Filomat 39:12 (2025), 4033–4042
https://doi.org/10.2298/FIL2512033L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Lie triple centralizers and generalized Lie triple derivations on triangular operator algebras by local actions

Xinzhuo Liu[a,*], **Jianhua Zhang**[a]

[a] *School of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, China*

Abstract. Let $\mathcal{U}$ be a triangular operator algebra, and $\phi : \mathcal{U} \to \mathcal{U}$ be a linear map. In this paper, under some mild conditions on $\mathcal{U}$, we prove that if ϕ satisfies

$$\phi([[U, V], W]) = [[\phi(U), V], W] = [[U, \phi(V)], W]$$

for any $U, V, W \in \mathcal{U}$ with $UV = UW = P$ being the standard idempotent(resp. $UV = UW = 0$), then there exist $\lambda \in \mathcal{Z}(\mathcal{U})$ and a linear map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ satisfying $\tau([[U, V], W]) = 0$ for any $U, V, W \in \mathcal{U}$ with $UV = UW = P$(resp.$UV = UW = 0$) such that $\phi(U) = \lambda U + \tau(U)$ for $U \in \mathcal{U}$. As an application, we give a characterization of generalized Lie triple derivations on $\mathcal{U}$.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over the complex field $\mathbb{C}$. By $B(\mathcal{X})$ we denote the algebra of all bounded linear operators on $\mathcal{X}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital subalgebras of $B(\mathcal{X})$ and $B(\mathcal{Y})$, respectively. Let $\mathcal{M} \subset B(\mathcal{Y}, \mathcal{X})$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, that is, for $a \in \mathcal{A}$, $a\mathcal{M} = 0$ implies $a = 0$, and for $b \in \mathcal{B}$, $\mathcal{M}b = 0$ implies $b = 0$. Under the usual matrix operations,

$$\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\} \subset B(\mathcal{X} \oplus \mathcal{Y})$$

is a triangular operator algebra with the unit $I = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & I_{\mathcal{B}} \end{pmatrix}$, where $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are the units of the algebra $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote

$$P_1 = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{B}} \end{pmatrix}, \mathcal{U}_{ij} = P_i \mathcal{U} P_j (1 \le i \le j \le 2),$$

2020 *Mathematics Subject Classification*. Primary 16W25; Secondary 46A66, 47L10.
Keywords. triangular operator algebras, Lie triple centralizer, generalized Lie triple derivations.
Received: 03 June 2024; Revised: 19 January 2025; Accepted: 10 March 2025
Communicated by Dijana Mosić
* Corresponding author: Xinzhuo Liu
Email addresses: liuxinzhuo2022@163.com (Xinzhuo Liu), jhzhang@snnu.edu.cn (Jianhua Zhang)
ORCID iDs: https://orcid.org/0009-0008-6242-6708 (Xinzhuo Liu), https://orcid.org/0000-0002-2612-3864 (Jianhua Zhang)

and P_1 is called the standard idempotent. It is clear that $\mathcal{U}$ can be represented as $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ and $\mathcal{U}_{12}$ is a faithful $(\mathcal{U}_{11}, \mathcal{U}_{22})$-bimodule. Let $\mathcal{Z}(\mathcal{U})$ be the center of $\mathcal{U}$. It follows from [7] that

$$\mathcal{Z}(\mathcal{U}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\}.$$

Let us define two natural projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a \text{ and } \pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b.$$

Then $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{B})$. There exists a unique algebra isomorphism $\eta : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U}))$ such that $am = m\eta(a)$ for all $m \in \mathcal{M}$.

Recall that a linear map $\phi : \mathcal{U} \to \mathcal{U}$ is called a centralizer if $\phi(UV) = \phi(U)V = U\phi(V)$ for all $U, V \in \mathcal{U}$, a linear map $\phi : \mathcal{U} \to \mathcal{U}$ is called a Lie centralizer if $\phi([U, V]) = [\phi(U), V]$ for all $U, V \in \mathcal{U}$, where $[U, V] = UV - VU$ is the Lie product of U and V. The structure of Lie centralizers on rings and operator algebras, has attracted some attention over past years. The relationship between a Lie centralizer $\phi : \mathcal{U} \to \mathcal{U}$ and the sum of a centralizer $\varphi : \mathcal{U} \to \mathcal{U}$ and a map $\zeta : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ has been studied(see [4], [8], [11] and references therein). For example, in [4], Fošner and Jing proved that under mild assumptions, every Lie centralizer ϕ from a triangular ring $\mathfrak{R}$ to itself is of standard form, that is, ϕ can be expressed through a centralizer $\varphi : \mathfrak{R} \to \mathfrak{R}$ and a linear mapping $\zeta : \mathfrak{R} \to \mathcal{Z}(\mathfrak{R})$ vanishing at commutators. Jabeen in [8] considered Lie centralizers on generalized matrix algebras.

There exist some important classes of mappings on algebras, such as Lie triple centralizers and Lie triple derivations, and their generalizations. A linear map $\phi : \mathcal{U} \to \mathcal{U}$ is called a Lie triple centralizer if

$$\phi([[U, V], W]) = [[\phi(U), V], W]$$

for all $U, V, W \in \mathcal{U}$. It can be easily checked that ϕ is a Lie triple centralizer on $\mathcal{U}$ if and only if $\phi([[U, V], W]) = [[U, \phi(V)], W]$ for all $U, V, W \in \mathcal{U}$. Obviously every Lie centralizer is a Lie triple centralizer, but the converse is generally not true. We say that a linear map $\sigma : \mathcal{U} \to \mathcal{U}$ is a Lie triple derivation if

$$\sigma([[A, B], C]) = [[\sigma(A), B], C] + [[A, \sigma(B)], C] + [[A, B], \sigma(C)]$$

for all $A, B, C \in \mathcal{U}$. A linear map $\Delta : \mathcal{U} \to \mathcal{U}$ is called a generalized Lie triple derivation associated with the Lie triple derivation σ

$$\Delta([[A, B], C]) = [[\Delta(A), B], C] + [[A, \sigma(B)], C] + [[A, B], \sigma(C)]$$

for all $A, B, C \in \mathcal{U}$ if and only if $\Delta - \sigma$ is a Lie triple centralizer(see [2]). In [2], Fadaee et al. gave the necessary and sufficient conditions for a Lie triple centralizer to be standard, and as an application, they characterized generalized Lie triple derivations. Xiao [13] proved that under mild assumptions, every Lie triple derivation δ on $\mathcal{U}$ is of standard form, that is, $\delta = d + \tau$, where d is a derivation from $\mathcal{U}$ to itself and τ is a linear map from $\mathcal{U}$ to $\mathcal{Z}(\mathcal{U})$ vanishing on all second commutators of $\mathcal{U}$. Recently, there have been a great interest in the study by local actions of Lie triple derivations and Lie triple centralizers. Liu[9] considered that Lie triple derivations on zero product on factor von Neumann algebras. Liu[10] showed that Lie triple derivations on projection product on von Neumann algebras. Let $\mathcal{M}$ be an arbitrary von Neumann algebra, Fadaee[3] proved that if an additive map $\phi : \mathcal{M} \to \mathcal{M}$ satisfies $\phi([[A, B], C]) = [[\phi(A), B], C] = [[A, \phi(B)], C]$ for any $A, B, C \in \mathcal{M}$ with $AB = 0$, then $\phi(A) = WA + \xi(A)$ for any $A \in \mathcal{M}$, where $W \in \mathcal{Z}(\mathcal{M})$ and $\xi : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is an additive mapping such that $\xi([[A, B], C]) = 0$ for any $A, B, C \in \mathcal{M}$ with $AB = 0$.

In this paper, we will consider the structure of a kind of Lie triple centralizer by local actions on triangular operator algebras. As an application, we give a characterization of generalized Lie triple derivations on $\mathcal{U}$.

2. Main result

The main result is the following theorem.

Theorem 2.1 Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular operator algebra satisfying
•$\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$
•$\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A, T], T] = 0, T \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B, T], T] = 0, T \in \mathcal{B}\}$

If $\phi : \mathcal{U} \to \mathcal{U}$ is a linear map satisfying

$$\phi([[U, V], W]) = [[\phi(U), V], W] = [[U, \phi(V)], W]$$

for all $U, V, W \in \mathcal{U}$ with $UV = UW = P_1$, then there exist $\lambda \in \mathcal{Z}(\mathcal{U})$ and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U) = \lambda U + \tau(U)$ for $U \in \mathcal{U}$, where $\tau([[U, V], W]) = 0$ for all $U, V, W \in \mathcal{U}$ with $UV = UW = P_1$.

Proof: We will complete the proof by several claims.

Claim 1 $\phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.
Since $P_1P_1 = P_1P_1 = P_1$, we have

$$0 = \phi([[P_1, P_1], P_1]) = [[\phi(P_1), P_1], P_1] = \phi(P_1)P_1 + P_1\phi(P_1) - 2P_1\phi(P_1)P_1 = P_1\phi(P_1)P_2,$$

and hence $\phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.
Claim 2 $\phi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.
For any $U_{12} \in \mathcal{U}_{12}$, since $(P_1 + U_{12})P_1 = (P_1 + U_{12})P_1 = P_1$, we have

$$\begin{aligned}\phi(U_{12}) =& \phi([[P_1 + U_{12}, P_1], P_1]) = [[P_1 + U_{12}, \phi(P_1)], P_1] \\ =& [[P_1, \phi(P_1)], P_1] + [[U_{12}, \phi(P_1)], P_1] = [[U_{12}, \phi(P_1)], P_1] = P_1\phi(P_1)U_{12} - U_{12}\phi(P_1)P_2.\end{aligned}$$

This implies that $P_1\phi(U_{12})P_1 = P_2\phi(U_{12})P_2 = 0$. Consequently, $\phi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

Claim 3 $\phi(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{22} (i = 1, 2)$.
For any invertible $A_{11} \in \mathcal{U}_{11}$, since $A_{11}^{-1}A_{11} = A_{11}^{-1}A_{11} = P_1$, we get

$$0 = \phi([[A_{11}^{-1}, A_{11}], A_{11}]) = [[A_{11}^{-1}, \phi(A_{11})], A_{11}] = A_{11}^{-1}\phi(A_{11})A_{11} - \phi(A_{11})P_1 - P_1\phi(A_{11}) + A_{11}\phi(A_{11})A_{11}^{-1}.$$

Multiplying the above equation by P_2 from the right, we obtain that $P_1\phi(A_{11})P_2 = 0$, and hence $\phi(A_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22}$. For any $U_{11} \in \mathcal{U}_{11}$, there exists an integer n such that $nP_1 - U_{11}$ is invertible. Let $U_{11} = nP_1 - (nP_1 - U_{11})$, by the above and Claim 1, we have $\phi(U_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22},\ U_{11} \in \mathcal{U}_{11}$.
For any $U_{22} \in \mathcal{U}_{22}$, since $(P_1 + U_{22})P_1 = (P_1 + U_{22})P_1 = P_1$, we get

$$\begin{aligned}0 =& \phi([[P_1 + U_{22}, P_1], P_1]) = \phi([[P_1, P_1], P_1]) + \phi([[U_{22}, P_1], P_1]) \\ =& \phi([[U_{22}, P_1], P_1]) = [[\phi(U_{22}), P_1], P_1] = \phi(U_{22})P_1 + P_1\phi(U_{22}) - 2P_1\phi(U_{22})P_1 = P_1\phi(U_{22})P_2,\end{aligned}$$

and so $\phi(U_{22}) \in \mathcal{U}_{11} + \mathcal{U}_{22},\ U_{22} \in \mathcal{U}_{22}$.

Claim 4 There exists a map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U_{ii}) - \tau(U_{ii}) \in \mathcal{U}_{ii}$, for all $U_{ii} \in \mathcal{U}_{ii}, i = 1, 2$.
For any invertible element $A_{11} \in \mathcal{U}_{11}$ and $A_{22} \in \mathcal{U}_{22}$, since $(A_{11}^{-1} + A_{22})A_{11} = (A_{11}^{-1} + A_{22})A_{11} = P_1$, we have

$$\begin{aligned}0 =& \phi([[A_{11}^{-1} + A_{22}, A_{11}], A_{11}]) = [[\phi(A_{11}^{-1} + A_{22}), A_{11}], A_{11}] = [[\phi(A_{11}^{-1}), A_{11}], A_{11}] + [[\phi(A_{22}), A_{11}], A_{11}] \\ =& [[\phi(A_{22}), A_{11}], A_{11}] = [[P_1\phi(A_{22})P_1 + P_2\phi(A_{22})P_2, A_{11}], A_{11}] = [[P_1\phi(A_{22})P_1, A_{11}], A_{11}]. \quad (2.1)\end{aligned}$$

Since $A_{11}(A_{11}^{-1} + A_{22}) = A_{11}(A_{11}^{-1} + A_{22}) = P_1$, we have

$$\begin{aligned}0 =& \phi([[A_{11}, A_{11}^{-1} + A_{22}], A_{11}^{-1} + A_{22}]) = [[\phi(A_{11}), A_{11}^{-1} + A_{22}], A_{11}^{-1} + A_{22}] \\ =& [[\phi(A_{11}), A_{11}^{-1}], A_{11}^{-1} + A_{22}] + [[\phi(A_{11}), A_{22}], A_{11}^{-1} + A_{22}]. \quad (2.2)\end{aligned}$$

Since $A_{11}A_{11}^{-1} = A_{11}(A_{11}^{-1} + A_{22}) = P_1$, we have

$$0 = \phi([[A_{11}, A_{11}^{-1}], A_{11}^{-1} + A_{22}]) = [[\phi(A_{11}), A_{11}^{-1}], A_{11}^{-1} + A_{22}]. \tag{2.3}$$

By Eqs. (2.2) and (2.3), then

$$0 = [[\phi(A_{11}), A_{22}], A_{11}^{-1} + A_{22}] = [[\phi(A_{11}), A_{22}], A_{11}^{-1}] + [[\phi(A_{11}), A_{22}], A_{22}] = [[P_2\phi(A_{11})P_2, A_{22}], A_{22}]. \tag{2.4}$$

By the condition of theorem 2.1 and Eqs. (2.1) and (2.4), we have $P_1\phi(A_{22})P_1 \in \mathcal{Z}(\mathcal{U}_{11}) = P_1\mathcal{Z}(\mathcal{U})P_1$, $P_2\phi(A_{11})P_2 \in \mathcal{Z}(\mathcal{U}_{22}) = P_2\mathcal{Z}(\mathcal{U})P_2$. For any $U_{11} \in \mathcal{U}_{11}$, there exists some numbers n such that $nP_1 - U_{11}$ is invertible. Then $P_1\phi(U_{22})P_1 \in \mathcal{Z}(\mathcal{U}_{11}) = P_1\mathcal{Z}(\mathcal{U})P_1$, $P_2\phi(U_{11})P_2 \in \mathcal{Z}(\mathcal{U}_{22}) = P_2\mathcal{Z}(\mathcal{U})P_2$.

For $U_{ii} \in \mathcal{U}_{ii}, i = 1, 2$, let $\tau_1(U_{11}) = P_2\phi(U_{11})P_2$, $\tau_2(U_{22}) = P_1\phi(U_{22})P_1$. For $U \in \mathcal{U}$, define the map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ as

$$\tau(U) = \tau_1(U_{11}) + \eta^{-1}(\tau_1(U_{11})) + \tau_2(U_{22}) + \eta(\tau_2(U_{22})).$$

It is obvious that $\tau(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{U})$. Then for any $U_{11} \in \mathcal{U}_{11}$, it follows that

$$\phi(U_{11}) - \tau(U_{11}) = P_1\phi(U_{11})P_1 + P_2\phi(U_{11})P_2 - \tau_1(\mathcal{U}_{11}) - \eta^{-1}(\tau_1(\mathcal{U}_{11})) = P_1\phi(U_{11})P_1 - \eta^{-1}(\tau_1(U_{11})) \in \mathcal{U}_{11}.$$

Similarly, we can obtain $\phi(U_{22}) - \tau(U_{22}) \in \mathcal{U}_{22}$.

Define a map $\varphi : \mathcal{U} \to \mathcal{U}$ as

$$\varphi(U) = \phi(U) - \tau(U)$$

for any $U \in \mathcal{U}$. It follows from claims 2 and 4 that $\varphi(U_{12}) \subseteq \mathcal{U}_{12}$, $\varphi(U_{ii}) = \phi(U_{ii}) - \tau(U_{ii}) \subseteq \mathcal{U}_{ii}$ with $i = 1, 2$ for all $U_{ii} \in \mathcal{U}_{ii}$, meanwhile, $\varphi(U_{12}) = \phi(U_{12})$, for all $U_{12} \in \mathcal{U}_{12}$.

Claim 5 For any $U_{ii} \in \mathcal{U}_{ii}$ $(i = 1, 2)$, we have

(a) $\varphi(U_{11}U_{12}) = \varphi(U_{11})U_{12} = U_{11}\varphi(U_{12})$;

(b) $\varphi(U_{12}U_{22}) = \varphi(U_{12})U_{22} = U_{12}\varphi(U_{22})$.

(a) For any invertible element $A_{11} \in \mathcal{U}_{11}$, and $U_{12} \in \mathcal{U}_{12}$. Since $(A_{11}^{-1} + A_{11}^{-1}U_{12})A_{11} = (A_{11}^{-1} + A_{11}^{-1}U_{12})A_{11} = P_1$, we have

$$\begin{aligned}&\varphi(A_{11}U_{12}) = \phi(A_{11}U_{12}) = \phi([[A_{11}^{-1} + A_{11}^{-1}U_{12}, A_{11}], A_{11}]) = [[\phi(A_{11}^{-1} + A_{11}^{-1}U_{12}), A_{11}], A_{11}]\\ =&[[\phi(A_{11}^{-1}), A_{11}], A_{11}] + [[\phi(A_{11}^{-1}U_{12}), A_{11}], A_{11}] = [[\phi(A_{11}^{-1}U_{12}), A_{11}], A_{11}].\end{aligned} \tag{2.5}$$

Replace U_{12} with $A_{11}U_{12}$ in Eqs. (2.5), then

$$\varphi(A_{11}A_{11}U_{12}) = [[\phi(A_{11}^{-1}A_{11}U_{12}), A_{11}], A_{11}] = [[\phi(U_{12}), A_{11}], A_{11}] = A_{11}A_{11}\phi(U_{12}) = A_{11}A_{11}\varphi(U_{12}).$$

For any $U_{11} \in \mathcal{U}_{11}$, there exists some numbers n such that we have $nP_1 - U_{11}$ is invertible. So

$$\varphi((nP_1 - U_{11})(nP_1 - U_{11})U_{12}) = (nP_1 - U_{11})(nP_1 - U_{11})\varphi(U_{12}),$$

and hence

$$\varphi(U_{11}U_{12}) = U_{11}\varphi(U_{12}) \tag{2.6}$$

for all $U_{ij} \in \mathcal{U}_{ij}$.

For any invertible element $A_{11} \in \mathcal{U}_{11}$ and $U_{12} \in \mathcal{U}_{12}$. We have

$$\begin{aligned}&\varphi(A_{11}U_{12}) = \phi(A_{11}U_{12}) = \phi([[A_{11}^{-1} + A_{11}^{-1}U_{12}, A_{11}], A_{11}])\\ =&[[A_{11}^{-1} + A_{11}^{-1}U_{12}, \phi(A_{11})], A_{11}] = [[A_{11}^{-1}, \phi(A_{11})], A_{11}] + [[A_{11}^{-1}U_{12}, \phi(A_{11})], A_{11}] = [[A_{11}^{-1}U_{12}, \phi(A_{11})], A_{11}]\\ =&[[A_{11}^{-1}U_{12}, \varphi(A_{11})], A_{11}] = A_{11}\varphi(A_{11})A_{11}^{-1}U_{12}.\end{aligned}$$

Replace U_{12} with $A_{11}U_{12}$, then $\varphi(A_{11}A_{11}U_{12}) = A_{11}\varphi(A_{11})U_{12}$.

For any $U_{11} \in \mathcal{U}_{11}$, we may find some numbers n such that $nP_1 - U_{11}$ is invertible. So

$$\varphi((nP_1 - U_{11})(nP_1 - U_{11})U_{12}) = (nP_1 - U_{11})\varphi(nP_1 - U_{11})U_{12}.$$

Thus,

$$n^2\varphi(U_{12}) - 2n\varphi(U_{11}U_{12}) + \varphi(U_{11}U_{11}U_{12}) = n^2\varphi(P_1)U_{12} - n\varphi(U_{11})U_{12} - nU_{11}\varphi(P_1)U_{12} + U_{11}\varphi(U_{11})U_{12}. \tag{2.7}$$

Replace n with $n+1$

$$\begin{aligned}&(n+1)^2\varphi(U_{12}) - 2(n+1)\varphi(U_{11}U_{12}) + \varphi(U_{11}U_{11}U_{12})\\ =&(n+1)^2\varphi(P_1)U_{12} - (n+1)\varphi(U_{11})U_{12} - (n+1)U_{11}\varphi(P_1)U_{12} + U_{11}\varphi(U_{11})U_{12}.\end{aligned} \tag{2.8}$$

By Eqs.(2.6) (2.7) and (2.8), then

$$2n\varphi(U_{12}) + \varphi(U_{12}) - 2\varphi(U_{11}U_{12}) = 2n\varphi(P_1)U_{12} + \varphi(P_1)U_{12} - \varphi(U_{11})U_{12} - U_{11}\varphi(P_1)U_{12}. \tag{2.9}$$

Replace n with $n+1$ in Eqs. (2.9)

$$2(n+1)\varphi(U_{12}) + \varphi(U_{12}) - 2\varphi(U_{11}U_{12}) = 2(n+1)\varphi(P_1)U_{12} + \varphi(P_1)U_{12} - \varphi(U_{11})U_{12} - U_{11}\varphi(P_1)U_{12}. \tag{2.10}$$

By Eqs. (2.9) and (2.10), then

$$\varphi(U_{12}) = \varphi(P_1)U_{12}. \tag{2.11}$$

By Eqs. (2.6), (2.7), (2.8), (2.11), then

$$\varphi(U_{11}U_{12}) = \varphi(U_{11})U_{12}.$$

We can prove that (a) is true.

(b) For any $U_{22} \in \mathcal{U}_{22}$, and $U_{12} \in \mathcal{U}_{12}$. Since $(P_1 + U_{12})(P_1 + U_{22} - U_{12}U_{22}) = (P_1 + U_{12})P_1 = P_1$, we have

$$\begin{aligned}&\varphi(U_{12}) = \phi(U_{12}) = \phi([[P_1 + U_{12}, P_1 + U_{22} - U_{12}U_{22}], P_1]) = [[P_1 + U_{12}, \phi(P_1) + \phi(U_{22}) - \phi(U_{12}U_{22})], P_1]\\ =&[[P_1 + U_{12}, \phi(P_1)], P_1] + [[P_1 + U_{12}, \phi(U_{22})], P_1] - [[P_1 + U_{12}, \phi(U_{12}U_{22})], P_1]\\ =&[[U_{12}, \phi(P_1)], P_1] + [[U_{12}, \phi(U_{22})], P_1] - [[P_1, \phi(U_{12}U_{22})], P_1]\\ =&[[U_{12}, \varphi(P_1)], P_1] + [[U_{12}, \varphi(U_{22})], P_1] - [[P_1, \varphi(U_{12}U_{22})], P_1] = P_1\varphi(P_1)U_{12} - U_{12}\varphi(U_{22}) + \varphi(U_{12}U_{22}).\end{aligned}$$

Then, $\varphi(U_{12}U_{22}) = U_{12}\varphi(U_{22})$ and $\varphi(U_{12}U_{22}) = \varphi(P_1U_{12}U_{22}) = \varphi(P_1)U_{12}U_{22} = \varphi(U_{12})U_{22}$. We can prove that (b) is true.

Claim 6 For any $A_{ii} \in \mathcal{U}_{ii}, B_{ii} \in \mathcal{U}_{ii}, S_{12} \in \mathcal{U}_{12} (i = 1, 2)$, we have
(a) $\varphi(A_{11}B_{11}) = \varphi(A_{11})B_{11} = A_{11}\varphi(B_{11})$;
(b) $\varphi(A_{22}B_{22}) = \varphi(A_{22})B_{22} = A_{22}\varphi(B_{22})$;
(a) For any $S_{12} \in \mathcal{U}_{12}$, by claim 5, on the one hand,

$$\varphi(A_{11}B_{11}S_{12}) = \varphi(A_{11}B_{11})S_{12}.$$

On the other hand,

$$\varphi(A_{11}B_{11}S_{12}) = A_{11}\varphi(B_{11}S_{12}) = A_{11}\varphi(B_{11})S_{12}.$$

Combining the above two equations, we have

$$(A_{11}\varphi(B_{11}) - \varphi(A_{11}B_{11}))S_{12} = 0.$$

Since $\mathcal{M}$ is faithful $(\mathcal{A},\mathcal{B})$-bimodule, we get $\varphi(A_{11}B_{11}) = A_{11}\varphi(B_{11})$.

$$\varphi(A_{11}B_{11}S_{12}) = \varphi(A_{11})B_{11}S_{12} = \varphi(A_{11}B_{11})S_{12}$$

that $\varphi(A_{11}B_{11}) = \varphi(A_{11})B_{11} = A_{11}\varphi(B_{11})$. We can show that (a) holds.

(b) It follows from Claims 5 that

$$\varphi(S_{12}A_{22}B_{22}) = S_{12}\varphi(A_{22}B_{22}).$$

On the other hand,

$$\varphi(S_{12}A_{22}B_{22}) = \varphi(S_{12}A_{22})B_{22} = S_{12}\varphi(A_{22})B_{22}$$

Combining the above two equations, we have

$$S_{12}(\varphi(A_{22}B_{22}) - A_{22}\varphi(B_{22})) = 0$$

Since $\mathcal{M}$ is faithful $(\mathcal{A},\mathcal{B})$-bimodule, we get $\varphi(A_{22}B_{22}) = A_{22}\varphi(B_{22})$. Similarly, we can obtain that $\varphi(A_{22}B_{22}) = \varphi(A_{22})B_{22}$, and hence $\varphi(A_{22}B_{22}) = \varphi(A_{22})B_{22} = A_{22}\varphi(B_{22})$.

We can show that (b) holds.

So, from steps 1-6, it follows that

$$\varphi(AB) = A\varphi(B) = \varphi(A)B.$$

for all $A, B \in \mathcal{U}$, then

$$\varphi(A) = \lambda A(\lambda \in \mathcal{Z}(\mathcal{U})).$$

Claim 7 $\tau([[U,V],W]) = 0$ for all $U,V,W \in \mathcal{U}$ with $UV = UW = P_1$.

For $UV = UW = P_1$, it follows that

$$\begin{aligned}\tau([[U,V],W]) &= \phi([[U,V],W]) - \varphi([[U,V],W]) = [[\phi(U),V],W] - \varphi([[U,V],W])\\ &=[[\varphi(U)+\tau(U),V],W] - \varphi([[U,V],W]) = [[\varphi(U),V],W] - \varphi([[U,V],W]) = 0.\end{aligned}$$

It follows from claim 1-7 that there exists a $\lambda \in \mathcal{Z}(\mathcal{U})$ and a linear map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U) = \lambda U + \tau(U)(U \in \mathcal{U})$, where $\tau([[U,V],W]) = 0$ for any $U,V,W \in \mathcal{U}$ with $UV = UW = P_1$.

Theorem 2.2 Let $\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular operator algebra satisfying

• $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$.

• $\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A,X],Y] = 0, X,Y \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B,X],Y] = 0, X,Y \in \mathcal{B}\}$.

If $\phi : \mathcal{U} \to \mathcal{U}$ is a linear map satisfying

$$\phi([[U,V],W]) = [[\phi(U),V],W] = [[U,\phi(V)],W]$$

for all $U,V,W \in \mathcal{U}$ with $UV = UW = 0$, then there exist $\lambda \in \mathcal{Z}(\mathcal{U})$ and $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U) = \lambda U + \tau(U)$ for $U \in \mathcal{U}$, where $\tau([[U,V],W]) = 0$ for all $U,V,W \in \mathcal{U}$ with $UV = UW = 0$.

Proof: We will use the same symbols with that in Theorem 2.1. We organize the proof in a series of claims.

Claim 1 $\phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.

Since $P_1P_2 = P_1P_2 = 0$, we have

$$0 = \phi([[P_1,P_2],P_2]) = [[\phi(P_1),P_2],P_2] = P_1\phi(P_1)P_2.$$

We obtain that $\phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.

Claim 2 $\phi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

For any $U_{12} \in \mathcal{U}_{12}$, since $U_{12}P_1 = U_{12}P_1 = 0$, we have

$$\phi(U_{12}) = \phi([[U_{12}, P_1], P_1]) = [[U_{12}, \phi(P_1)], P_1] = -U_{12}\phi(P_1) + P_1\phi(P_1)U_{12}.$$

This implies that $P_1\phi(U_{12})P_1 = P_2\phi(U_{12})P_2 = 0$. Consequently, $\phi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

Claim 3 $\phi(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{22}(i = 1, 2)$.

For any $U_{11} \in \mathcal{U}_{11}$, since $U_{11}P_2 = U_{11}P_2 = 0$, we get

$$0 = \phi([[U_{11}, P_2], P_2]) = [[\phi(U_{11}), P_2], P_2] = P_1\phi(U_{11})P_2,$$

and hence $\phi(U_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22}$. Similarly, $\phi(U_{22}) \in \mathcal{U}_{11} + \mathcal{U}_{22},\ U_{22} \in \mathcal{U}_{22}$.

Claim 4 There exists a map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U_{ii}) - \tau(U_{ii}) \in \mathcal{U}_{ii}$, for all $U_{ii} \in \mathcal{U}_{ii}, i = 1, 2$.

For any $U_{ii} \in \mathcal{U}_{ii}$, since $U_{22}U_{11} = U_{22}U_{12} = 0$, we have

$$0 = \phi([[U_{22}, U_{11}], U_{12}]) = [[\phi(U_{22}), U_{11}], U_{12}],$$

so $[\phi(U_{22}), U_{11}] \in \mathcal{Z}(\mathcal{U})$. Then, we obtain that $[P_1\phi(U_{22})P_1, U_{11}] \in \mathcal{Z}(\mathcal{U}_{11})$. Similarly, we obtain that $[U_{22}, P_2\phi(U_{11})P_2] \in \mathcal{Z}(\mathcal{U}_{22})$. By the condition of theorem 2.2, we have $P_1\phi(U_{22})P_1 \in \mathcal{Z}(\mathcal{U}_{11}) = P_1\mathcal{Z}(\mathcal{U})P_1$, $P_2\phi(U_{11})P_2 \in \mathcal{Z}(\mathcal{U}_{22}) = P_2\mathcal{Z}(\mathcal{U})P_2$.

For $U_{ii} \in \mathcal{U}_{ii}, i = 1, 2$, let $\tau_1(U_{11}) = P_2\phi(U_{11})P_2$, $\tau_2(U_{22}) = P_1\phi(U_{22})P_1$. For $U \in \mathcal{U}$, define the map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ as

$$\tau(U) = \tau_1(U_{11}) + \eta^{-1}(\tau_1(U_{11})) + \tau_2(U_{22}) + \eta(\tau_2(U_{22})).$$

It is obvious that $\tau(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{U})$. Then for any $U_{11} \in \mathcal{U}_{11}$, it follows that

$$\phi(U_{11}) - \tau(U_{11}) = P_1\phi(U_{11})P_1 + P_2\phi(U_{11})P_2 - \tau_1(\mathcal{U}_{11}) - \eta^{-1}(\tau_1(\mathcal{U}_{11})) = P_1\phi(U_{11})P_1 - \eta^{-1}(\tau_1(U_{11})) \in \mathcal{U}_{11}.$$

Similarly, we can obtain $\phi(U_{22}) - \tau(U_{22}) \in \mathcal{U}_{22}$.

Define a map $\varphi : \mathcal{U} \to \mathcal{U}$ as

$$\varphi(U) = \phi(U) - \tau(U)$$

for any $U \in \mathcal{U}$. It follows from claims 2 and 4 that $\varphi(U_{12}) \subseteq \mathcal{U}_{12}$, $\varphi(U_{ii}) = \phi(U_{ii}) - \tau(U_{ii}) \subseteq \mathcal{U}_{ii}$ with $i = 1, 2$ for all $U_{ii} \in \mathcal{U}_{ii}$, meanwhile, $\varphi(U_{12}) = \phi(U_{12})$, for all $U_{12} \in \mathcal{U}_{12}$.

Claim 5 For any $U_{ii} \in \mathcal{U}_{ii}$ $(i = 1, 2)$, we have

(a) $\varphi(U_{11}U_{12}) = \varphi(U_{11})U_{12} = U_{11}\varphi(U_{12})$;

(b) $\varphi(U_{12}U_{22}) = \varphi(U_{12})U_{22} = U_{12}\varphi(U_{22})$.

(a) For any invertible element $U_{11} \in \mathcal{U}_{11}$, and $U_{12} \in \mathcal{U}_{12}$. Since $U_{12}U_{11} = U_{12}P_1 = 0$, we have

$$\varphi(U_{11}U_{12}) = \phi(U_{11}U_{12}) = \phi([[U_{12}, U_{11}], P_1]) = [[\phi(U_{12}), U_{11}], P_1] = U_{11}\phi(U_{12}) = U_{11}\varphi(U_{12})$$

and,

$$\varphi(U_{11}U_{12}) = \phi(U_{11}U_{12}) = \phi([[U_{12}, U_{11}], P_1]) = [[U_{12}, \phi(U_{11})], P_1] = [[U_{12}, \varphi(U_{11})], P_1] = \varphi(U_{11})U_{12}$$

We can show that (a) holds.

(b) Similarly, we can show that (b) holds.

Claim 6 For any $A_{ii} \in \mathcal{U}_{ii}, B_{ii} \in \mathcal{U}_{ii}, S_{12} \in \mathcal{U}_{12}(i = 1, 2)$, we have

(a) $\varphi(A_{11}B_{11}) = \varphi(A_{11})B_{11} = A_{11}\varphi(B_{11})$;

(b) $\varphi(A_{22}B_{22}) = \varphi(A_{22})B_{22} = A_{22}\varphi(B_{22})$;

(a)For any $S_{12} \in \mathcal{U}_{12}$, by claim 5, on the one hand,

$$\varphi(A_{11}B_{11}S_{12}) = \varphi(A_{11}B_{11})S_{12}.$$

on the other hand,

$$\varphi(A_{11}B_{11}S_{12}) = A_{11}\varphi(B_{11}S_{12}) = A_{11}\varphi(B_{11})S_{12}.$$

Combining the above two equations, we have

$$(A_{11}\varphi(B_{11}) - \varphi(A_{11}B_{11}))S_{12} = 0.$$

Since $\mathcal{M}$ is faithful $(\mathcal{A}, \mathcal{B})$-bimodule, we get $\varphi(A_{11}B_{11}) = A_{11}\varphi(B_{11})$.

It follows from

$$\varphi(A_{11}B_{11}S_{12}) = \varphi(A_{11})B_{11}S_{12} = \varphi(A_{11}B_{11})S_{12}$$

that $\varphi(A_{11}B_{11}) = \varphi(A_{11})B_{11} = A_{11}\varphi(B_{11})$. We can show that (a) holds.

(b) Similarly, we can show that (b) holds.

So, from steps 1-6, it follows that

$$\varphi(AB) = A\varphi(B) = \varphi(A)B.$$

for all $A, B \in \mathcal{U}$, then

$$\varphi(A) = \lambda A(\lambda \in \mathcal{Z}(\mathcal{U})).$$

Claim 7 $\tau([[U, V], W]) = 0$ for all $U, V, W \in \mathcal{U}$ with $UV = UW = 0$.

For $UV = UW = 0$, it follows that

$$\begin{aligned}\tau([[U, V], W]) &= \phi([[U, V], W]) - \varphi([[U, V], W]) = [[\phi(U), V], W] - \varphi([[U, V], W]) \\ &= [[\varphi(U) + \tau(U), V], W] - \varphi([[U, V], W]) = [[\varphi(U), V], W] - \varphi([[U, V], W]) = 0.\end{aligned}$$

Hence there exists a $\lambda \in \mathcal{Z}(\mathcal{U})$ and a linear map $\tau : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $\phi(U) = \lambda U + \tau(U)(U \in \mathcal{U})$, where $\tau([[U, V], W]) = 0$ for any $U, V, W \in \mathcal{U}$ with $UV = UW = 0$.

3. Application

Application 1: characterization of generalized Lie triple derivations by acting on idempotent products.

Theorem 3.1

Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular operator algebra satisfying

(1)$\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$,

(2)$\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A, T], T] = 0, T \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B, T], T] = 0, T \in \mathcal{B}\}$.

Suppose that a linear map $\sigma : \mathcal{U} \to \mathcal{U}$

$$\sigma([[A, B], C]) = [[\sigma(A), B], C] + [[A, \sigma(B)], C] + [[A, B], \sigma(C)]$$

for all $A, B, C \in \mathcal{U}$ with $AB = AC = P_1$, then σ is of the form $\sigma = \varphi + h$, where $\varphi : \mathcal{U} \to \mathcal{U}$ is a derivation, a linear map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ vanishing on $[[A, B], C]$ for all $A, B, C \in \mathcal{U}$ with $AB = AC = P_1$.

Theorem 3.2 Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular operator algebra satisfying

•$\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$,

•$\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A, T], T] = 0, T \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B, T], T] = 0, T \in \mathcal{B}\}$.

Suppose that a linear map $\Delta : \mathcal{U} \to \mathcal{U}$

$$\Delta([[A, B], C]) = [[\Delta(A), B], C] + [[A, \sigma(B)], C] + [[A, B], \sigma(C)]$$

for all $A, B, C \in \mathcal{U}$ with $AB = AC = P_1$. Then, there exist $\lambda \in \mathcal{Z}(\mathcal{U})$, a derivation φ and $h_1 : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that

$$\Delta(A) = \varphi(A) + h_1(A) + \lambda A$$

for $A \in \mathcal{U}$, where $h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{U}$ with $AB = AC = P_1$.
Proof: According to the theorem 3.1, there are linear maps φ and h on $\mathcal{U}$,

$$\varphi : \mathcal{U} \to \mathcal{U} \quad h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$$

$h([A,B],C]) = 0$ with $AB = AC = P_1$. By assumption, for a Lie triple centralizer $\phi = \Delta - \sigma$ on $\mathcal{U}$, we have

$$A,B,C \in \mathcal{U}, AB = AC = P_1 \Rightarrow \phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C]$$

It follows from the result of this paper that there exist $\lambda \in \mathcal{Z}(\mathcal{U})$ and a linear map τ on $\mathcal{U}$ such that $\phi(A) = \lambda A + \tau(A)$, where $\tau(A) \in \mathcal{Z}(\mathcal{U})$ for all $A \in \mathcal{U}$ and $\tau([[A,B],C]) = 0, AB = AC = P_1$. Suppose that $h_1 = \tau + h$, thus $h_1 : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a linear map where $h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{U}$ with $AB = AC = P_1$. Thus, we have

$$\Delta(A) = \sigma(A) + \phi(A) = \varphi(A) + h(A) + \lambda A + \tau(A) = \varphi(A) + h_1(A) + \lambda A$$

for all $A \in \mathcal{U}$, this completes the proof.

Application 2: characterization of generalized Lie triple derivations by acting on zero products.
Theorem 3.3 Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular operator algebra satisfying
(1)$\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$,
(2)$\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A,X],Y] = 0, X,Y \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B,X],Y] = 0, X,Y \in \mathcal{B}\}$.
Suppose that a linear map $\sigma : \mathcal{U} \to \mathcal{U}$,

$$\sigma([[A,B],C]) = [[\sigma(A),B],C] + [[A,\sigma(B)],C] + [[A,B],\sigma(C)]$$

for all $A,B,C \in \mathcal{U}$ with $AB = AC = 0$, then σ is of the form $\sigma = \varphi + h$, where $\varphi : \mathcal{U} \to \mathcal{U}$ is a derivation, a linear map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ vanishing on $[[A,B],C]$ for all $A,B,C \in \mathcal{U}$ with $AB = AC = 0$.
Theorem 3.4 Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular operator algebra satisfying
•$\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$,
•$\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} | [[A,X],Y] = 0, X,Y \in \mathcal{A}\}$, $\mathcal{Z}(\mathcal{B}) = \{B \in \mathcal{B} | [[B,X],Y] = 0, X,Y \in \mathcal{B}\}$.
Suppose that a linear map $\Delta : \mathcal{U} \to \mathcal{U}$,

$$\Delta([[A,B],C]) = [[\Delta(A),B],C] + [[A,\sigma(B)],C] + [[A,B],\sigma(C)]$$

for all $A,B,C \in \mathcal{U}$ with $AB = AC = 0$. Then, there exist $\lambda \in \mathcal{Z}(\mathcal{U})$, a derivation φ and $h_1 : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that

$$\Delta(A) = \varphi(A) + h_1(A) + \lambda A.$$

for $A \in \mathcal{U}$, where $h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{U}$ with $AB = AC = 0$.
Proof: According to the theorem 3.3, there exist linear maps φ and h on $\mathcal{U}$

$$\varphi : \mathcal{U} \to \mathcal{U} \quad h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$$

$h([A,B],C]) = 0$ with $AB = AC = 0$. By assumption, for a Lie triple centralizer $\phi = \Delta - \sigma$ on $\mathcal{U}$, we have

$$A,B,C \in \mathcal{U}, AB = AC = 0 \Rightarrow \phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C]$$

It follows from the result of this paper that there exist $\lambda \in \mathcal{Z}(\mathcal{U})$ and a linear map τ on $\mathcal{U}$ such that $\phi(A) = \lambda A + \tau(A)$, where $\tau(A) \in \mathcal{Z}(\mathcal{U})$ for all $A \in \mathcal{U}$ and $\tau([[A,B],C]) = 0, AB = AC = 0$. Suppose that $h_1 = \tau + h$, thus $h_1 : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a linear map where $h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{U}$ with $AB = AC = 0$. Thus, we have

$$\Delta(A) = \sigma(A) + \phi(A) = \varphi(A) + h(A) + \lambda A + \tau(A) = \varphi(A) + h_1(A) + \lambda A$$

for all $A \in \mathcal{U}$, this completes the proof.

References

[1] C. R. Zhang, J. H. zhang, General form of two classes of derivable maps on triangular algebras. in press.
[2] B. Fadaee, H. Ghahramani, W. Jing, Lie triple centralizers on generalized matrix algebras, Quast. Math. **46**(2023): 281–300 .
[3] B. Fadaee, H. Ghahramani, Additive mappings on von Neumann algebras acting as Lie triple centralizer via local actions and related mappings, Acta Sci. Math. **21**(2024): 1–18.
[4] A. Fošner, W. Jing, Lie centralizers on triangular rings and nest algebras, Adv Oper Theory, **2**(2019): 342-350.
[5] S. Goodarzi, A. Khotanloo, Lie Centralizers and generalized Lie derivations on prime rings by local actions. Commun. Algebra, **51**(12), 5277–5286 (2023).
[6] H. Ghahramani, A. H. Mokhtari, and F. Wei, Lie centralizers and commutant preserving maps on generalized matrix algebras. J Algebra Appl. **23**(2024), 2450106.
[7] X. Zhao, Nonlinear Lie triple derivations by local actions on triangular algebras, J. Algebra Appl. **22**(2023), 2350059.
[8] A. Jabeen, Lie (Jordan) centralizers on generalized matrix algebras, Commu Algebra, **49**(2021) 278-291 .
[9] L. Liu, Lie triple derivations on factor von Neumann algebras, B Korean Math Soc , **52**(2015), 581–591.
[10] L. Liu, Lie triple derivations on von Neumann algebras, Chinese Ann Math B, **52**(2018) , 817–828.
[11] L. Liu, On nonlinear Lie centralizers of generalized matrix algebras, Linear Multilinear A, **70**(2022) 2693–2705.
[12] L. Liu, K. Gao, Characterizations of Lie centralizers of triangular algebras, Linear Multilinear A, **71**(2023): 2375–2391.
[13] Z. Xiao, F. Wei, Lie triple derivations of triangular algebras, Linear Algebra Appl. **437**(2012): 1234–1249.