_2 + P_2 \circ (\delta(H_{11}) + \delta(H_{12})) \bullet P_2 + P_2 \circ (H_{11} + H_{12}) \bullet \delta(P_2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\delta(P_2 \circ (H_{11} + H_{12}) \bullet P_2)\\
&= \delta(P_2) \circ (H_{11} + H_{12}) \bullet P_2 + P_2 \circ \delta(H_{11} + H_{12}) \bullet P_2 + P_2 \circ (H_{11} + H_{12}) \bullet \delta(P_2).
\end{aligned}$$

From the last two expressions, we get $P_2 \circ M \bullet P_2 = 0$. This gives $M_{12} = M_{22} = 0$. Now, since $(P_2 - P_1) \circ H_{12} \bullet P_1 = 0$, then we have

$$\begin{aligned}
&\delta(P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1 + (P_2 - P_1) \circ \delta(H_{11} + H_{12}) \bullet P_1 + (P_2 - P_1) \circ (H_{11} + H_{12}) \bullet \delta(P_1)\\
&= \delta((P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1)\\
&= \delta((P_2 - P_1) \circ H_{11} \bullet P_1) + \delta((P_2 - P_1) \circ H_{12} \bullet P_1)\\
&= \delta(P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1 + (P_2 - P_1) \circ (\delta(H_{11}) + \delta(H_{12})) \bullet P_1\\
&+ (P_2 - P_1) \circ (H_{11} + H_{12}) \bullet \delta(P_1).
\end{aligned}$$

This implies that $(P_2 - P_1) \circ M \bullet P_1 = 0$ and hence $M_{11} = 0$. Therefore, $M = M_{11} + M_{12} + M_{22} = 0$. Thus,

$$\delta(H_{11} + H_{12}) = \delta(H_{11}) + \delta(H_{12}).$$

Similarly, one can easily obtain $\delta(H_{12} + H_{22}) = \delta(H_{12}) + \delta(H_{22})$.

Claim 2.8. *For any $H_{11} \in \mathcal{H}_{11}, H_{12} \in \mathcal{H}_{12}$ and $H_{22} \in \mathcal{H}_{22}$, we have*

$$\delta(H_{11} + H_{12} + H_{22}) = \delta(H_{11}) + \delta(H_{12}) + \delta(H_{22}).$$

It is sufficient to show that $M = \delta(H_{11} + H_{12} + H_{22}) - \delta(H_{11}) - \delta(H_{12}) - \delta(H_{22}) = 0$. It follows from Claim 2.7 and $P_1 \circ H_{22} \bullet P_1 = 0$ that

$$\begin{aligned}
&\delta(P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet P_1)\\
&= \delta(P_1 \circ (H_{11} + H_{12}) \bullet P_1) + \delta(P_1 \circ H_{22} \bullet P_1)\\
&= \delta(P_1) \circ (H_{11} + H_{12} + H_{22}) \bullet P_1 + P_1 \circ (\delta(H_{11}) + \delta(H_{12}) + \delta(H_{22})) \bullet P_1\\
&+ P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet \delta(P_1).
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\delta(P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet P_1)\\
&= \delta(P_1) \circ (H_{11} + H_{12} + H_{22}) \bullet P_1 + P_1 \circ \delta(H_{11} + H_{12} + H_{22}) \bullet P_1\\
&+ P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet \delta(P_1).
\end{aligned}$$

From the last two expressions, we conclude that $P_1 \circ M \bullet P_1 = 0$. This gives $M_{11} = M_{12} = 0$. Next, since $(P_2 - P_1) \circ H_{11} \bullet P_2 = 0$, then reasoning as above, we obtain $M_{22} = 0$ and thus, $M = 0$. Hence the result.

Claim 2.9. *For any $G_{12}, H_{12} \in \mathcal{H}_{12}$, we have*

$$\delta(G_{12} + H_{12}) = \delta(G_{12}) + \delta(H_{12}).$$

For any $U_{12}, V_{12} \in \mathcal{A}_{12}$, assume that $G_{12} = U_{12} + U_{12}^* \in \mathcal{H}_{12}$ and $H_{12} = V_{12} + V_{12}^* \in \mathcal{H}_{12}$. Thus

$$\begin{aligned}
&(P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*)\\
&= (U_{12} + U_{12}^*) + (V_{12} + V_{12}^*) + (U_{12}V_{12}^* + V_{12}U_{12}^* + U_{12}^*V_{12} + V_{12}^*U_{12})\\
&= G_{12} + H_{12} + G_{12}H_{12}^* + H_{12}G_{12}^*.
\end{aligned}$$

Note that $G_{12}H_{12}^* + H_{12}G_{12}^* = U_{12}V_{12}^* + V_{12}U_{12}^* + U_{12}^*V_{12} + V_{12}^*U_{12} = W_{11} + W_{22}$, where $W_{11} = U_{12}V_{12}^* + V_{12}U_{12}^* \in \mathcal{H}_{11}$ and $W_{22} = U_{12}^*V_{12} + V_{12}^*U_{12} \in \mathcal{H}_{22}$. Since $U_{12} + U_{12}^*, V_{12} + V_{12}^* \in \mathcal{H}_{12}$, then it follows from Claims 2.4 (*i*) and 2.8 that

$$\begin{aligned}
&\delta(G_{12} + H_{12}) + \delta(W_{11}) + \delta(W_{22}) = \delta(G_{12} + H_{12} + W_{11} + W_{22})\\
&= \delta(G_{12} + H_{12} + G_{12}H_{12}^* + H_{12}G_{12}^*) = \delta\Big((P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*)\Big)\\
&= \Big(\delta(P_1) + \delta(U_{12} + U_{12}^*)\Big) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*) + (P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet \Big(\delta(P_2) + \delta(V_{12} + V_{12}^*)\Big)\\
&= \delta\Big(P_1 \circ \frac{1}{2}I \bullet P_2\Big) + \delta\Big(P_1 \circ \frac{1}{2}I \bullet (V_{12} + V_{12}^*)\Big) + \delta\Big((U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet P_2\Big)\\
&+ \delta\Big((U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (V_{12} + V_{12}^*)\Big) = \delta(G_{12}) + \delta(H_{12}) + \delta(G_{12}H_{12}^* + H_{12}G_{12}^*)\\
&= \delta(G_{12}) + \delta(H_{12}) + \delta(W_{11}) + \delta(W_{22}).
\end{aligned}$$

Thus, we have $\delta(G_{12} + H_{12}) = \delta(G_{12}) + \delta(H_{12})$. Hence the claim.

Claim 2.10. *For any $G_{ii}, H_{ii} \in \mathcal{H}_{ii}$ $(i = 1, 2)$, we have*

(*i*) $\delta(G_{11} + H_{11}) = \delta(G_{11}) + \delta(H_{11})$;

(*ii*) $\delta(G_{22} + H_{22}) = \delta(G_{22}) + \delta(H_{22})$.

To prove (i), we shall show that $M = \delta(G_{11} + H_{11}) - \delta(G_{11}) - \delta(H_{11}) = 0$. We can write

$$\begin{aligned}&\delta(P_2 \circ (G_{11} + H_{11}) \bullet P_2) = \delta(P_2 \circ G_{11} \bullet P_2) + \delta(P_2 \circ H_{11} \bullet P_2)\\&= \delta(P_2) \circ (G_{11} + H_{11}) \bullet P_2 + P_2 \circ (\delta(G_{11}) + \delta(H_{11})) \bullet P_2 + P_2 \circ (G_{11} + H_{11}) \bullet \delta(P_2).\end{aligned}$$

Alternatively, we obtain

$$\begin{aligned}&\delta(P_2 \circ (G_{11} + H_{11}) \bullet P_2)\\&= \delta(P_2) \circ (G_{11} + H_{11}) \bullet P_2 + P_2 \circ \delta(G_{11} + H_{11}) \bullet P_2 + P_2 \circ (G_{11} + H_{11}) \bullet \delta(P_2).\end{aligned}$$

Thus, we get $P_2 \circ M \bullet P_2 = 0$. This gives us that $M_{12} = M_{22} = 0$. It remains to show that $M_{11} = 0$. Observe next that, for any $U_{12} \in \mathcal{A}_{12}$, $U = U_{12} + U_{12}^* \in \mathcal{H}_{12}$. Then $U \circ G_{11} \bullet \frac{1}{2}I$, $U \circ H_{11} \bullet \frac{1}{2}I \in \mathcal{H}_{12}$. Therefore, it follows from Claims 2.4 (i) and 2.9 that

$$\begin{aligned}&\delta(U) \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I + U \circ \delta(G_{11} + H_{11}) \bullet \frac{1}{2}I\\&= \delta\left(U \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I\right) = \delta\left(U \circ G_{11} \bullet \frac{1}{2}I\right) + \delta\left(U \circ H_{11} \bullet \frac{1}{2}I\right)\\&= \delta(U) \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I + U \circ \left(\delta(G_{11}) + \delta(H_{11})\right) \bullet \frac{1}{2}I.\end{aligned}$$

Thus, we get $U \circ M \bullet \frac{1}{2}I = 0$. This leads to $M_{11} = 0$, which gives the desired result.

Remark 2.11. *It follows from Claims 2.7–2.10 that δ is additive on $\mathcal{H}$.*

Claim 2.12. $\delta(I) = \delta(iI) = 0$.

From Claims 2.4, 2.6 and Remark 2.11, we get

$$\delta(I) = \delta\left(\frac{1}{2}I + \frac{1}{2}I\right) = \delta\left(\frac{1}{2}I\right) + \delta\left(\frac{1}{2}I\right) = 0$$

and

$$\delta(iI) = i\delta(I) = 0.$$

Claim 2.13. $\delta(K)^* = -\delta(K)$, $\delta(iK) = i\delta(K)$, $\delta(K_1 + K_2) = \delta(K_1) + \delta(K_2)$ *for all* $K, K_1, K_2 \in \mathcal{K}$.

Since for any $K \in \mathcal{K}$, $K \circ I \bullet I = 0$, then from Claims 2.3, 2.12 and the hypothesis, we have

$$0 = \delta(K \circ I \bullet I) = \delta(K) \circ I \bullet I = 2\delta(K) + 2\delta(K)^*.$$

Therefore, we obtain $\delta(K)^* = -\delta(K)$ for all $K \in \mathcal{K}$.

In view of Remark 2.11 and Claim 2.12, we have

$$4\delta(iK) = \delta(4iK) = \delta(K \circ iI \bullet I) = \delta(K) \circ iI \bullet I = 4i\delta(K).$$

Thus, $\delta(iK) = i\delta(K)$ for all $K \in \mathcal{K}$.

Let $K_1, K_2 \in \mathcal{K}$. Then, in view of Remark 2.11 and $\delta(iK) = i\delta(K)$, we have

$$i\delta(K_1 + K_2) = \delta(i(K_1 + K_2)) = \delta(iK_1) + \delta(iK_2) = i(\delta(K_1) + \delta(K_2)).$$

This implies that

$$\delta(K_1 + K_2) = \delta(K_1) + \delta(K_2)$$

for all $K_1, K_2 \in \mathcal{K}$.

Claim 2.14. *For any $A, B \in \mathcal{A}$, $\delta(A+B) = \delta(A) + \delta(B)$ and $\delta(iA) = i\delta(A)$.*

Let $K_1, K_2 \in \mathcal{K}$. Then, using Claims 2.4 and 2.13, we get

$$\begin{aligned} -i\delta(K_1) &= \delta(-iK_1) = \delta\left(K_1 + iK_2 \circ \frac{1}{2}I \bullet \frac{1}{2}iI\right) = \delta(K_1 + iK_2) \circ \frac{1}{2}I \bullet \frac{1}{2}iI \\ &= -\frac{1}{2}i\delta(K_1 + iK_2) + \frac{1}{2}i\delta(K_1 + iK_2)^* \end{aligned} \tag{10}$$

and

$$\begin{aligned} i\delta(K_2) &= \delta(iK_2) = \delta\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet (K_1 + iK_2)\right) = \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta(K_1 + iK_2) \\ &= \frac{1}{2}\delta(K_1 + iK_2) + \frac{1}{2}\delta(K_1 + iK_2)^*. \end{aligned} \tag{11}$$

From Equations (10) and (11), we obtain

$$\delta(K_1 + iK_2) = \delta(K_1) + i\delta(K_2). \tag{12}$$

Next, suppose that $A, B \in \mathcal{A}$ such that $A = S_1 + iS_2$ and $B = K_1 + iK_2$ for $S_1, S_2, K_1, K_2 \in \mathcal{K}$. So, from Equation (12) and Claim 2.13, we have

$$\begin{aligned} \delta(A+B) &= \delta((S_1 + K_1) + i(S_2 + K_2)) = \delta(S_1) + \delta(K_1) + i\delta(S_2) + i\delta(K_2) \\ &= \delta(S_1 + iS_2) + \delta(K_1 + iK_2) = \delta(A) + \delta(B). \end{aligned}$$

Also, we have

$$\delta(iA) = \delta\big(i(S_1 + iS_2)\big) = i\big(\delta(S_1) + i\delta(S_2)\big) = i\delta(A).$$

Claim 2.15. *For any $A \in \mathcal{A}$, $\delta(A^*) = \delta(A)^*$.*

Since $\delta\left(\frac{1}{2}I\right) = 0$, then we have

$$\delta\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet A\right) = \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta(A).$$

This implies that

$$\delta(A + A^*) = \delta(A) + \delta(A)^*.$$

This gives

$$\delta(A^*) = \delta(A)^*.$$

Claim 2.16. *δ is an additive $*$-derivation on $\mathcal{A}$.*

From Claims 2.14 and 2.15, δ is additive with $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$. To complete the proof of Theorem 2.1, it remains to show that δ satisfies the Leibniz rule on $\mathcal{A}$, i.e., $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. For any $A, B \in \mathcal{A}$, we have

$$\begin{aligned} \delta(AB + B^*A^*) &= \delta\left(\frac{1}{2}I \circ B^* \bullet A\right) = \frac{1}{2}I \circ \delta(B^*) \bullet A + \frac{1}{2}I \circ B^* \bullet \delta(A) \\ &= \delta(B^*)A^* + A\delta(B^*)^* + B^*\delta(A)^* + \delta(A)B \\ &= \delta(B^*)A^* + A\delta(B) + B^*\delta(A)^* + \delta(A)B. \end{aligned} \tag{13}$$

Replacing B by iB in Equation (13) and using Claim 2.15, we obtain

$$\delta(iAB - iB^*A^*) = \delta(-iB^*)A^* + A\delta(iB) - iB^*\delta(A)^* + i\delta(A)B$$

$$= i\Big(- \delta(B^*)A^* + A\delta(B) - B^*\delta(A)^* + \delta(A)B\Big).$$

This implies that

$$\delta(AB - B^*A^*) = -\delta(B^*)A^* + A\delta(B) - B^*\delta(A)^* + \delta(A)B. \tag{14}$$

Adding Equations (13) and (14), we obtain

$$\delta(AB) = \delta(A)B + A\delta(B).$$

Hence, the proof of Theorem 2.1 is completed.

Let $\mathfrak{H}$ be a complex Hilbert space and $\mathcal{B}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathfrak{H})$ is said to be a standard operator algebra if it contains all finite rank operators on $\mathfrak{H}$.

Recall that a von Neumann algebra $\mathcal{A}$ is a weakly closed self-adjoint algebra of operators on $\mathfrak{H}$, contains the identity operator I. $\mathcal{A}$ is said to be a factor von Neumann algebra if its centre is trivial. As the factor von Neumann algebras and standard operator algebras are prime $*$-algebras, the following results are the immediate consequences of Theorem 2.1.

Corollary 2.17. *Let $\mathcal{A}$ ba a factor von Neumann algebra with $dim(\mathcal{A}) \geq 2$. Then a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathcal{A}$, is an additive $*$-derivation.

Corollary 2.18. *Let $\mathfrak{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathfrak{H}$ containing the identity operator I. If $\mathcal{A}$ is closed under the adjoint operation, then every nonlinear mixed bi-skew Jordan triple derivation i.e., a map $\delta : \mathcal{A} \to \mathcal{B}(\mathfrak{H})$ satisfying*

$$\delta(A \circ B \bullet C) = \delta(A) \circ B \bullet C + A \circ \delta(B) \bullet C + A \circ B \bullet \delta(C)$$

for all $A, B, C \in \mathcal{A}$, is an additive $*$-derivation.

3. Non-linear mixed bi-skew Jordan triple higher derivations on prime $*$-algebras

In this section, we show that a non-linear mixed bi-skew Jordan triple higher derivation on a prime $*$-algebra $\mathcal{A}$, is an additive $*$-higher derivation on $\mathcal{A}$. Precisely, we prove the following:

Theorem 3.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unity I and a non-trivial projection. Let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a non-linear mixed bi-skew Jordan triple higher derivation on $\mathcal{A}$ i.e., $\delta_0 = id_{\mathcal{A}}$ (the identity map on $\mathcal{A}$) and*

$$\delta_n(A \circ B \bullet C) = \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n}} \delta_p(A) \circ \delta_q(B) \bullet \delta_r(C) \tag{15}$$

for all $A, B, C \in \mathcal{A}$ and for each $n \in \mathbb{N}$. Then Δ is an additive $*$-higher derivation on $\mathcal{A}$.

We shall use the method of mathematical induction on $n \in \mathbb{N}$ to prove Theorem 3.1. The following series of claims establishes the proof.

Claim 3.2. $\delta_n(0) = 0$ *for each $n \in \mathbb{N}$.*

For $n = 0$ the result is obvious and for $n = 1$, it is true by Claim 2.3. Using the induction hypothesis, suppose that the result holds for $m \leq n-1$, i.e., $\delta_m(0) = 0$. We show that it holds for $m = n$. We have

$$\begin{aligned}\delta_n(0) &= \delta_n(0 \circ 0 \bullet 0)\\ &= \delta_n(0) \circ 0 \bullet 0 + 0 \circ \delta_n(0) \bullet 0 + 0 \circ 0 \bullet \delta_n(0) + \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(0) \circ \delta_q(0) \bullet \delta_r(0)\\ &= 0.\end{aligned}$$

Claim 3.3.

(*i*) $\delta_n(\frac{1}{2}I) = 0$, *for each* $n \in \mathbb{N}$ *with* $n \geq 1$;

(*ii*) $\delta_n(-\frac{1}{2}I) = 0$, *for each* $n \in \mathbb{N}$ *with* $n \geq 1$;

(*iii*) $\delta_n(\frac{1}{2}iI) = 0$, *for each* $n \in \mathbb{N}$ *with* $n \geq 1$.

The results hold for $n = 1$ by Claim 2.4. Let us assume that they are true for $m \leq n-1$, i.e., $\delta_m(\frac{1}{2}I) = 0$, $\delta_m(-\frac{1}{2}I) = 0$ and $\delta_m(\frac{1}{2}iI) = 0$. We shall show that they hold for $m = n$.

(*i*) Let $A = B = C = \frac{1}{2}I$ in Equation (15). Then, we can write

$$\begin{aligned}
\delta_n\left(\frac{1}{2}I\right) &= \delta_n\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet \frac{1}{2}I\right) \\
&= \delta_n\left(\frac{1}{2}I\right) \circ \frac{1}{2}I \bullet \frac{1}{2}I + \frac{1}{2}I \circ \delta_n\left(\frac{1}{2}I\right) \bullet \frac{1}{2}I + \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta_n\left(\frac{1}{2}I\right) \\
&+ \sum_{\substack{p+q+r=n \\ 0\leq p,q,r\leq n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(\frac{1}{2}I\right) \\
&= \frac{3}{2}\left(\delta_n\left(\frac{1}{2}I\right) + \delta_n\left(\frac{1}{2}I\right)^*\right). \qquad (16)
\end{aligned}$$

From Equation (16) it is evident that $\delta_n\left(\frac{1}{2}I\right)$ is self-adjoint and hence again from Equation (16), we get $\delta_n\left(\frac{1}{2}I\right) = 0$.

(*ii*) If we put $A = B = \frac{1}{2}I$ and $C = -\frac{1}{2}I$ in Equation (15), then we get

$$\begin{aligned}
\delta_n\left(-\frac{1}{2}I\right) &= \delta_n\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet -\frac{1}{2}I\right) \\
&= \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta_n\left(-\frac{1}{2}I\right) + \sum_{\substack{p+q+r=n \\ 0\leq p,q,r\leq n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(-\frac{1}{2}I\right) \\
&= \frac{1}{2}\left(\delta_n\left(-\frac{1}{2}I\right) + \delta_n\left(-\frac{1}{2}I\right)^*\right). \qquad (17)
\end{aligned}$$

From Equation (17) we observe that $\delta_n\left(-\frac{1}{2}I\right)$ is self-adjoint, i.e.,

$$\delta_n\left(-\frac{1}{2}I\right)^* = \delta_n\left(-\frac{1}{2}I\right). \qquad (18)$$

Also

$$\begin{aligned}
0 &= \delta_n\left(\frac{1}{2}I\right) \\
&= \delta_n\left(\frac{1}{2}I \circ -\frac{1}{2}I \bullet -\frac{1}{2}I\right) \\
&= \frac{1}{2}I \circ \delta_n\left(-\frac{1}{2}I\right) \bullet -\frac{1}{2}I + \frac{1}{2}I \circ -\frac{1}{2}I \bullet \delta_n\left(-\frac{1}{2}I\right) + \sum_{\substack{p+q+r=n \\ 0\leq p,q,r\leq n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(-\frac{1}{2}I\right) \bullet \delta_r\left(-\frac{1}{2}I\right) \\
&= -\left(\delta_n\left(-\frac{1}{2}I\right) + \delta_n\left(-\frac{1}{2}I\right)^*\right).
\end{aligned}$$

Thus, we get

$$\delta_n\left(-\frac{1}{2}I\right)^* = -\delta_n\left(-\frac{1}{2}I\right). \tag{19}$$

From Equations (18) and (19), we obtain $\delta_n\left(-\frac{1}{2}I\right) = 0$.

(*iii*) Let $A = B = \frac{1}{2}iI$ and $C = \frac{1}{2}I$ in Equation (15). Then, we have

$$\begin{aligned}
0 &= \delta_n\left(-\frac{1}{2}I\right) \\
&= \delta_n\left(\frac{1}{2}iI \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right) \\
&= \delta_n\left(\frac{1}{2}iI\right) \circ \frac{1}{2}iI \bullet \frac{1}{2}I + \frac{1}{2}iI \circ \delta_n\left(\frac{1}{2}iI\right) \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}iI\right) \circ \delta_q\left(\frac{1}{2}iI\right) \bullet \delta_r\left(\frac{1}{2}I\right) \\
&= i\left(\delta_n\left(\frac{1}{2}iI\right) - \delta_n\left(\frac{1}{2}iI\right)^*\right).
\end{aligned}$$

This gives

$$\delta_n\left(\frac{1}{2}iI\right)^* = \delta_n\left(\frac{1}{2}iI\right). \tag{20}$$

On the other hand, we have

$$\begin{aligned}
0 &= \delta_n\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet \frac{1}{2}iI\right) \\
&= \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta_n\left(\frac{1}{2}iI\right) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(\frac{1}{2}iI\right) \\
&= \frac{1}{2}\left(\delta_n\left(\frac{1}{2}iI\right)^* + \delta_n\left(\frac{1}{2}iI\right)\right).
\end{aligned}$$

This implies that

$$\delta_n\left(\frac{1}{2}iI\right)^* = -\delta_n\left(\frac{1}{2}iI\right). \tag{21}$$

From Equations (20) and (21), we obtain $\delta_n\left(\frac{1}{2}iI\right) = 0$.

Claim 3.4. *For any $H \in \mathcal{H}$, $\delta_n(H)^* = \delta_n(H)$ for each $n \in \mathbb{N}$.*

Observe that $H = H \circ \frac{1}{2}I \bullet \frac{1}{2}I$. It follows from Claim 3.3 (*i*) that

$$\begin{aligned}
\delta_n(H) &= \delta_n\left(H \circ \frac{1}{2}I \bullet \frac{1}{2}I\right) \\
&= \delta_n(H) \circ \frac{1}{2}I \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p(H) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(\frac{1}{2}I\right) \\
&= \frac{1}{2}\left(\delta_n(H) + \delta_n(H)^*\right).
\end{aligned}$$

Thus, $\delta_n(H)^* = \delta_n(H)$.

Claim 3.5. *For any $H \in \mathcal{H}$ and $n \in \mathbb{N}$, we have*

(*i*) $\delta_n(-iH) = -i\delta_n(H)$;
(*ii*) $\delta_n(iH) = i\delta_n(H)$.

Observe that $-iH \circ \frac{1}{2}I \bullet \frac{1}{2}I = H \circ \frac{1}{2}iI \bullet \frac{1}{2}I = 0$. It follows from Claims 3.2 and 3.3 that

$$\delta_n\left(-iH \circ \frac{1}{2}I \bullet \frac{1}{2}I\right) = \delta_n\left(H \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right).$$

This implies that

$$\begin{aligned}&\delta_n(-iH) \circ \frac{1}{2}I \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(-iH) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(\frac{1}{2}I\right)\\ &= \delta_n(H) \circ \frac{1}{2}iI \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(H) \circ \delta_q\left(\frac{1}{2}iI\right) \bullet \delta_r\left(\frac{1}{2}I\right).\end{aligned}$$

It follows that

$$\delta_n(-iH) \circ \frac{1}{2}I \bullet \frac{1}{2}I = \delta_n(H) \circ \frac{1}{2}iI \bullet \frac{1}{2}I.$$

Using Claim 3.4, we obtain

$$\delta_n(-iH) + \delta_n(-iH)^* = 0. \tag{22}$$

Next, since $H = -iH \circ \frac{1}{2}iI \bullet \frac{1}{2}I = \frac{1}{2}I \circ H \bullet \frac{1}{2}I$, then

$$\delta_n\left(-iH \circ \frac{1}{2}iI \bullet \frac{1}{2}I\right) = \delta_n\left(\frac{1}{2}I \circ H \bullet \frac{1}{2}I\right).$$

Hence

$$\begin{aligned}&\delta_n(-iH) \circ \frac{1}{2}iI \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(-iH) \circ \delta_q\left(\frac{1}{2}iI\right) \bullet \delta_r\left(\frac{1}{2}I\right)\\ &= \frac{1}{2}I \circ \delta_n(H) \bullet \frac{1}{2}I + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q(H) \bullet \delta_r\left(\frac{1}{2}I\right).\end{aligned}$$

This gives

$$\delta_n(-iH) - \delta_n(-iH)^* = -2i\delta_n(H). \tag{23}$$

Addition of Equations (22) and (23) leads to

$$\delta_n(-iH) = -i\delta_n(H).$$

Similarly, one can easily obtain that $\delta_n(iH) = i\delta_n(H)$.

Claim 3.6. *For any $H_{ii} \in \mathcal{H}_{ii}, (i = 1, 2)$, $H_{12} \in \mathcal{H}_{12}$ and $n \in \mathbb{N}$, we have*

$$\delta_n(H_{ii} + H_{12}) = \delta_n(H_{ii}) + \delta_n(H_{12}).$$

By Claim 2.7 the result is true for $n = 1$. Using the induction hypothesis, suppose that it holds for $m \leq n-1$, i.e., $\delta_m(H_{ii} + H_{12}) = \delta_m(H_{ii}) + \delta_m(H_{12})$. We have to show that it is true for $m = n$. Let $M = \delta_n(H_{11} + H_{12}) - \delta_n(H_{11}) - \delta_n(H_{12})$. So, by Claim 3.4, we have $M^* = M$. Proving the claim, we have to show that $M = 0$. We prove the Claim for $i = 1$ and the case for $i = 2$ can be proved analogously. We can write

$$\begin{aligned}
&\delta_n\Big(P_2 \circ (H_{11} + H_{12}) \bullet P_2\Big)\\
&= \delta_n(P_2 \circ H_{11} \bullet P_2) + \delta_n(P_2 \circ H_{12} \bullet P_2)\\
&= \delta_n(P_2) \circ (H_{11} + H_{12}) \bullet P_2 + P_2 \circ (\delta_n(H_{11}) + \delta_n(H_{12})) \bullet P_2\\
&+ P_2 \circ (H_{11} + H_{12}) \bullet \delta_n(P_2) + \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2) \circ (\delta_q(H_{11}) + \delta_q(H_{12})) \bullet \delta_r(P_2).
\end{aligned}$$

On the other hand, we write

$$\begin{aligned}
&\delta_n(P_2 \circ (H_{11} + H_{12}) \bullet P_2)\\
&= \delta_n(P_2) \circ (H_{11} + H_{12}) \bullet P_2 + P_2 \circ \delta_n(H_{11} + H_{12}) \bullet P_2 + P_2 \circ (H_{11} + H_{12}) \bullet \delta_n(P_2)\\
&+ \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2) \circ \delta_q(H_{11} + H_{12}) \bullet \delta_r(P_2).
\end{aligned}$$

From the last two expressions, we get $P_2 \circ M \bullet P_2 = 0$. This gives $M_{12} = M_{22} = 0$. Now, since $(P_2 - P_1) \circ H_{12} \bullet P_1 = 0$, then we have

$$\begin{aligned}
&\delta_n(P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1 + (P_2 - P_1) \circ \delta_n(H_{11} + H_{12}) \bullet P_1\\
&+ (P_2 - P_1) \circ (H_{11} + H_{12}) \bullet \delta_n(P_1) + \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2 - P_1) \circ \delta_q(H_{11} + H_{12}) \bullet \delta_r(P_1)\\
&= \delta_n\Big((P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1\Big)\\
&= \delta_n\Big((P_2 - P_1) \circ H_{11} \bullet P_1\Big) + \delta_n\Big((P_2 - P_1) \circ H_{12} \bullet P_1\Big)\\
&= \delta_n(P_2 - P_1) \circ (H_{11} + H_{12}) \bullet P_1 + (P_2 - P_1) \circ (\delta_n(H_{11}) + \delta_n(H_{12})) \bullet P_1\\
&+ (P_2 - P_1) \circ (H_{11} + H_{12}) \bullet \delta_n(P_1) + \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2 - P_1) \circ (\delta_q(H_{11}) + \delta_q(H_{12})) \bullet \delta_r(P_1).
\end{aligned}$$

This implies that $(P_2 - P_1) \circ M \bullet P_1 = 0$ and hence $M_{11} = 0$. Therefore, $M = 0$, i.e.,

$$\delta_n(H_{11} + H_{12}) = \delta_n(H_{11}) + \delta_n(H_{12}).$$

Similarly, one can easily obtain that $\delta_n(H_{12} + H_{22}) = \delta_n(H_{12}) + \delta_n(H_{22})$.

Claim 3.7. *For any $H_{11} \in \mathcal{H}_{11}, H_{12} \in \mathcal{H}_{12}, H_{22} \in \mathcal{H}_{22}$ and $n \in \mathbb{N}$, we have*

$$\delta_n(H_{11} + H_{12} + H_{22}) = \delta_n(H_{11}) + \delta_n(H_{12}) + \delta_n(H_{22}).$$

For $n = 0$ it is trivial and for $n = 1$ it holds by Claim 2.8. In view of the induction hypothesis, let the result hold for $m \leq n-1$, i.e., $\delta_m(H_{11} + H_{12} + H_{22}) = \delta_m(H_{11}) + \delta_m(H_{12}) + \delta_m(H_{22})$. We have to show that it also holds for $m = n$. It is sufficient to show that $M = \delta_n(H_{11} + H_{12} + H_{22}) - \delta_n(H_{11}) - \delta_n(H_{12}) - \delta_n(H_{22}) = 0$. It follows from Claim 3.6 and $P_1 \circ H_{22} \bullet P_1 = 0$ that

$$\delta_n\Big(P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet P_1\Big)$$

$$
\begin{aligned}
&= \delta_n\Big(P_1 \circ (H_{11} + H_{12}) \bullet P_1\Big) + \delta_n\Big(P_1 \circ H_{22} \bullet P_1\Big)\\
&= \delta_n(P_1) \circ (H_{11} + H_{12} + H_{22}) \bullet P_1 + P_1 \circ (\delta_n(H_{11}) + \delta_n(H_{12}) + \delta_n(H_{22})) \bullet P_1\\
&+ P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet \delta_n(P_1) + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(P_1) \circ (\delta_q(H_{11}) + \delta_q(H_{12}) + \delta_q(H_{22})) \bullet \delta_r(P_1).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\delta_n\Big(P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet P_1\Big)\\
&= \delta_n(P_1) \circ (H_{11} + H_{12} + H_{22}) \bullet P_1 + P_1 \circ \delta_n(H_{11} + H_{12} + H_{22}) \bullet P_1\\
&+ P_1 \circ (H_{11} + H_{12} + H_{22}) \bullet \delta_n(P_1) + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(P_1) \circ (\delta_q(H_{11} + H_{12} + H_{22})) \bullet \delta_r(P_1).
\end{aligned}
$$

From the last two expressions we conclude that $P_1 \circ M \bullet P_1 = 0$. From this, we obtain $M_{11} = M_{12} = 0$. Next, since $(P_2 - P_1) \circ H_{11} \bullet P_2 = 0$, then reasoning as above, we get $M_{22} = 0$ and thus, $M = 0$. Hence the result.

Claim 3.8. *For any $G_{12}, H_{12} \in \mathcal{H}_{12}$ and $n \in \mathbb{N}$, we have*

$$\delta_n(G_{12} + H_{12}) = \delta_n(G_{12}) + \delta_n(H_{12}).$$

For any $U_{12}, V_{12} \in \mathcal{A}_{12}$, assume that $G_{12} = U_{12} + U_{12}^* \in \mathcal{H}_{12}$ and $H_{12} = V_{12} + V_{12}^* \in \mathcal{H}_{12}$. Thus

$$
\begin{aligned}
&(P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*)\\
&= (U_{12} + U_{12}^*) + (V_{12} + V_{12}^*) + (U_{12}V_{12}^* + V_{12}U_{12}^* + U_{12}^*V_{12} + V_{12}^*U_{12})\\
&= G_{12} + H_{12} + G_{12}H_{12}^* + H_{12}G_{12}^*.
\end{aligned}
$$

Note that $G_{12}H_{12}^* + H_{12}G_{12}^* = U_{12}V_{12}^* + V_{12}U_{12}^* + U_{12}^*V_{12} + V_{12}^*U_{12} = W_{11} + W_{22}$, where $W_{11} = U_{12}V_{12}^* + V_{12}U_{12}^* \in \mathcal{H}_{11}$ and $W_{22} = U_{12}^*V_{12} + V_{12}^*U_{12} \in \mathcal{H}_{22}$. Since $U_{12} + U_{12}^*, V_{12} + V_{12}^* \in \mathcal{H}_{12}$, then it follows from Claims 3.3 and 3.7 that

$$
\begin{aligned}
&\delta_n(G_{12} + H_{12}) + \delta_n(W_{11}) + \delta_n(W_{22})\\
&= \delta_n(G_{12} + H_{12} + W_{11} + W_{22})\\
&= \delta_n(G_{12} + H_{12} + G_{12}H_{12}^* + H_{12}G_{12}^*)\\
&= \delta_n\left((P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*)\right)\\
&= \left(\delta_n(P_1) + \delta_n(U_{12} + U_{12}^*)\right) \circ \frac{1}{2}I \bullet (P_2 + V_{12} + V_{12}^*)\\
&+ (P_1 + U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet \left(\delta_n(P_2) + \delta_n(V_{12} + V_{12}^*)\right)\\
&+ \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p(P_1 + U_{12} + U_{12}^*) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r(P_2 + V_{12} + V_{12}^*)\\
&= \delta_n\left(P_1 \circ \frac{1}{2}I \bullet P_2\right) + \delta_n\left(P_1 \circ \frac{1}{2}I \bullet (V_{12} + V_{12}^*)\right)\\
&+ \delta_n\left((U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet P_2\right) + \delta_n\left((U_{12} + U_{12}^*) \circ \frac{1}{2}I \bullet (V_{12} + V_{12}^*)\right)\\
&= \delta_n(G_{12}) + \delta_n(H_{12}) + \delta_n(G_{12}H_{12}^* + H_{12}G_{12}^*)\\
&= \delta_n(G_{12}) + \delta_n(H_{12}) + \delta_n(W_{11}) + \delta_n(W_{22}).
\end{aligned}
$$

Thus, we obtain $\delta_n(G_{12} + H_{12}) = \delta_n(G_{12}) + \delta_n(H_{12})$. Hence the claim.

Claim 3.9. *For any $G_{ii}, H_{ii} \in \mathcal{H}_{ii}$ $(i = 1, 2)$ and $n \in \mathbb{N}$, we have*

(*i*) $\delta_n(G_{11} + H_{11}) = \delta_n(G_{11}) + \delta_n(H_{11})$;
(*ii*) $\delta_n(G_{22} + H_{22}) = \delta_n(G_{22}) + \delta_n(H_{22})$.

For $n = 1$, the result holds by Claim 2.10. Let us assume that it holds for all $m \leq n - 1$. We shall show that it also holds for $m = n$. To prove (*i*), we shall show that $M = \delta_n(G_{11} + H_{11}) - \delta_n(G_{11}) - \delta_n(H_{11}) = 0$. We can write

$$\begin{aligned}
&\delta_n\Big(P_2 \circ (G_{11} + H_{11}) \bullet P_2\Big)\\
&= \delta_n(P_2 \circ G_{11} \bullet P_2) + \delta_n(P_2 \circ H_{11} \bullet P_2)\\
&= \delta_n(P_2) \circ (G_{11} + H_{11}) \bullet P_2 + P_2 \circ (\delta_n(G_{11}) + \delta_n(H_{11})) \bullet P_2 + P_2 \circ (G_{11} + H_{11}) \bullet \delta_n(P_2)\\
&+ \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2) \circ (\delta_q(G_{11}) + \delta_q(H_{11})) \bullet \delta_r(P_2).
\end{aligned}$$

Alternatively, we write

$$\begin{aligned}
&\delta_n\Big(P_2 \circ (G_{11} + H_{11}) \bullet P_2\Big)\\
&= \delta_n(P_2) \circ (G_{11} + H_{11}) \bullet P_2 + P_2 \circ \delta_n(G_{11} + H_{11}) \bullet P_2 + P_2 \circ (G_{11} + H_{11}) \bullet \delta_n(P_2)\\
&+ \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(P_2) \circ \delta_q(G_{11} + H_{11}) \bullet \delta_r(P_2).
\end{aligned}$$

Thus, we get $P_2 \circ M \bullet P_2 = 0$. This gives us that $M_{12} = M_{22} = 0$. It remains to show that $M_{11} = 0$. Observe next that, for any $U_{12} \in \mathcal{A}_{12}$, $U = U_{12} + U_{12}^* \in \mathcal{H}_{12}$. Then $U \circ G_{11} \bullet \frac{1}{2}I$, $U \circ H_{11} \bullet \frac{1}{2}I \in \mathcal{H}_{12}$. Therefore, it follows from Claims 3.3 and 3.8 that

$$\begin{aligned}
&\delta_n(U) \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I + U \circ \delta_n(G_{11} + H_{11}) \bullet \frac{1}{2}I\\
&+ \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(U) \circ \delta_q(G_{11} + H_{11}) \bullet \delta_r\left(\frac{1}{2}I\right)\\
&= \delta_n\left(U \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I\right)\\
&= \delta_n\left(U \circ G_{11} \bullet \frac{1}{2}I\right) + \delta_n\left(U \circ H_{11} \bullet \frac{1}{2}I\right)\\
&= \delta_n(U) \circ (G_{11} + H_{11}) \bullet \frac{1}{2}I + U \circ \Big(\delta_n(G_{11}) + \delta_n(H_{11})\Big) \bullet \frac{1}{2}I\\
&+ \sum_{\substack{p+q+r=n\\ 0\leq p,q,r\leq n-1}} \delta_p(U) \circ \Big(\delta_q(G_{11}) + \delta_q(H_{11})\Big) \bullet \delta_r\left(\frac{1}{2}I\right).
\end{aligned}$$

Using the similar arguments as used above, we get $U \circ M \bullet \frac{1}{2}I = 0$. This leads to $M_{11} = 0$, which completes the proof of part (*i*). In the similar manner, part (*ii*) can be proved easily. Therefore, we obtain the desired result.

Remark 3.10. *It follows from Claims 3.6–3.9 that δ_n is additive on $\mathcal{H}$.*

Claim 3.11. $\delta_n(I) = \delta_n(iI) = 0$ *for each $n \in \mathbb{N}$ with $n \geq 1$.*

From Claims 3.3, 3.5 and Remark 3.10, we get

$$\delta_n(I) = \delta_n\left(\frac{1}{2}I + \frac{1}{2}I\right) = \delta_n\left(\frac{1}{2}I\right) + \delta_n\left(\frac{1}{2}I\right) = 0$$

and

$$\delta_n(iI) = i\delta_n(I) = 0.$$

Claim 3.12. *$\delta_n(K)^* = -\delta_n(K)$, $\delta_n(iK) = i\delta_n(K)$, $\delta_n(K_1 + K_2) = \delta_n(K_1) + \delta_n(K_2)$ for all $K, K_1, K_2 \in \mathcal{K}$.*

Since for any $K \in \mathcal{K}$, $K \circ I \bullet I = 0$, then from Claim 3.11 and the hypothesis, we have

$$\begin{aligned} 0 &= \delta_n(K \circ I \bullet I) \\ &= \delta_n(K) \circ I \bullet I + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p(K) \circ \delta_q(I) \bullet \delta_r(I) \\ &= 2\delta_n(K) + 2\delta_n(K)^*. \end{aligned}$$

Therefore, we have $\delta_n(K)^* = -\delta_n(K)$ for all $K \in \mathcal{K}$.

In view of Remark 3.10 and Claim 3.11, we have

$$\begin{aligned} 4\delta_n(iK) &= \delta_n(4iK) \\ &= \delta_n(K \circ iI \bullet I) \\ &= \delta_n(K) \circ iI \bullet I + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p(K) \circ \delta_q(iI) \bullet \delta_r(I) \\ &= 4i\delta_n(K). \end{aligned}$$

Thus, $\delta_n(iK) = i\delta_n(K)$ for all $K \in \mathcal{K}$.

Now, let $K_1, K_2 \in \mathcal{K}$. Then, in view of Remark 3.10 and $\delta_n(iK) = i\delta_n(K)$, we have

$$i\delta_n(K_1 + K_2) = \delta_n\big(i(K_1 + K_2)\big) = \delta_n(iK_1) + \delta_n(iK_2) = i\big(\delta_n(K_1) + \delta_n(K_2)\big).$$

This gives

$$\delta_n(K_1 + K_2) = \delta_n(K_1) + \delta_n(K_2)$$

for all $K_1, K_2 \in \mathcal{K}$.

Claim 3.13. *For any $A, B \in \mathcal{A}$ and $n \in \mathbb{N}$, $\delta_n(A + B) = \delta_n(A) + \delta_n(B)$ and $\delta_n(iA) = i\delta_n(A)$.*

Let $K_1, K_2 \in \mathcal{K}$. Then, from Claim 3.3 and Remark 3.10, we have

$$\begin{aligned} -i\delta_n(K_1) &= \delta_n(-iK_1) \\ &= \delta_n\left((K_1 + iK_2) \circ \frac{1}{2}I \bullet \frac{1}{2}iI\right) \\ &= \delta_n(K_1 + iK_2) \circ \frac{1}{2}I \bullet \frac{1}{2}iI + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \delta_p(K_1 + iK_2) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r\left(\frac{1}{2}iI\right) \\ &= -\frac{1}{2}i\delta_n(K_1 + iK_2) + \frac{1}{2}i\delta_n(K_1 + iK_2)^* \end{aligned} \tag{24}$$

and

$$i\delta_n(K_2) = \delta_n(iK_2)$$

$$
\begin{aligned}
&= \delta_n\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet (K_1 + iK_2)\right)\\
&= \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta_n(K_1 + iK_2) + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r(K_1 + iK_2)\\
&= \frac{1}{2}\delta_n(K_1 + iK_2) + \frac{1}{2}\delta_n(K_1 + iK_2)^*. \qquad (25)
\end{aligned}
$$

From Equations (24) and (25), we obtain

$$\delta_n(K_1 + iK_2) = \delta_n(K_1) + i\delta_n(K_2). \qquad (26)$$

Next, suppose that $A, B \in \mathcal{A}$ such that $A = S_1 + iS_2$ and $B = K_1 + iK_2$ for $S_1, S_2, K_1, K_2 \in \mathcal{K}$. So, from Equation (26) and Claim 3.12, we have

$$
\begin{aligned}
\delta_n(A + B) &= \delta_n\Big((S_1 + K_1) + i(S_2 + K_2)\Big)\\
&= \delta_n(S_1) + \delta_n(K_1) + i\delta_n(S_2) + i\delta_n(K_2)\\
&= \delta_n(S_1 + iS_2) + \delta_n(K_1 + iK_2)\\
&= \delta_n(A) + \delta_n(B).
\end{aligned}
$$

Thus, we have

$$\delta_n(iA) = \delta_n\Big(i(S_1 + iS_2)\Big) = i\Big(\delta_n(S_1) + i\delta_n(S_2)\Big) = i\delta_n(A)$$

for all $A \in \mathcal{A}$.

Claim 3.14. *For any $A \in \mathcal{A}$ and $n \in \mathbb{N}$, we have $\delta_n(A^*) = \delta_n(A)^*$.*

Since $\delta_n\left(\frac{1}{2}I\right) = 0$ for all $n \geq 1$, we have

$$\delta_n\left(\frac{1}{2}I \circ \frac{1}{2}I \bullet A\right) = \frac{1}{2}I \circ \frac{1}{2}I \bullet \delta_n(A) + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q\left(\frac{1}{2}I\right) \bullet \delta_r(A).$$

This implies that

$$\delta_n(A + A^*) = \delta_n(A) + \delta_n(A)^*.$$

It gives

$$\delta_n(A^*) = \delta_n(A)^*.$$

Claim 3.15. *$\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is an additive $*$-higher derivation on $\mathcal{A}$.*

It has been proved in Claims 3.13 and 3.14 that $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$, is additive with $\delta_n(A^*) = \delta_n(A)^*$ for all $A \in \mathcal{A}$ and for each $n \geq 1$. Now, for any $A, B \in \mathcal{A}$ and for each $n \geq 1$, we have

$$
\begin{aligned}
&\delta_n(AB + B^*A^*) = \delta_n(\frac{1}{2}I \circ B^* \bullet A)\\
&= \frac{1}{2}I \circ \delta_n(B^*) \bullet A + \frac{1}{2}I \circ B^* \bullet \delta_n(A) + \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n-1}} \delta_p\left(\frac{1}{2}I\right) \circ \delta_q(B^*) \bullet \delta_r(A)\\
&= \delta_n(B^*)A^* + A\delta_n(B^*)^* + B^*\delta_n(A)^* + \delta_n(A)B + \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} \Big\{\delta_q(B^*)\delta_r(A)^* + \delta_r(A)\delta_q(B^*)^*\Big\}
\end{aligned}
$$

$$= \delta_n(B^*)A^* + A\delta_n(B) + B^*\delta_n(A)^* + \delta_n(A)B + \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} \left\{\delta_q(B)^*\delta_r(A)^* + \delta_r(A)\delta_q(B)\right\}. \tag{27}$$

Replacing B by iB in Equation (27) and using Claim 3.14, we obtain

$$\begin{aligned}
&\delta_n(iAB - iB^*A^*)\\
&= \delta_n(-iB^*)A^* + A\delta_n(iB) - iB^*\delta_n(A)^* + i\delta_n(A)B + \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} \left\{\delta_q(iB)^*\delta_r(A)^* + \delta_r(A)\delta_q(iB)\right\}\\
&= i\Big(-\delta_n(B^*)A^* + A\delta_n(B) - B^*\delta_n(A)^* + \delta_n(A)B\Big) + \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} i\{-\delta_q(B)^*\delta_r(A)^* + \delta_r(A)\delta_q(B)\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\delta_n(AB - B^*A^*) = &-\delta_n(B^*)A^* + A\delta_n(B) - B^*\delta_n(A)^* + \delta_n(A)B\\
&+ \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} \{-\delta_q(B)^*\delta_r(A)^* + \delta_r(A)\delta_q(B)\}.
\end{aligned} \tag{28}$$

Adding Equations (27) and (28), we obtain

$$\delta_n(AB) = \delta_n(A)B + A\delta_n(B) + \sum_{\substack{q+r=n\\ 1\le q,r\le n-1}} \delta_r(A)\delta_q(B) = \sum_{\substack{q+r=n\\ 0\le q,r\le n}} \delta_r(A)\delta_q(B).$$

Therefore, $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is an additive $*$-higher derivation on $\mathcal{A}$. This completes the proof of Theorem 3.1.

Applying Theorem 3.1 on some special classes of prime $*$-algebras such as factor von Neumann algebras and standard operator algebras, we have the following results:

Corollary 3.16. *Let $\mathcal{A}$ be a factor von Neumann algebra with $dim(\mathcal{A}) \ge 2$. Let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a non-linear mixed bi-skew Jordan triple higher derivation on $\mathcal{A}$ i.e., $\delta_0 = id_{\mathcal{A}}$ (the identity map on $\mathcal{A}$) and*

$$\delta_n(A \circ B \bullet C) = \sum_{\substack{p+q+r=n\\ 0\le p,q,r\le n}} \delta_p(A) \circ \delta_q(B) \bullet \delta_r(C)$$

for all $A, B, C \in \mathcal{A}$ and $n \in \mathbb{N}$. Then Δ is an additive $*$-higher derivation on $\mathcal{A}$.

Corollary 3.17. *Let $\mathfrak{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathfrak{H}$ containing the identity operator I. If $\mathcal{A}$ is closed under the adjoint operation, then every non-linear mixed bi-skew Jordan triple higher derivation $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}(\mathfrak{H})$, is an additive $*$-higher derivation.*

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