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Representations and cohomology theory of relative Rota-Baxter alternative algebras

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Abstract. We introduce the concept of relative Rota-Baxter alternative algebras. The representations and low dimensional cohomology theory of relative Rota-Baxter alternative algebras are studied. Some applications in deformation and abelian extension are given.

1. Introduction

Alternative algebra is an important class of non-associative algebras [23]. For a general theory of alternative algebras, see references [5, 21, 25, 29]. It is closely related to Lie algebras, Jordan algebras, and Malcev algebras. So far, many scholars have made systematic research on the structure and representation of alternative algebras. Zorn studied the structural theory of finite-dimensional alternating algebras in his articles [30, 31]. Schafer and Jacobson systematically studied the representation of alternative algebra in [14] and [22], respectively. Recently, Ni and Bai introduced pre-alternative algebras and pre-alternative bialgebras in the article [19], and they gave the relationship between Rota-Baxter operators on alternative algebras and the solutions of Yang-Baxter equations. Other authors investigated the theory of cohomology and deformation in alternative algebras [11], Hom-alternative algebras [12] and BiHom-alternative algebras [7].

The notion of Rota-Baxter operators was introduced by Baxter [3] in 1960s. Subsequently, the connection between the Yang-Baxter equation and zero-weight Rota-Baxter operators on Lie algebras was discovered in [4]. Cartier [6] and Rota [20] investigated the connection between Rota-Baxter operators and combinatorics. In [10], Das defined the cohomology of a Rota-Baxter family algebra. Moreover, Aguiar observed that solutions to the associative Yang-Baxter equation correlate with weight zero Rota-Baxter operators on associative algebras [1]. In the year 1999, Kupershmidt [17] introduced the notion of relative Rota-Baxter operators (also called O-operators or Kupershmidt operators) on Lie algebras in the study of classical r-matrices. Relative Rota-Baxter operator on Lie algebras was further studied in [18, 24, 26], on associative algebras in [2], on Leibniz algebras in [9] and on 3-Lie algebras in [16, 27]. Recently, deformations and cohomology theory for relative Rota-Baxter operators on Lie algebras was developed in [15]. Representations and cohomology of a relative Rota-Baxter associative algebra was studied in [8]. Deformation theory and homotopy theory of Rota-Baxter algebras in the view of operad was given in [28].

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Due to the importance of alternative algebras and Rota-Baxter operators, we introduce the notion of relative Rota-Baxter operators on alternative algebras in this paper. This will be called relative Rota-Baxter alternative algebras. Now it is a natural question what is the representation and cohomology theory for this type of algebra? In this paper, we will give an partial answer to this question. We investigate the representation theory for this type of alternative algebras. As applications, we will study the infinitesimal deformation and abelian extension theory in detail.

The organization of this paper is as follows. In Section 2, we review some basic facts about alternative algebras and relative Rota-Baxter operators. As an example, we compute all relative Rota-Baxter operators on a given 4-dimensional alternative algebra over a 4-dimensional vector space. In Section 3, we investigate the representation theory of relative Rota-Baxter alternative algebras. We show that given a relative Rota-Baxter alternative algebra and a bimodule, one can construct a new relative Rota-Baxter alternative algebra on their direct sum space which is called the semidirect product. We also study the second cohomology group of relative Rota-Baxter alternative algebras. In Section 4, we will study infinitesimal deformations of relative Rota-Baxter alternative algebras. In Section 5, we prove that the equivalent classes of abelian extensions are one-to-one correspondence to the elements of the second cohomology groups.

Throughout the rest of this paper, we work over a fixed field of characteristic 0. A linear map φ from a vector space W to V is denoted by $\varphi : W \to V$ or (W, V, φ). The space of linear maps from a vector space W to V and tensor product of vector spaces V and W are denoted by $\mathrm{Hom}(W, V)$ and $V \otimes W$ respectively.

2. Preliminaries

In this section, we recall some definitions and fix some notations about alternative algebras and relative Rota-Baxter operators.

Definition 2.1. *An alternative algebra is a vector space A with a multiplication $A \times A \to A : (x, y) \mapsto x \cdot y$ such that the following identities hold:*

$$(x, y, z) = -(y, x, z), \quad (x, y, z) = -(x, z, y),$$

where $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of the elements $x, y, z \in A$. Note that the above identities are equivalent to the following identities:

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) + (y \cdot x) \cdot z - y \cdot (x \cdot z) = 0,$$
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) + (x \cdot z) \cdot y - x \cdot (z \cdot y) = 0.$$

An algebra A is left alternative (right alternative) if it satisfies the left alternative law $(x \cdot x) \cdot y = x \cdot (x \cdot y)$ (the right alternative law $y \cdot (x \cdot x) = (y \cdot x) \cdot x$). An alternative algebra is one which is both left and right alternative.

Definition 2.2. *Let A be an alternative algebra and V be a vector space. Then V is called a left A-module if there is a linear map $\cdot : A \otimes V \to V, (x, v) \to x \cdot v$ such that*

$$(x \cdot y) \cdot v - x \cdot (y \cdot v) + (y \cdot x) \cdot v - y \cdot (x \cdot v) = 0, \tag{1}$$

a right A-module if there is a linear map $\cdot : V \otimes A \to V, (v, x) \to v \cdot x$ such that

$$(v \cdot x) \cdot y - v \cdot (x \cdot y) + (v \cdot y) \cdot x - v \cdot (y \cdot x) = 0, \tag{2}$$

and an A-bimodule if further these two maps satisfying the following conditions:

$$(v \cdot x) \cdot y - v \cdot (x \cdot y) + (x \cdot v) \cdot y - x \cdot (v \cdot y) = 0, \tag{3}$$
$$(x \cdot y) \cdot v - x \cdot (y \cdot v) + (x \cdot v) \cdot y - x \cdot (v \cdot y) = 0, \tag{4}$$

for all $x, y \in A$ and $v \in V$.

Definition 2.3. *Let V be a bimodule of an alternative algebra A. A linear map $T : V \to A$ is called a relative Rota-Baxter operator associated to V if for all $u, v \in V$,*

$$T(u) \cdot T(v) = T(T(u) \cdot v + u \cdot T(v)). \tag{5}$$

In this case, (V, A, T) is called a relative Rota-Baxter alternative algebra.

A morphism between two relative Rota-Baxter alternative algebras (V, A, T) and (V', A', T') is a pair of linear maps $p_1 : V \to V'$ and $p_0 : A \to A'$ such that $p_0(x \cdot y) = p_0(x) \cdot' p_0(y)$ and $T' \circ p_1 = p_0 \circ T$.

Let V be a bimodule of A. Then the direct sum $A \oplus V$ carries an alternative algebra structure given by

$$(x, u) \cdot (y, v) := \big(x \cdot y,\ x \cdot v + u \cdot y\big),$$

for $x, y \in A$ and $u, v \in V$. This is called the semidirect product which is denoted by $A \ltimes V$. With this notation, we have the following characterization of relative Rota-Baxter operators.

Proposition 2.4. *A linear map $T : V \to A$ is a relative Rota-Baxter operator if and only if the graph $Gr(T) = \{(T(u), u) | \ u \in V\}$ is a subalgebra of the semidirect product alternative algebra $A \ltimes V$.*

Proof. For any $u, v \in V$, we have

$$(T(u), u) \cdot (T(v), v) = \big(T(u) \cdot T(v),\ T(u) \cdot v + u \cdot T(v)\big).$$

The right hand side is belongs to $Gr(T)$ if and only if T is a relative Rota-Baxter operator. Hence the result follows. □

Example 2.5. *Let A be an alternative algebra of dimension 4 with basis $\{e_0, e_1, e_2, e_3\}$ given by the following multiplication (the unspecified products are zeros):*

$$e_0 \cdot e_0 = e_0,\ \ e_0 \cdot e_1 = e_1,\ \ e_2 \cdot e_0 = e_2,\ \ e_2 \cdot e_3 = e_1,\ \ e_3 \cdot e_0 = e_3,\ \ e_3 \cdot e_2 = -e_1.$$

Let V be a bimodule of A, $\{v_0, v_1, v_2, v_3\}$ is a set of basis for V. A bimodule of A over V is given by the following table (see [13]):

$\cdot$	e_0	e_1	e_2	e_3	v_0	v_1	v_2	v_3
e_0	e_0	e_1	0	0	v_0	0	v_2	v_3
e_1	0	0	0	0	0	v_0	0	0
e_2	e_2	0	0	e_1	0	$-v_3$	0	0
e_3	e_3	0	$-e_1$	0	0	v_2	0	0
v_0	v_0	0	0	0	0	0	0	0
v_1	v_1	0	v_3	$-v_2$	0	0	0	0
v_2	0	0	v_0	0	0	0	0	0
v_3	0	0	0	v_0	0	0	0	0

Table 1: Multiplication of A and V

Proposition 2.6. *Let $T : V \to A$ be a relative Rota-Baxter operator for the 4-dimension alternative algebra in last Example 2.5. Then the corresponding matrices for the relative Rota-Baxter operators are the following:*

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & c \end{pmatrix},$$

$$T_5=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\a&b&0&c\end{pmatrix},T_6=\begin{pmatrix}0&0&0&0\\0&0&0&0\\a&b&0&0\\c&\frac{bc}{a}&0&0\end{pmatrix},T_7=\begin{pmatrix}0&0&0&0\\0&a&b&0\\0&c&0&0\\0&d&0&0\end{pmatrix},T_8=\begin{pmatrix}0&0&0&0\\0&a&b&c\\0&d&0&0\\0&0&0&0\end{pmatrix},$$

$$T_9=\begin{pmatrix}0&0&0&0\\0&0&0&0\\a&b&c&0\\0&0&0&0\end{pmatrix},T_{10}=\begin{pmatrix}0&0&0&0\\0&a&0&0\\0&b&c&0\\0&0&0&0\end{pmatrix},T_{11}=\begin{pmatrix}0&0&0&0\\0&a&0&0\\0&b&0&0\\0&c&0&0\end{pmatrix},T_{12}=\begin{pmatrix}0&0&0&0\\0&b&0&a\\0&c&d&0\\0&a&0&0\end{pmatrix},$$

$$T_{13}=\begin{pmatrix}0&0&0&0\\a&b&c&d\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

where a, b, c, d, e are arbitrary parameters.

Proof. We give a sketch of proof as follows. Denote the corresponding matrix under the basis $\{v_0, v_1, v_2, v_3\}$ and $\{e_0, e_1, e_2, e_3\}$ by

$$T\begin{pmatrix}v_0\\v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}r_{11}&r_{12}&r_{13}&r_{14}\\r_{21}&r_{22}&r_{23}&r_{24}\\r_{31}&r_{32}&r_{33}&r_{34}\\r_{41}&r_{42}&r_{43}&r_{44}\end{pmatrix}\begin{pmatrix}e_0\\e_1\\e_2\\e_3\end{pmatrix},$$

where

$$\begin{aligned}T(v_0)&=r_{11}e_0+r_{12}e_1+r_{13}e_2+r_{14}e_3,\\T(v_1)&=r_{21}e_0+r_{22}e_1+r_{23}e_2+r_{24}e_3,\\T(v_2)&=r_{31}e_0+r_{32}e_1+r_{33}e_2+r_{34}e_3,\\T(v_3)&=r_{41}e_0+r_{42}e_1+r_{43}e_2+r_{44}e_3.\end{aligned}$$

Now, for any $v_i, v_j \in V$, since T is a relative Rota-Baxter operator, we have the equality

$$T(v_0)\cdot T(v_0)=T(T(v_0)\cdot v_0+v_0\cdot T(v_0)).$$

By the table in Example 2.5, we get

$$\begin{aligned}&T(v_0)\cdot T(v_0)\\=&(r_{11}e_0+r_{12}e_1+r_{13}e_2+r_{14}e_3)\cdot(r_{11}e_0+r_{12}e_1+r_{13}e_2+r_{14}e_3)\\=&r_{11}^2e_0+r_{11}r_{12}e_1+r_{11}r_{13}e_2+r_{11}r_{14}e_3,\end{aligned}$$

$$\begin{aligned}&T(T(v_0)\cdot v_0+v_0\cdot T(v_0))\\=&T((r_{11}e_0+r_{12}e_1+r_{13}e_2+r_{14}e_3)\cdot v_0+v_0\cdot(r_{11}e_0+r_{12}e_1+r_{13}e_2+r_{14}e_3))\\=&2r_{11}^2e_0+2r_{11}r_{12}e_1+2r_{11}r_{13}e_2+2r_{11}r_{14}e_3,\end{aligned}$$

Thus we have $r_{11}^2=0, r_{11}r_{12}=0, r_{11}r_{13}=0, r_{11}r_{14}=0$. This imply that $r_{11}=0$.

Similarly, by calculating $T(v_i)\cdot T(v_j)=T(T(v_i)\cdot v_j+v_i\cdot T(v_j))(0\le i,j\le 3)$, we get the result. The details are omitted. □

Proposition 2.7. *Let $T : V \to A$ be a relative Rota-Baxter operator. Then V carries an alternative algebra structure by*

$$u \cdot_T v := T(u) \cdot v + u \cdot T(v).$$

Proof. For any $u, v, w \in V$, since A is an alternative algebra, we have

$$\begin{aligned}
& (u \cdot_T v) \cdot_T w - u \cdot_T (v \cdot_T w) + (v \cdot_T u) \cdot_T w - v \cdot_T (u \cdot_T w) \\
= \ & (T(u) \cdot v + u \cdot T(v)) \cdot_T w - u \cdot_T (T(v) \cdot w + v \cdot T(w)) \\
& +(T(v) \cdot u + v \cdot T(u)) \cdot_T w - v \cdot_T (T(u) \cdot w + u \cdot T(w)) \\
= \ & \underline{T(T(u) \cdot v + u \cdot T(v))} \cdot w + (T(u) \cdot v + u \cdot T(v)) \cdot T(w) \\
& -T(u) \cdot (T(v) \cdot w + v \cdot T(w)) - u \cdot \underline{T(T(v) \cdot w + v \cdot T(w))} \\
& +\underline{T(T(v) \cdot u + v \cdot T(u))} \cdot w + (T(v) \cdot u + v \cdot T(u)) \cdot T(w) \\
& -T(v) \cdot (T(u) \cdot w + u \cdot T(w)) - v \cdot \underline{T(T(u) \cdot w + u \cdot T(w))} \\
= \ & \underline{(T(u) \cdot T(v))} \cdot w + (T(u) \cdot v) \cdot T(w) + (u \cdot T(v)) \cdot T(w) \\
& -T(u) \cdot (T(v) \cdot w) - T(u) \cdot (v \cdot T(w)) - u \cdot \underline{(T(v) \cdot T(w))} \\
& +\underline{(T(v) \cdot T(u))} \cdot w + (T(v) \cdot u) \cdot T(w) + (v \cdot T(u)) \cdot T(w) \\
& -T(v) \cdot (T(u) \cdot w) - T(v) \cdot (u \cdot T(w)) - v \cdot \underline{(T(u) \cdot T(w))} \\
= \ & 0,
\end{aligned}$$

where in the third equality we use the fact T is a relative Rota-Baxter operator satisfying (5) and in the fourth equality we use the fact V is a bimodule of A satisfying (1) and (3). Thus we obtain

$$(u \cdot_T v) \cdot_T w - u \cdot_T (v \cdot_T w) + (v \cdot_T u) \cdot_T w - v \cdot_T (u \cdot_T w) = 0.$$

Similar computations show that

$$(u \cdot_T v) \cdot_T w - u \cdot_T (v \cdot_T w) + (u \cdot_T w) \cdot_T v - u \cdot_T (w \cdot_T v) = 0.$$

This finishes the proof. □

3. Representation and cohomology

In this section, the representation and cohomology theory of a relative Rota-Baxter alternative algebra are given.

Definition 3.1. *Let (M, A, T) be a relative Rota-Baxter alternative algebra. A bimodule of (M, A, T) is an object (W, V, φ) such that the following conditions are satisfied:*

(i) V and W are bimodules of $(A, \cdot)$ respectively;

(ii) there exists linear maps $\triangleright : V \otimes M \to W$ and $\triangleleft : M \otimes V \to W$ such that:

$$\varphi(w) \cdot T(m) = \varphi(\varphi(w) \triangleright m + w \cdot T(m)), \tag{6}$$

$$T(m) \cdot \varphi(w) = \varphi(T(m) \cdot w + m \triangleleft \varphi(w)); \tag{7}$$

(iii) these bimodule structure satisfying the following compatibility conditions

$$(x \cdot v) \triangleright m - x \cdot (v \triangleright m) + (v \cdot x) \triangleright m - v \triangleright (x \cdot m) = 0, \tag{8}$$

$$(m \cdot x) \triangleleft v - m \triangleleft (x \cdot v) + (m \triangleleft v) \cdot x - m \triangleleft (v \cdot x) = 0, \tag{9}$$

$$(x \cdot v) \triangleright m - x \cdot (v \triangleright m) + (x \cdot m) \triangleleft v - x \cdot (m \triangleleft v) = 0, \tag{10}$$

$$(v \cdot x) \triangleright m - v \triangleright (x \cdot m) + (v \triangleright m) \cdot x - v \triangleright (m \cdot x) = 0, \tag{11}$$

$$(m \triangleleft v) \cdot x - m \triangleleft (v \cdot x) + (v \triangleright m) \cdot x - v \triangleright (m \cdot x) = 0, \tag{12}$$

$$(m \cdot x) \triangleleft v - m \triangleleft (x \cdot v) + (x \cdot m) \triangleleft v - x \cdot (m \triangleleft v) = 0, \tag{13}$$

where $x \in A$, $m \in M$, $v \in V$ and $w \in W$.

For example, let $(W, V, \varphi) = (M, A, T)$, then we get the adjoint representation of (M, A, T) on itself with $\varphi = T$, $\triangleleft : M \otimes A \to M, m \triangleleft x \triangleq m \cdot x$ and $\triangleright : A \otimes M \to M, x \triangleright m \triangleq x \cdot m$.

We construct semidirect product of a relative Rota-Baxter alternative algebra (M, A, T) with its bimodule (W, V, φ).

Proposition 3.2. *Given a bimodule of a relative Rota-Baxter alternative algebra* (M, A, T) *on* (W, V, φ)*. Define on* $(M \oplus W, A \oplus V, \widehat{T} = T + \varphi)$ *the following maps*

$$\begin{cases} \widehat{T}(m+w) & \triangleq & T(m) + \varphi(w), \\ (x+v) \cdot (x' + v') & \triangleq & x \cdot x' + x \cdot v' + v \cdot x', \\ (x+v) \cdot (m+w) & \triangleq & x \cdot m + x \cdot w + v \triangleright m, \\ (m+w) \cdot (x+v) & \triangleq & m \cdot x + w \cdot x + m \triangleleft v. \end{cases} \tag{14}$$

Then $(M \oplus W, A \oplus V, \widehat{T})$ *is a Rota-Baxter alternative algebra, which is called the semidirect product of the alternative algebra of* (M, A, T) *and* (W, V, φ)*.*

Proof. We verify that $\widehat{T}$ is a relative Rota-Baxter operator. We are going to verify the equality

$$\widehat{T}(m+w) \cdot \widehat{T}(m'+w') = \widehat{T}(\widehat{T}(m+w) \cdot (m'+w') + (m+w) \cdot \widehat{T}(m'+w')). \tag{15}$$

The left hand side of (15) is

$$\begin{aligned} \widehat{T}(m+w) \cdot \widehat{T}(m'+w') &= (T(m) + \varphi(w)) \cdot (T(m') + \varphi(w')) \\ &= T(m) \cdot T(m') + T(m) \cdot \varphi(w') + \varphi(w) \cdot T(m'), \end{aligned}$$

and the right hand side of (15) is

$$\begin{aligned} & \widehat{T}(\widehat{T}(m+w) \cdot (m'+w') + (m+w) \cdot \widehat{T}(m'+w')) \\ = \; & \widehat{T}((T(m) + \varphi(w)) \cdot (m'+w') + (m+w) \cdot (T(m') + \varphi(w'))) \\ = \; & \widehat{T}(T(m) \cdot m' + T(m) \cdot w' + \varphi(w) \triangleright m' + m \cdot T(m') + w \cdot T(m') + m \triangleleft \varphi(w')) \\ = \; & T(T(m) \cdot m') + \varphi(T(m) \cdot w') + \varphi(\varphi(w) \triangleright m') \\ & + T(m \cdot T(m')) + \varphi(w \cdot T(m')) + \varphi(m \triangleleft \varphi(w')). \end{aligned}$$

Thus the two sides of (15) are equal to each other by conditions (6) and (7).

Next, we verify that $M \oplus W$ carries a bimodule structure over alternative algebra $A \oplus V$. The left module condition is

$$(x+v, y+v', m+w) + (y+v', x+v, m+w) = 0. \tag{16}$$

By direct computations, we have

$$\begin{aligned} & (x+v, y+v', m+w) + (y+v', x+v, m+w) \\ = \; & ((x+v) \cdot (y+v')) \cdot (m+w) - (x+v) \cdot ((y+v') \cdot (m+w)) \\ & + ((y+v') \cdot (x+v)) \cdot (m+w) - (y+v') \cdot ((x+v) \cdot (m+w)) \\ = \; & (x \cdot y + x \cdot v' + v \cdot y) \cdot (m+w) - (x+v)(y \cdot m + y \cdot w + v' \triangleright m) \\ & + (y \cdot x + y \cdot v' + v \cdot x) \cdot (m+w) - (y+v')(x \cdot m + x \cdot w + v' \triangleright m) \\ = \; & (x \cdot y) \cdot m + (x \cdot y) \cdot w + (x \cdot v') \triangleright m + (v \cdot y) \triangleright m \\ & - x \cdot (y \cdot m) - x \cdot (y \cdot w) - x \cdot (v' \triangleright m) - v \triangleright (y \cdot m) \\ & + (y \cdot x) \cdot m + (y \cdot x) \cdot w + (y \cdot v) \triangleright m + (v' \cdot x) \triangleright m \\ & - y \cdot (x \cdot m) - y \cdot (x \cdot w) - y \cdot (v \triangleright m) - v' \triangleright (x \cdot m) \\ = \; & 0. \end{aligned}$$

Thus the two sides of (16) are equal to each other if and only if (8) holds. Similar computations show that $M \oplus V$ carries a right module structure over alternative algebra $A \oplus W$ if and only if (9) holds.

Finally, the bimodule conditions are satisfied if and only if (10), (11), (12), and (13) hold. This completes the proof. □

Now we define a low dimensional cohomology for relative Rota-Baxter alternative algebras as follows. We denote by $C^n = C^n((A, M, T), (W, V, \varphi))$ the space of all n-cochains. Let

$$C^1 = \mathrm{Hom}(M, W) \oplus \mathrm{Hom}(A, V),$$

$$C^2 = \mathrm{Hom}(M, V) \oplus \mathrm{Hom}(A \otimes A, V) \oplus \mathrm{Hom}(A \otimes M, W) \oplus \mathrm{Hom}(M \otimes A, W),$$

and

$$C^3 = \mathrm{Hom}(M \otimes M, V) \oplus \mathrm{Hom}(\otimes^3 A, V) \oplus \mathrm{Hom}(A \otimes A \otimes M, W) \oplus \mathrm{Hom}(M \otimes A \otimes A, W).$$

For $n = 1$, let $(N_1, N_0) \in C^1$, we define the map $D_1 : C^1 \to C^2$ by

$$\begin{aligned}
D_1(N_1, N_0)(m) &= \varphi \circ N_1(m) - N_0 \circ T(m),\\
D_1(N_1, N_0)(x, y) &= N_0(x) \cdot y + x \cdot N_0(y) - N_0(xy),\\
D_1(N_1, N_0)(x, m) &= N_0(x) \triangleright m + x \cdot N_1(m) - N_1(x \cdot m),\\
D_1(N_1, N_0)(m, x) &= N_1(m) \cdot x + m \triangleleft N_0(x) - N_1(m \cdot x).
\end{aligned}$$

Thus a 1-cocycle is $(N_1, N_0) \in \mathrm{Hom}(M, W) \oplus \mathrm{Hom}(A, V)$, such that

$$\varphi \circ N_1(m) - N_0 \circ T(m) = 0, \tag{17}$$

$$N_0(x)y + xN_0(y) - N_0(xy) = 0, \tag{18}$$

$$N_0(x) \triangleright m + x \cdot N_1(m) - N_1(x \cdot m) = 0, \tag{19}$$

$$N_1(m) \cdot x + m \triangleleft N_0(x) - N_1(m \cdot x) = 0. \tag{20}$$

For $n = 2$, let $(\theta, \omega, \mu, \nu) \in C^2$, we define the map $D_2 : C^2 \to C^3$ by

$$\begin{aligned}
D_2(\theta, \omega, \mu, \nu)(m, n) &= \theta(m) \cdot T(n) + T(m) \cdot \theta(n) - \theta(T(m) \cdot n) + \theta(m \cdot T(n))\\
&\quad -\varphi(\theta(m) \triangleright n + \mu(T(m), n) + \nu(m, T(n)) + m \triangleleft \theta(n)),\\
D_2^l(\theta, \omega, \mu, \nu)(x, y, z) &= \omega(x, y) \cdot z + \omega(x \cdot y, z) - x \cdot \omega(y, z) - \omega(x, y \cdot z)\\
&\quad +\omega(y, x) \cdot z + \omega(y \cdot x, z) - y \cdot \omega(x, z) - \omega(y, x \cdot z),\\
D_2^r(\theta, \omega, \mu, \nu)(x, y, z) &= \omega(x, y) \cdot z + \omega(x \cdot y, z) - x \cdot \omega(y, z) - \omega(x, y \cdot z)\\
&\quad +\omega(x, z) \cdot y + \omega(x \cdot z, y) - x \cdot \omega(z, y) - \omega(x, z \cdot y),\\
D_2^l(\theta, \omega, \mu, \nu)(x, y, m) &= \omega(x, y) \triangleright m + \mu(x \cdot y, m) - x \cdot \mu(y, m) - \mu(x, y \cdot m)\\
&\quad +\omega(y, x) \triangleright m + \mu(y \cdot x, m) - y \cdot \mu(x, m) - \mu(y, x \cdot m),\\
D_2^l(\theta, \omega, \mu, \nu)(m, x, y) &= m \triangleleft \omega(x, y) + \nu(m, x \cdot y) - \nu(m, x) \cdot y - \nu(m \cdot x, y)\\
&\quad +m \triangleleft \omega(y, x) + \nu(m, y \cdot x) - \nu(m, y) \cdot x - \nu(m \cdot y, x),\\
D_2^r(\theta, \omega, \mu, \nu)(x, y, m) &= \omega(x, y) \triangleright m + \mu(x \cdot y, m) - x \cdot \mu(y, m) - \mu(x, y \cdot m)\\
&\quad +\mu(x, m) \cdot y + \nu(x \cdot m, y) - x \cdot \nu(m, y) - \mu(x, m \cdot y),\\
D_2^r(\theta, \omega, \mu, \nu)(m, x, y) &= m \triangleleft \omega(x, y) + \nu(m, x \cdot y) - \nu(m, x) \cdot y - \nu(m \cdot x, y)\\
&\quad +x \cdot \nu(m, y) + \mu(x, m \cdot y) - \mu(x, m) \cdot y - \nu(x \cdot m, y).
\end{aligned}$$

Thus a 2-coboundary is $(\theta, \omega, \mu, \nu) \in C^2$ such that $(\theta, \omega, \mu, \nu) = D_1(N_1, N_0)$, i.e.

$$\theta(m) = \varphi \circ N_1(m) - N_0 \circ T(m), \tag{21}$$

$$\omega(x, y) = N_0(x) \cdot y + x \cdot N_0(y) - N_0(xy), \tag{22}$$

$$\mu(x, m) = N_0(x) \triangleright m + x \cdot N_1(m) - N_1(x \cdot m), \tag{23}$$

$$\nu(m, x) = N_1(m) \cdot x + m \triangleleft N_0(x) - N_1(m \cdot x), \tag{24}$$

and a 2-cocycle is $(\theta, \omega, \mu, \nu) \in C^2$ such that

$$\begin{aligned}
\theta(m)\cdot T(n) + T(m)\cdot\theta(n) - \theta(T(m)\cdot n) + \theta(m\cdot T(n)) & \\
-\varphi(\theta(m)\triangleright n + \mu(T(m),n) + \nu(m,T(n)) + m\triangleleft\theta(n)) &= 0, \quad (25)\\
\omega(x,y)\cdot z + \omega(x\cdot y,z) - x\cdot\omega(y,z) - \omega(x,y\cdot z) & \\
+\omega(y,x)\cdot z + \omega(y\cdot x,z) - y\cdot\omega(x,z) - \omega(y,x\cdot z) &= 0, \quad (26)\\
\omega(x,y)\cdot z + \omega(x\cdot y,z) - x\cdot\omega(y,z) - \omega(x,y\cdot z) & \\
+\omega(x,z)\cdot y + \omega(x\cdot z,y) - x\cdot\omega(z,y) - \omega(x,z\cdot y) &= 0, \quad (27)\\
\omega(x,y)\triangleright m + \mu(x\cdot y,m) - x\cdot\mu(y,m) - \mu(x,y\cdot m) & \\
+\omega(y,x)\triangleright m + \mu(y\cdot x,m) - y\cdot\mu(x,m) - \mu(y,x\cdot m) &= 0, \quad (28)\\
m\triangleleft\omega(x,y) + \nu(m,x\cdot y) - \nu(m,x)\cdot y - \nu(m\cdot x,y) & \\
+m\triangleleft\omega(y,x) + \nu(m,y\cdot x) - \nu(m,y)\cdot x - \nu(m\cdot y,x) &= 0, \quad (29)\\
\omega(x,y)\triangleright m + \mu(x\cdot y,m) - x\cdot\mu(y,m) - \mu(x,y\cdot m) & \\
+\mu(x,m)\cdot y + \nu(x\cdot m,y) - x\cdot\nu(m,y) - \mu(x,m\cdot y) &= 0, \quad (30)\\
m\triangleleft\omega(x,y) + \nu(m,x\cdot y) - \nu(m,x)\cdot y - \nu(m\cdot x,y) & \\
+x\cdot\nu(m,y) + \mu(x,m\cdot y) - \mu(x,m)\cdot y - \nu(x\cdot m,y) &= 0. \quad (31)
\end{aligned}$$

Proposition 3.3. *The space of 2-coboundaries is contained in the space of 2-cocycles, i.e.* $D^2 = D_2 \circ D_1 = 0$.

Proof. We should verify that if $(\theta, \omega, \mu, \nu) = D_1(N_1, N_0)$, then $D_2(\theta, \omega, \mu, \nu) = D_2 D_1(N_1, N_0) = 0$. Assume $(\theta, \omega, \mu, \nu)$ is given by (21)–(24), we will verify that it must satisfy (25)–(31). We compute (25) as follows:

$$\begin{aligned}
& \theta(m)\cdot T(n) + T(m)\cdot\theta(n) - \theta(T(m)\cdot n) - \theta(m\cdot T(n)) \\
& -\varphi(\theta(m)\triangleright n) - \varphi(\mu(T(m),n)) - \varphi(m\triangleleft\theta(n)) - \varphi(\nu(m,T(n))) + \omega(T(m),T(n)) \\
= \; & (\varphi\circ N_1(m) - N_0\circ T(m))\cdot T(n) + T(m)\cdot(\varphi\circ N_1(n) - N_0\circ T(n)) \\
& -(\varphi\circ N_1(T(m)\cdot n) - N_0\circ T(T(m)\cdot n)) - (\varphi\circ N_1(m\cdot T(n)) - N_0\circ T(m\cdot T(n))) \\
& -\varphi((\varphi\circ N_1(m) - N_0\circ T(m))\triangleright n) - \varphi(N_0(T(m))\triangleright n + T(m)\cdot N_1(n) - N_1(T(m)\cdot n)) \\
& -\varphi(m\triangleleft(\varphi\circ N_1(n) - N_0\circ T(n))) - \varphi(N_1(m)\cdot T(n) + m\triangleleft N_0(T(n)) - N_1(m\cdot T(n))) \\
& +N_0(T(m))\cdot T(n) + T(m)\cdot N_0(T(n)) - N_0(T(m)T(n)) \\
= \; & (\varphi\circ N_1(m))\cdot T(n) - (N_0\circ T(m))\cdot T(n) + T(m)\cdot(\varphi\circ N_1(n)) - T(m)\cdot(N_0\circ T(n)) \\
& -\varphi\circ N_1(T(m)\cdot n) + N_0\circ T(T(m)\cdot n) - \varphi\circ N_1(m\cdot T(n)) + N_0\circ T(m\cdot T(n)) \\
& -\varphi((\varphi\circ N_1(m))\triangleright n) + \varphi((N_0\circ T(m))\triangleright n) - \varphi(N_0(T(m))\triangleright n) - \varphi(T(m)\cdot N_1(n)) \\
& +\varphi(N_1(T(m)\cdot n)) - \varphi(m\triangleleft(\varphi\circ N_1(n))) + \varphi(m\triangleleft(N_0\circ T(n))) - \varphi(N_1(m)\cdot T(n)) \\
& -\varphi(m\triangleleft N_0(T(n))) + \varphi(N_1(m\cdot T(n))) + N_0(T(m))\cdot T(n) + T(m)\cdot N_0(T(n)) \\
& -N_0(T(m)T(n)) \\
= \; & 0,
\end{aligned}$$

where we use the fact that A is a relative Rota-Baxter alternative algebra and

$$\varphi(m\triangleleft\varphi\circ N_1(n)) = T(m)\cdot(\varphi\circ N_1(n)) - \varphi(T(m)\cdot N_1(n)),$$

$$\varphi(\varphi\circ N_1(m)\triangleright n) = (\varphi\circ N_1(m))\cdot T(n) - \varphi(N_1(m)\cdot T(n)).$$

The other equalities can be checked similarly. This completes the proof. □

Denote the set of 2-cocycles by $\mathbf{Z}^2((M,A,T),(W,V,\varphi))$, the set of 2-coboundaries by $\mathbf{B}^2((M,A,T),(W,V,\varphi))$ and the second cohomology group by $\mathbf{H}^2((M,A,T),(W,V,\varphi))$.

Definition 3.4. *The second cohomology group of* (M,A,T) *with coefficients in* (W,V,φ) *is defined as the quotient*

$$\mathbf{H}^2((M,A,T),(W,V,\varphi)) = \mathbf{Z}^2((M,A,T),(W,V,\varphi))/\mathbf{B}^2((M,A,T),(W,V,\varphi)).$$

4. Infinitesimal deformations

Let (M, A, T) be a relative Rota-Baxter alternative algebra and $\theta : M \to A,\ \omega : A \otimes A \to A,\ \mu : A \otimes M \to M,\ \nu : M \otimes A \to M$ be linear maps. Consider a λ-parametrized family of linear operations:

$$\begin{aligned} T_\lambda(m) &\triangleq T(m) + \lambda\theta(m), \\ x \cdot_\lambda y &\triangleq x \cdot y + \lambda\omega(x, y), \\ x \cdot_\lambda m &\triangleq x \cdot m + \lambda\mu(x, m), \\ m \cdot_\lambda x &\triangleq m \cdot x + \lambda\nu(m, x). \end{aligned}$$

If $(M_\lambda, A_\lambda, T_\lambda)$ forms a relative Rota-Baxter alternative algebra, then we say that $(\theta, \omega, \mu, \nu)$ generates a 1-parameter infinitesimal deformation of (M, A, T).

Theorem 4.1. *With the notations above, $(\theta, \omega, \mu, \nu)$ generates a 1-parameter infinitesimal deformation of (M, A, T) if and only if the following conditions hold:*

(i) *$(\theta, \omega, \mu, \nu)$ is a 2-cocycle of (M, A, T) with coefficients in the adjoint representation;*

(ii) *(M, A, θ) is a relative Rota-Baxter alternative algebra with multiplication ω on A and bimodule structure of A on M by μ, ν.*

Proof. If $(M_\lambda, A_\lambda, T_\lambda)$ is a relative Rota-Baxter alternative algebra, then T_λ is a relative Rota-Baxter operater. Thus we have

$$\begin{aligned} & T_\lambda(m) \cdot_\lambda T_\lambda(n) - T_\lambda(T_\lambda(m) \cdot_\lambda n + n \cdot_\lambda T_\lambda(m)) \\ = \ & (T(m) + \lambda\theta(m)) \cdot_\lambda (T(n) + \lambda\theta(n)) - T_\lambda((T(m) + \lambda\theta(m)) \cdot_\lambda n + n \cdot_\lambda (T(m) + \lambda\theta(m))) \\ = \ & T(m) \cdot T(n) + \lambda T(m) \cdot \theta(n) + \lambda\theta(m) \cdot T(n) + \lambda^2\theta(m) \cdot \theta(n) - T_\lambda((T(m) + \lambda\theta(m)) \cdot n \\ & + \lambda\mu((T(m) + \lambda\theta(m)), n) + n \cdot (T(m) + \lambda\theta(m)) + \lambda\nu(n, (T(m) + \lambda\theta(m)))) \\ & + \lambda^3\omega(T(m) + \theta(m), T(n) + \theta(n)) \\ = \ & T(m) \cdot T(n) + \lambda T(m) \cdot \theta(n) + \lambda\theta(m) \cdot T(n) + \lambda^2\theta(m) \cdot \theta(n) \\ & - T(T(m) \cdot n) - \lambda T(\theta(m) \cdot n) - \lambda T(\mu(T(m), n)) - \lambda^2 T(\mu(\theta(m), n)) \\ & - T(n \cdot T(m)) - \lambda T(n \cdot \theta(m)) - \lambda T(\nu(n, (T(m)))) - \lambda^2 T(\nu(n, \theta(m))) \\ & - \lambda\theta((T(m) \cdot n) - \lambda^2\theta(\theta(m) \cdot n) - \lambda^2\theta(\mu(T(m), n)) - \lambda^3\theta(\mu(\theta(m), n)) \\ & - \lambda\theta(n \cdot T(m)) - \lambda^2\theta(n \cdot \theta(m)) - \lambda^2\theta(\nu(n, (T(m)))) - \lambda^3\theta(\nu(n, \theta(m))) \\ & + \lambda^3\omega(T(m) + \theta(m), T(n) + \theta(n)) \\ = \ & 0, \end{aligned}$$

which implies that

$$\begin{aligned} T(m) \cdot \theta(n) + \theta(m) \cdot T(n) + T\nu(m, x) & \\ - \omega(T(m), x) - T(\theta(m) \cdot n) - T(\mu(T(m), n)) & \\ - T(n \cdot \theta(m)) - T(\nu(n, (T(m)))) - \theta((T(m) \cdot n) - \theta(n \cdot T(m)) &= 0, \qquad (32) \\ \theta(m) \cdot \theta(n) - T(\mu(\theta(m), n)) - T(\nu(n, \theta(m))) - \theta(\theta(m) \cdot n) & \\ - \theta(\mu(T(m), n)) - \theta(n \cdot \theta(m)) - \theta(\nu(n, (T(m)))) &= 0, \qquad (33) \\ \omega(T(m) + \theta(m), T(n) + \theta(n)) + \theta(\mu((\theta(m)), n)) - \theta(\nu(n, \theta(m))) &= 0. \qquad (34) \end{aligned}$$

Since A_λ is an alternative algebra, we get

$$\begin{aligned}
& (x \cdot_\lambda y) \cdot_\lambda z - x \cdot_\lambda (y \cdot_\lambda z) + (y \cdot_\lambda x) \cdot_\lambda z - y \cdot_\lambda (x \cdot_\lambda z) \\
= \; & (x \cdot y + \lambda\omega(x,y)) \cdot z - x \cdot (y \cdot z + \lambda\omega(y,z)) \\
& +(y \cdot x + \lambda\omega(y,x)) \cdot z - y \cdot (x \cdot z + \lambda\omega(x,z)) \\
= \; & (x \cdot y) \cdot z + \lambda(\omega(x,y) \cdot z) + \lambda\omega(x \cdot y, z) + \lambda^2\omega(\omega(x,y),z) \\
& -x \cdot (y \cdot z) - \lambda(x \cdot \omega(y,z)) - \lambda\omega(x, y \cdot z) - \lambda^2\omega(x,\omega(y,z)) \\
& +(y \cdot x) \cdot z + \lambda(\omega(y,x) \cdot z) + \lambda\omega(y \cdot x, z) + \lambda^2\omega(\omega(y,x),z) \\
& -y \cdot (x \cdot z) - \lambda(y \cdot \omega(x,z)) - \lambda\omega(y, x \cdot z) - \lambda^2\omega(y,\omega(x,z)) \\
= \; & 0,
\end{aligned}$$

$$\begin{aligned}
& (x \cdot y) \cdot z - x \cdot (y \cdot z) + (x \cdot z) \cdot y - x \cdot (z \cdot y) \\
= \; & (x \cdot y + \lambda\omega(x,y)) \cdot z - x \cdot (y \cdot z + \lambda\omega(y,z)) \\
& +(x \cdot z + \lambda\omega(x,z)) \cdot y - x \cdot (z \cdot y + \lambda\omega(z,y)) \\
= \; & (x \cdot y) \cdot z + \lambda(\omega(x,y) \cdot z) + \lambda\omega(x \cdot y, z) + \lambda^2\omega(\omega(x,y),z) \\
& -x \cdot (y \cdot z) - \lambda(x \cdot \omega(y,z)) - \lambda\omega(x, y \cdot z) - \lambda^2\omega(x,\omega(y,z)) \\
& +(x \cdot z) \cdot y + \lambda(\omega(x,z) \cdot y) + \lambda\omega(x \cdot z, y) + \lambda^2\omega(\omega(x,z),y) \\
& -x \cdot (z \cdot y) - \lambda(x \cdot \omega(z,y)) - \lambda\omega(x, z \cdot y) - \lambda^2\omega(x,\omega(z,y)) \\
= \; & 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
\omega(x,y) \cdot z + \omega(x \cdot y, z) - x \cdot \omega(y,z) - \omega(x, y \cdot z) & \\
+\omega(y,x) \cdot z + \omega(y \cdot x, z) - y \cdot \omega(x,z) - \omega(y, x \cdot z) &= 0, \quad (35)\\
\omega(\omega(x,y),z) - \omega(x,\omega(y,z)) + \omega(\omega(y,x),z) - \omega(y,\omega(x,z)) &= 0, \quad (36)\\
\omega(x,y) \cdot z + \omega(x \cdot y, z) - x \cdot \omega(y,z) - \omega(x, y \cdot z) & \\
+\omega(x,z) \cdot y + \omega(x \cdot z, y) - x\omega(z,y) - \omega(x, z \cdot y) &= 0, \quad (37)\\
\omega(\omega(x,y),z) - \omega(x,\omega(y,z)) + \omega(\omega(x,z),y) - \omega(x,\omega(z,y)) &= 0. \quad (38)
\end{aligned}$$

Since M_λ is a left module of A_λ, we have

$$\begin{aligned}
& (x \cdot_\lambda y) \cdot_\lambda m - x \cdot_\lambda (y \cdot_\lambda m) + (y \cdot_\lambda x) \cdot_\lambda m - y \cdot_\lambda (x \cdot_\lambda m) \\
= \; & (x \cdot y + \lambda\omega(x,y)) \cdot m - x \cdot (y \cdot m + \lambda\mu(y,m)) \\
& +(y \cdot x + \lambda\omega(y,x)) \cdot m - y \cdot (x \cdot m + \lambda\mu(x,m)) \\
= \; & (x \cdot y) \cdot m + \lambda(\omega(x,y)) \cdot m + \lambda\mu(x \cdot y, m) + \lambda^2\mu(\omega(x,y),m) \\
& -x \cdot (y \cdot m) - \lambda(x \cdot \mu(y,m)) - \lambda\mu(x, y \cdot m) - \lambda^2\mu(x,\mu(y,m)) \\
& (y \cdot x) \cdot m + \lambda(\omega(y,x)) \cdot m + \lambda\mu(y \cdot x, m) + \lambda^2\mu(\omega(y,x),m) \\
& -y \cdot (x \cdot m) - \lambda(y \cdot \mu(x,m)) - \lambda\mu(y, x \cdot m) - \lambda^2\mu(y,\mu(x,m)) \\
= \; & 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
\omega(x,y) \cdot m + \mu(x \cdot y, m) - x \cdot \mu(y,m) - \mu(x, y \cdot m) & \\
+(\omega(y,x)) \cdot m + \mu(y \cdot x, m) - y \cdot \mu(x,m) - \mu(y, x \cdot m) &= 0, \quad (39)\\
\mu(\omega(x,y),m) - \mu(x,\mu(y,m)) + \mu(\omega(y,x),m) - \mu(y,\mu(x,m)) &= 0. \quad (40)
\end{aligned}$$

Similar computations show that

$$\begin{aligned} & m\cdot\omega(x,y)+\nu(m,xy)-\nu(m,x)\cdot y-\nu(m\cdot x,y) \\ & +m\cdot\omega(y,x)+\nu(m,yx)-\nu(m,y)\cdot x-\nu(m\cdot y,x) &=& 0, \quad (41)\\ & \nu(m,\omega(x,y))-\nu(\nu(m,x),y)+\nu(m,\omega(y,x))-\nu(\nu(m,y),x) &=& 0. \quad (42)\end{aligned}$$

By (33), (35), (37), (39) and (41), we find that (θ,ω,μ,ν) is a 2-cocycle of (M,A,T) with the coefficients in the adjoint representation. Furthermore, by (32), (34), (36), (38), (40) and (42), (M,A,θ) is a relative Rota-Baxter alternative algebra with multiplication ω on A and bimodule structures of A on M by μ,ν. □

5. Abelian extensions

Definition 5.1. *Let (M,A,T) and (W,V,φ) be two relative Rota-Baxter alternative algebras. An extension of (M,A,T) through (W,V,φ) is a relative Rota-Baxter alternative algebra $(\widehat{M},\widehat{A},\widehat{T})$ such that the following short exact sequence satisfying $\mathrm{Im}(i_0)=\mathrm{Ker}(p_0)$, $\mathrm{Im}(i_1)=\mathrm{Ker}(p_1)$ and the diagram commute*

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & W & \xrightarrow{i_1} & \widehat{M} & \xrightarrow{p_1} & M & \xrightarrow{0} & 0\\ {\scriptstyle 0}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\widehat{T}}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle 0}\downarrow\\ 0 & \xrightarrow{0} & V & \xrightarrow{i_0} & \widehat{A} & \xrightarrow{p_0} & A & \xrightarrow{0} & 0 \end{array} \quad (43)$$

It is called an abelian extension if (W,V,φ) with trivial multiplication. In this case, we denoted it by $\widehat{\mathrm{E}}=(\widehat{M},\widehat{A},\widehat{T})$ or simply by $\widehat{\mathrm{E}}$.

A splitting $\sigma=(\sigma_0,\sigma_1):(M,A,T)\to(\widehat{M},\widehat{A},\widehat{T})$ consists of linear maps $\sigma_0:A\to\widehat{A}$ and $\sigma_1:M\to\widehat{M}$ such that $p_0\circ\sigma_0=\mathrm{id}_A$, $p_1\circ\sigma_1=\mathrm{id}_M$ and $\widehat{T}\circ\sigma_1=\sigma_0\circ T$.

Two extensions $\widehat{\mathrm{E}}:0\longrightarrow(W,V,\varphi)\stackrel{i}{\longrightarrow}(\widehat{M},\widehat{A},\widehat{T})\stackrel{p}{\longrightarrow}(M,A,T)\longrightarrow 0$ and $\widetilde{\mathrm{E}}:0\longrightarrow(W,V,\varphi)\stackrel{j}{\longrightarrow}(\widetilde{M},\widetilde{A},\widetilde{T})\stackrel{q}{\longrightarrow}(M,A,T)\longrightarrow 0$ are called equivalent to each other, if there exists a morphism $F:(\widehat{M},\widehat{A},\widehat{T})\longrightarrow(\widetilde{M},\widetilde{A},\widetilde{T})$ such that $F\circ i=j$, $q\circ F=p$.

Let $(\widehat{M},\widehat{A},\widehat{T})$ be an extension of (M,A,T) by (W,V,φ) and $\sigma:(M,A,T)\to(\widehat{M},\widehat{A},\widehat{T})$ be a splitting. Define the following maps:

$$\left\{\begin{array}{lllrcl} \cdot: & A\otimes W & \longrightarrow & W, & x\cdot w & \triangleq \sigma_0(x)\cdot w,\\ \cdot: & W\otimes A & \longrightarrow & W, & w\cdot x & \triangleq w\cdot\sigma_0(x),\\ \cdot: & V\otimes A & \longrightarrow & V, & v\cdot x & \triangleq v\cdot\sigma_0(x),\\ \cdot: & A\otimes V & \longrightarrow & V, & x\cdot v & \triangleq \sigma_0(x)\cdot v,\\ \triangleright: & V\otimes M & \longrightarrow & W, & v\triangleright m & \triangleq v\cdot\sigma_1(m),\\ \triangleleft: & M\otimes V & \longrightarrow & W, & m\triangleleft v & \triangleq \sigma_1(m)\cdot v, \end{array}\right. \quad (44)$$

for all $x,y,z\in A$, $m\in M$. Note that the multiplications in the right hand side are in fact $(\widehat{M},\widehat{A},\widehat{T})$, but we denote it by $\cdot$ instead of $\widehat{\cdot}$ for simplicity.

Proposition 5.2. *With the above notations, (W,V,φ) is a bimodule of (M,A,T). Furthermore, this bimodule structure does not depend on the choice of the splitting σ. Moreover, equivalent abelian extensions give the same bimodule on (W,V,φ).*

Proof. First, we show that the bimodule is well-defined. Since $\mathrm{Ker}p_0\cong V$, then for $v\in V$, we have $p_0(v)=0$. By the fact that (p_1,p_0) is a morphism between $(\widehat{M},\widehat{A},\widehat{T})$ and (M,A,T), we get

$$p_0(x\cdot v)=p_0(\sigma_0(x)\cdot v)=p_0\sigma_0(x)\cdot p_0(v)=p_0\sigma_0(x)\cdot 0=0.$$

Thus $x \cdot v \in \ker p_0 \cong V$. Similar computations show that

$$p_0(x \cdot w) = p_0(\sigma_0(x) \cdot w) = p_0\sigma_0(x) \cdot p_0(w) = p_0\sigma_0(x) \cdot 0 = 0,$$

$$p_0(m \triangleleft v) = p_0(\sigma_1(m) \cdot v) = p_0\sigma_1(m) \cdot p_0(v) = p_0\sigma_0(m) \cdot 0 = 0.$$

Thus $x \cdot w, m \triangleleft v \in \mathrm{Ker} p_0 = V$.

We will show that these maps are independent of the choice of σ. In fact, if we choose another splitting $\sigma' : A \to \widehat{A}$, then $p_0(\sigma_0(x) - \sigma'_0(x)) = x - x = 0$, $p_1(\sigma_1(m) - \sigma'_1(m)) = m - m = 0$, i.e. $\sigma_0(x) - \sigma'_0(x) \in \mathrm{Ker} p_0 = V$, $\sigma_1(m) - \sigma'_1(m) \in \mathrm{Ker} p_1 = W$. Thus we get $\sigma_0(x) \cdot (w + v) = \sigma'_0(x) \cdot (w + v)$ and $\sigma_1(m) \cdot v = \sigma'_1(m) \cdot v$, which implies that the maps in (44) are independent on the choice of σ. Therefore the bimodule structures are well-defined.

Secondly, we check that (W, V, φ) is indeed a bimodule of (M, A, T). Let φ to be the restriction of the map $\widehat{T}$ on W, i.e. $\varphi(w) = \widehat{T}|_W(w)$. For $\triangleleft : V \otimes M \to W$, we get

$$\begin{aligned}
&= T(m) \cdot \varphi(w)\\
&= \sigma_0 T(m) \cdot \varphi(w)\\
&= \widehat{T}(\sigma_1(m)) \cdot \widehat{T}(w)\\
&= \widehat{T}(\widehat{T}\sigma_1(m) \cdot w + \sigma_1(m) \cdot \widehat{T}(w))\\
&= \widehat{T}(\sigma_0 T(m) \cdot w + \sigma_1(m) \cdot \widehat{T}(w))\\
&= \varphi(T(m) \cdot w + m \triangleleft \varphi(w)).
\end{aligned}$$

Thus we get

$$T(m) \cdot \varphi(w) = \varphi(T(m) \cdot w + m \triangleleft \varphi(w)).$$

Similarly, we obtain

$$\varphi(w) \cdot T(m) = \varphi(\varphi(w) \triangleright m + w \cdot T(m)).$$

Since (W, V, φ) is an abelian relative Rota-Baxter alternative algebra, we also get

$$\begin{aligned}
&(x \cdot w) \triangleright m - x \cdot (w \triangleright m) + (w \cdot x) \triangleright m - w \triangleright (x \cdot m)\\
= \;&(\sigma_0(x) \cdot w) \triangleright m - x \cdot (w \cdot \sigma_1(m)) + (w \cdot \sigma_0(x)) \triangleright m - w \triangleright (\sigma_0(x) \cdot m)\\
= \;&(\sigma_0(x) \cdot w) \cdot \sigma_1(m) - \sigma_0(x) \cdot (w \cdot \sigma_1(m)) + (w \cdot \sigma_0(x)) \cdot \sigma_1(m) - w \cdot (\sigma_0(x) \cdot \sigma_1(m))\\
= \;&0
\end{aligned}$$

and

$$\begin{aligned}
&(m \cdot x) \triangleleft w - m \triangleleft (x \cdot w) + (m \triangleleft w) \cdot x - m \triangleleft (w \cdot x)\\
= \;&(m \cdot \sigma_0(x)) \triangleleft w - m \triangleleft (\sigma_0(x) \cdot w) + (\sigma_1(m) \cdot w) \cdot x - m \triangleleft (w \cdot \sigma_0(x))\\
= \;&(\sigma_1(m) \cdot \sigma_0(x)) \cdot w - \sigma_1(m) \cdot (\sigma_0(x) \cdot w) + (\sigma_1(m) \cdot w) \cdot \sigma_0(x) - \sigma_1(m) \cdot (w \cdot \sigma_0(x))\\
= \;&0.
\end{aligned}$$

This verify the bimodule conditions (8) and (9). Similarly, one can verify the bimodule conditions (10)–(13).

Therefore, (W, V, φ) is a bimodule of (M, A, T). Finally, suppose that $\widehat{\mathrm{E}}$ and $\widetilde{\mathrm{E}}$ are equivalent abelian extensions, and $F : (\widehat{M}, \widehat{A}, \widehat{T}) \longrightarrow (\widetilde{M}, \widetilde{A}, \widetilde{T})$ be the morphism. Choose linear sections σ and σ' of p and q. Then we have $q_0 F_0 \sigma_0(x) = p_0 \sigma_0(x) = x = q_0 \sigma'_0(x)$, thus $F_0\sigma_0(x) - \sigma'_0(x) \in \mathrm{Ker} q_0 = V$. Therefore, we obtain

$$\sigma'_0(x) \cdot v = F_0\sigma_0(x) \cdot v = F_0(\sigma_0(x) \cdot v) = \sigma_0(x) \cdot v,$$

which implies that equivalent abelian extensions give the same module structures on V. Similarly, we can show that equivalent abelian extensions also give the same (W, V, φ). Therefore, equivalent abelian extensions also give the same representation. The proof is finished. □

Let $\sigma : (M, A, T) \to (\widehat{M}, \widehat{A}, \widehat{T})$ be a splitting of an abelian extension. Define the following linear maps:

$$\begin{cases} \theta: & M & \longrightarrow & V, & \theta(m) & \triangleq & \widehat{T}\sigma_1(m) - \sigma_0(T(m)), \\ \omega: & A\otimes A & \longrightarrow & V, & \omega(x,y) & \triangleq & \sigma_0(x)\cdot\sigma_0(y) - \sigma_0(x\cdot y), \\ \mu: & A\otimes M & \longrightarrow & W, & \mu(x,m) & \triangleq & \sigma_0(x)\cdot\sigma_1(m) - \sigma_1(x\cdot m), \\ \nu: & M\otimes A & \longrightarrow & W, & \nu(m,x) & \triangleq & \sigma_1(m)\cdot\sigma_0(x) - \sigma_1(m\cdot x), \end{cases} \tag{45}$$

for all $x, y, z \in A$, $m \in M$.

Theorem 5.3. *With the above notations, $(\theta, \omega, \mu, \nu)$ is a 2-cocycle of (M, A, T) with coefficients in (W, V, φ).*

Proof. Since $\widehat{T}$ is a a relative Rota-Baxter operater, we have the equality

$$\begin{aligned}
& \widehat{T}(\sigma_1(m))\cdot\widehat{T}(\sigma_1(n)) - \widehat{T}(\widehat{T}(\sigma_1(m))\cdot\sigma_1(n) + \sigma_1(m)\cdot\widehat{T}(\sigma_1(n))) \\
= \ & (\theta(m) + \sigma_0(T(m)))(\theta(n) + \sigma_0(T(n))) \\
& -\widehat{T}((\theta(m) + \sigma_0(T(m)))\cdot\sigma_1(n) + \sigma_1(m)\cdot(\theta(n) + \sigma_0(T(n)))) \\
= \ & \theta(m)\cdot\sigma_0(T(n)) + \sigma_0(T(m))\theta(n) + \sigma_0(T(m))\cdot\sigma_0(T(n)) \\
& -\widehat{T}(\theta(m)\cdot\sigma_1(n) + \sigma_0(T(m))\cdot\sigma_1(n) + \sigma_1(m)\cdot\theta(n) + \sigma_1(m)\cdot\sigma_0(T(n))) \\
= \ & \theta(m)\cdot\sigma_0(T(n)) + \sigma_0(T(m))\theta(n) + \sigma_0(T(m))\cdot\sigma_0(T(n)) \\
& -\widehat{T}(\theta(m)\cdot\sigma_1(n) + \mu(T(m),n) + \sigma_1(T(m)\cdot n) + \sigma_1(m)\cdot\theta(n) + \nu(m,T(n)) + \sigma_1(m\cdot T(n)) \\
= \ & \theta(m)\cdot T(n) + T(m)\cdot\theta(n) + \omega(T(m),T(n)) + \sigma_0(T(T(m)\cdot n + m\cdot T(n))) \\
& -\varphi(\theta(m)\triangleright n + \mu(T(m),n) + m\triangleleft\theta(n) + \nu(m,T(n))) \\
& -\theta(T(m)\cdot n) - \sigma_0(T(T(m)\cdot n)) - \theta(m\cdot T(n)) - \sigma_0(T(m\cdot T(n))) \\
= \ & \theta(m)\cdot T(n) + T(m)\cdot\theta(n) + \omega(T(m),T(n)) - \theta(T(m)\cdot n) - \theta(m\cdot T(n)) \\
& -\varphi(\theta(m)\triangleright n) - \varphi(\mu(T(m),n)) - \varphi(m\triangleleft\theta(n)) - \varphi(\nu(m,T(n))) \\
= \ & 0.
\end{aligned}$$

Thus we obtain the 2-cocycle condition (25).

Since $(\widehat{A}, \widehat{\alpha}_A)$ is an alternative algebra, we get

$$\begin{aligned}
& (\sigma_0(x)\cdot\sigma_0(y))\cdot\sigma_0(z) - \sigma_0(x)\cdot(\sigma_0(y)\cdot\sigma_0(z)) \\
& +(\sigma_0(y)\cdot\sigma_0(x))\cdot\sigma_0(z) - \sigma_0(y)\cdot(\sigma_0(x)\cdot\sigma_0(z)) \\
= \ & (\omega(x,y) + \sigma_0(x\cdot y))\cdot\sigma_0(z) - \sigma_0(x)\cdot(\omega(y,z) + \sigma_0(y\cdot z)) \\
& +(\omega(y,x) + \sigma_0(y\cdot x))\cdot\sigma_0(z) - \sigma_0(y)\cdot(\omega(x,z) + \sigma_0(x\cdot z)) \\
= \ & \omega(x,y)\cdot z + \sigma_0(x\cdot y)\cdot\sigma_0(z) - x\cdot\omega(y,z) - \sigma_0(x)\cdot\sigma_0(y\cdot z)) \\
& +\omega(y,x)\cdot z + \sigma_0(y\cdot x)\cdot\sigma_0(z) - y\cdot\omega(x,z) - \sigma_0(\cdot y)\cdot\sigma_0(x\cdot z)) \\
= \ & \omega(x,y)\cdot z + \omega(x\cdot y,z) + \sigma_0((x\cdot y)\cdot z) - x\cdot\omega(y,z) - \omega(x,y\cdot z) - \sigma_0(x\cdot(y\cdot z)) \\
& +\omega(y,x)\cdot z + \omega(y\cdot x,z) + \sigma_0((y\cdot x)\cdot z) - y\cdot\omega(x,z) - \omega(y,x\cdot z) - \sigma_0((y\cdot x)\cdot\cdot z) \\
= \ & \omega(x,y)\cdot z + \omega(x\cdot y,z) - x\cdot\omega(y,z) - \omega(x,y\cdot z) \\
& +\omega(y,x)\cdot z + \omega(y\cdot x,z) - y\cdot\omega(x,z) - \omega(y,x\cdot z) \\
= \ & 0.
\end{aligned}$$

Thus we obtain the 2-cocycle condition (26). Similarly, since $\widehat{M}$ is a bimodule of $\widehat{A}$, we have

$$\begin{aligned}
& (\sigma_0(x)\cdot\sigma_0(y))\cdot\sigma_1(m)-\sigma_0(x)\cdot(\sigma_0(y)\cdot\sigma_1(m)) \\
& +(\sigma_0(y)\cdot\sigma_0(x))\cdot\sigma_1(m)-\sigma_0(y)\cdot(\sigma_0(x)\cdot\sigma_1(m)) \\
= & (\omega(x,y)+\sigma_0(x\cdot y))\cdot\sigma_1(m)-\sigma_0(x)\cdot(\mu(y,m)+\sigma_1(y\cdot m)) \\
& +(\omega(y,x)+\sigma_0(y\cdot x))\cdot\sigma_1(m)-\sigma_0(y)\cdot(\mu(x,m)+\sigma_1(x\cdot m)) \\
= & \omega(x,y)\triangleright m+\sigma_0(x\cdot y)\cdot\sigma_1(m)-x\cdot\mu(y,m)-\sigma_0(x)\cdot\sigma_1(y\cdot m) \\
& +\omega(y,x)\triangleright m+\sigma_0(y\cdot x)\cdot\sigma_1(m)-y\cdot\mu(x,m)-\sigma_0(y)\sigma_1(x\cdot m) \\
= & \omega(x,y)\triangleright m+\mu(x\cdot y,m)+\sigma_1(x\cdot y\cdot m)-x\cdot\mu(y,m)-\mu(x,y\cdot m)-\sigma_1(x\cdot(y\cdot m)) \\
& +\omega(y,x)\triangleright m+\mu(y\cdot x,m)+\sigma_1(y\cdot x\cdot m)-y\cdot\mu(x,m)-\mu(y,x\cdot m)-\sigma_1(y\cdot(x\cdot m)) \\
= & \omega(x,y)\triangleright m+\mu(x\cdot y,m)-x\cdot\mu(y,m)-\mu(x,y\cdot m) \\
& +\omega(y,x)\triangleright m+\mu(y\cdot x,m)-y\cdot\mu(x,m)-\mu(y,x\cdot m) \\
= & 0.
\end{aligned}$$

Thus, we obtain the 2-cocycle condition (28). Similar computations show the 2-cocycle conditions (27) and (29)-(31). Therefore, by the above discussion, we obtain that (θ,ω,μ,ν) is a 2-cocycle of (M,A,T) with coefficients in (W,V,φ). This completes the proof. □

Now we define a relative Rota-Baxter alternative algebra on $(M\oplus W, A\oplus V, \widehat{T})$ using the 2-cocycle given above. More precisely, we have

$$\begin{cases}
\widehat{T}(m+w) \triangleq T(m)+\theta(w)+\varphi(w), \\
(x+v)(x'+v') \triangleq x\cdot x'+\omega(x,x')+x\cdot v'+v\cdot x', \\
(x+v)\cdot(m+w) \triangleq x\cdot m+\mu(x,m)+x\cdot w+v\triangleright m, \\
(m+w)\cdot(x+v) \triangleq m\cdot x+\nu(m,x)+w\cdot x+m\triangleleft v,
\end{cases} \tag{46}$$

for all $x,y,z\in A$, $m,n\in M$, $v\in V$ and $w\in W$. Thus any extension $\widehat{E}$ given by (43) is isomorphic to

$$\begin{array}{ccccccccc}
0 & \xrightarrow{0} & W & \xrightarrow{i_1} & M\oplus W & \xrightarrow{p_1} & M & \xrightarrow{0} & 0 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\widehat{T}}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle 0}\downarrow \\
0 & \xrightarrow{0} & V & \xrightarrow{i_0} & A\oplus V & \xrightarrow{p_0} & A & \xrightarrow{0} & 0.
\end{array} \tag{47}$$

Theorem 5.4. *There is a one-to-one correspondence between the equivalent classes of abelian extensions and the elements in the second cohomology group* $\mathbf{H}^2((M,A,T),(W,V,\varphi))$.

Proof. We have known from the above discussion that abelian extensions of relative Rota-Baxter alternative algebras are correspond to 2-cocylces and vice verse. Let E′ be another abelian extension determined by the 2-cocycle $(\theta',\omega',\mu',\nu')$. We are going to show that E and E′ are equivalent if and only if 2-cocycles (θ,ω,μ,ν) and $(\theta',\omega',\mu',\nu')$ are in the same cohomology class.

Since F is an equivalence of abelian extensions, there exist two linear maps $N_0 : A\longrightarrow V$ and $N_1 : M\longrightarrow W$ such that

$$F_0(x+v)=x+N_0(x)+v,\quad F_1(m+w)=m+N_1(m)+w. \tag{48}$$

By the equality

$$\widehat{T}'F_1(m) = F_0\widehat{T}(m),$$

we get

$$\theta(m)-\theta'(m)=\varphi N_1(m)-N_0T(m). \tag{49}$$

Furthermore, by the equality

$$F_0(x \cdot y + \omega(x, y)) = F_0(x) \cdot F_0(y),$$

we get

$$\omega(x, y) - \omega'(x, y) = x \cdot N_0(y) + N_0(x) \cdot y - N_0(xy). \tag{50}$$

Similarly, by the equality

$$F_1(x \cdot m + \mu(x, m)) = F_0(x) \cdot F_1(m),$$

we obtain that

$$\mu(x, m) - \nu'(x, m) = N_0(x) \triangleright m + x \cdot N_1(m) - N_1(x \cdot m). \tag{51}$$

Similarly, we get

$$\nu(x, m) - \nu'(x, m) = N_1(m) \cdot x + m \triangleleft N_0(x) - N_1(m \cdot x). \tag{52}$$

By (49)–(52), we deduce that $(\psi, \omega, \mu, \nu) - (\psi', \omega', \mu', \nu') = D(N_0, N_1)$. Thus, they are in the same cohomology class.

Conversely, if $(\theta, \omega, \mu, \nu)$ and $(\theta', \omega', \mu', \nu')$ are in the same cohomology class, assume that $(\theta, \omega, \mu, \nu) - (\theta', \omega', \mu', \nu') = D(N_0, N_1)$. Then we define (F_0, F_1) by (48). Similar as the above proof, we can show that (F_0, F_1) is an equivalence. We omit the details. □

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