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On one identity between characteristic determinant and norming constants and its application to traces, II

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Abstract. Our aim is to prove one relation between characteristic determinant and norms of orthogonal eigenvectors of eigenvalue problem for fourth order differential operator equation.That relation as appears plays a crucial role in deriving the regularized trace formulas.

1. Introduction

Consider the next eigenvalue problem:

$$l_0[y] \equiv y^{IV}(t) + Ay(t) = \lambda y(t) \tag{1}$$

$$y(0) = y''(0) = 0, \tag{2}$$

$$-y'''(1) = \lambda Q_1 y(1), \tag{3}$$

$$y''(1) = \lambda Q_2 y'(1) \tag{4}$$

in $L_2(H,(0,1))$ with an abstract separable Hilbert space H (see [7]). Coefficients A, Q_1, Q_2 of the problem are unbounded operators in H. They are assumed to be self-adjoint, positive-definite, moreover, $A^{-1} \in \sigma_\infty$. Denote the eigenvalues and eigenvectors of A by $\gamma_1 \le \gamma_2 \le \dots$ and $\phi_1, \phi_2, \dots$, respectively. For fourth order differential operator equations with λ in only one boundary condition and without unbounded operators Q_1, Q_2 we refer to our works [5, 9, 10, 14]. For higher order differential operator equation and without spectral parameter in boundary condition refer to [9, 13]. Sturm-Liouville operator equation cases were treated for example, in [13, 14] with sum or able or nonsummable coefficients refer to [1–6, 8, 11, 12].

In $\mathcal{H} = L_2(H,(0,1)) \oplus H^2$, define operator L_0:

$$D(L_0) = \{Y = \{y(t), y_1, y_2) \in \mathcal{H},\ y(t) \in W_4(0,1),$$

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$$y_1 = Q_1 y(1),\ y_2 = Q_2 y'(1),\ y(1) \in D(Q_1),\ y'(1) \in D(Q_2),$$
$$y'''(1),\ y''(1) \in H,\ y(0) = y''(0) = 0\} \tag{5}$$
$$L_0 Y = \{l_0[y], -y'''(1), y'(1)\} \tag{6}$$

Recall here that $W_4(0,b)$ is a closure of $C^4(H_1,[0,b])$ in the norm:

$$\|y(t)\|^2_{W_4(0,b)} = \|y(t)\|^2_{L_2(H_1,(0,b))} + \|y^{IV}(t)\|_{L_2(H,(0,b))},$$

where H_j ($j > 0$) is a closure of $D(A)$ with respect to norm produced by the scalar product

$$(u,u)_{H_j} = (A^j u, A^j u)_H.$$

Setting:

$$l_1[y] \equiv y^{IV}(t) + Ay(t) + q(t)y(t),$$

where $q^*(t) = q(t)$ is bounded for each $t \in [0,1]$, define operator L_1 with the same domain as in (5) and

$$L_1 Y = \{l_1[y], -y'''(1), y'(1)\}$$

It is known that spectrum of L_0 is discrete. Because of $cosC, Q_1 A^{-\frac{3}{4}}$ and $Q_2 A^{-\frac{1}{2}} \in \sigma_\infty$, and $q(t)$ is bounded spectrum of L_1 is also discrete.

Solutions of problems(1), (2) are:

$$y(t) = sh\sqrt[4]{\lambda I - A}\, t\, F_1 + \sin\sqrt[4]{\lambda I - A}\, t\, F_2, \tag{7}$$

where I is an identity operator in H and $F_1, F_2 \in H_{3/4}$.

For computing the asymptotics of eigenvalues of L_0, take $Q_1 = Q_2 = A^\alpha$, restricting α to the interval $(0, 1/2)$ and let to hold the conditions $Q_1 A^{-3/4}, Q_2 A^{-1/2} \in \sigma_\infty$.

Putting in (3), (4) vector-function $y(t)$ defined by (7) and using the spectral expansion of operator A:

$$A = \sum_{k=1}^{\infty} \gamma_k(\cdot, \phi_k)\phi_k$$

also denoting

$$\sqrt[4]{\lambda - \gamma_k} = z,\ (F_1, \phi_k) = c_{1k},\quad (F_2, \phi_k) = c_{2k}$$

one has

$$-z^3\, chz\ c_{1k} + z^3 cosz\ c_{2k} = (z^4 + \gamma_k) shz \gamma_k^\alpha\ c_{1k} + (z^4 + \gamma_k) sinz \gamma_k^\alpha\ c_{2k} \tag{8}$$
$$zshz\ \ c_{1k} - zsinz\ c_{2k} = (z^4 + \gamma_k) chz \gamma_k^\alpha\ c_{1k} + (z^4 + \gamma_k) cosz \gamma_k^\alpha\ c_{2k} \tag{9}$$

Equations (8) and (9) form a system of linear algebraic equations in c_{1k}, c_{2k}, which has non-zero roots if and only if the determinant of coefficients (so called the characteristic determinant) is zero:

$$\Delta(z) = \begin{vmatrix} -z^3 chz - \left(z^4 + \gamma_k\right) shz\ \gamma_k^\alpha & z^3 \cos z - \left(z^4 + \gamma_k\right) \sin z\ \gamma_k^\alpha \\ zshz - \left(z^4 + \gamma_k\right) chz\ \gamma_k^\alpha & -zsinz - \left(z^4 + \gamma_k\right) \cos z\ \gamma_k^\alpha \end{vmatrix} = 0 \tag{10}$$

or

$$tgz = \frac{-2z^3\left(z^4 + \gamma_k\right)\ \gamma_k^\alpha + z^4 thz - \left(z^4 + \gamma_k\right)^2 \gamma_k^{2\alpha} thz}{z^4 + 2z\left(z^4 + \gamma_k\right)\ \gamma_k^\alpha thz - \left(z^4 + \gamma_k\right)^2 \gamma_k^{2\alpha}} \tag{11}$$

2. Orthonormal eigenvectors of the operator L_0

Reminding that by $\{\varphi_\kappa\}$ we denote orthonormal eigenvectors of the operator A, then orthogonal eigenvectors of L_0 will have the next form

$$Y_{k,j} = \Big\{c_{1k,j} sh \sqrt[4]{\lambda_{k,j} - \gamma_k} t\varphi_k + c_{2k,j} sin \sqrt[4]{\lambda_{k,j} - \gamma_k} t\varphi_k, c_{1k,j}\gamma_k{}^{\alpha} sh \sqrt[4]{\lambda_{k,j} - \gamma_k}\varphi_k +$$
$$+ c_{2k,j}\,\gamma_k{}^{\alpha} sin \sqrt[4]{\lambda_{k,j} - \gamma_k}\varphi_k\ , c_{1k,j} \sqrt[4]{\lambda_{k,j} - \gamma_k}\gamma_k{}^{\alpha} ch \sqrt[4]{\lambda_{k,j} - \gamma_k}\varphi_k +$$
$$+ c_{2k,j}\ \sqrt[4]{\lambda_{k,j} - \gamma_k}\gamma_k^{\alpha} cos \sqrt[4]{\lambda_{k,j} - \gamma_k}\varphi_k\Big\},\ k = \overline{1,\infty},\ j = \overline{1,\infty}. \tag{12}$$

Denote

$$\sqrt[4]{\lambda_{k,j} - \gamma_k} = z_{k,j}.$$

Coefficients $c_{1k,j}$ and $c_{2k,j}$ are the values of c_{1k} and c_{2k} in relations (8), (9) obtained by taking there $\lambda = \lambda_{k,j}$. From (9)

$$c_{1k} = \frac{zsinz + \left(z^4 + \gamma_k\right) cosz\ \gamma_k^{\alpha}}{zshz - \left(z^4 + \gamma_k\right) chz\ \gamma_k^{\alpha}} c_{2k} \tag{13}$$

We denote the multiplier at c_{2k} by $H(z)$:

$$c_{1k} = H(z)c_{2k} \tag{14}$$

Then $c_{1k,j}$ denotes the value of c_{1k} at $z = z_{k,j}$:

$$c_{1k,j} = H(z_{k,j})c_{2k,j} \tag{15}$$

Assuming for shortcut of notations $c_{2k,j} = c_{k,j}$ with all above in mind we have the following expressions for $Y_{k,j}$:

$$Y_{k,j} = c_{k,j}\Big\{\sin z_{k,j} t\varphi_k\ + H(z_{k,j}) shz_{k,j} t\varphi_k, \gamma_k^{\alpha} sinz_{k,j}\varphi_k +$$
$$+ H(z_{k,j})\gamma_k^{\alpha} shz_{k,jk}\varphi_k,\ z_{k,j}\gamma_k^{\alpha}\cos z_{k,j}\varphi_k + H(z_{k,j}) z_{k,j}\gamma_k^{\alpha} chz_{k,j}\varphi_k\Big\} =$$
$$= c_{k,j}\varphi_k\Big\{\sin z_{k,j} t + H\left(z_{k,j}\right) shz_{k,j} t, \gamma_k^{\alpha}\sin z_{k,j} + H(z_{k,j})\gamma_k^{\alpha} shz_{k,j}, z_{k,j}\gamma_k^{\alpha}\cos z_{k,j} +$$
$$+ H(z_{k,j}) z_{k,j}\gamma_k^{\alpha} chz_{k,j}\Big\},\ k = \overline{1,\infty},\ j = \overline{1,\infty} \tag{16}$$

Introduce the direct sum space $\Lambda = L_2\left((0,1)\right) \oplus C^2$, ($C$ is a complex space) with a scalar product of the elements $u = (u\,(t)\,, u_1, u_2)\,, v = (v\,(t)\,, v_1, v_2)$ defined as

$$(u, v)_\Lambda = \int_0^1 u\bar{v}dt + \gamma_k^{-\alpha} u_1\bar{v}_1 + \gamma_k^{-\alpha} u_2\bar{v}_2$$

Thus,

$$Y_{k,j} = c_{k,j}\Psi_{k,j}\varphi_k \tag{17}$$

where by $\Psi_{k,j} \in \Lambda$ we denote the vector in parenthesis in the right side of (16).

Set

$$\Phi_{k,j} = \Psi_{k,j}\varphi_k, \tag{18}$$

obviously $\Phi_{k,j} \in \mathcal{H}$. $\Upsilon_{k,j}$ will form an orthonormal system of eigenvectors by putting in place of $c_{k,j}$ the norming constants:

$$c_{k,j}{}^2 = \frac{1}{\left\|\Phi_{k,j}\right\|^2_{\mathcal{H}}} = \frac{1}{\left\|\Psi_{k,j}\right\|^2_{\Lambda}}. \tag{19}$$

It was used here that, $\left(\Phi_{k,j}, \Phi_{k,j}\right)_{\mathcal{H}} = \left(\Psi_{k,j}, \Psi_{k,j}\right)_{\Lambda} \cdot (\varphi_k, \varphi_k) = \left(\Psi_{k,j}, \Psi_{k,j}\right)_{\Lambda}$ since $(\varphi_k, \varphi_k) = 1$.

Thus,

$$\begin{aligned}
\left\|\Phi_{k,j}\right\|_1^2 = &\int_0^1 \sin^2 z_{k,j} t dt + H(z_{k,j})^2 \int_0^1 sh^2 z_{k,j} t dt + \\
&+2H(z_{k,j}) \int_0^1 sh z_{k,j} t sin z_{k,j} t dt + H(z_{k,j})^2 \gamma_k{}^2 2\alpha sh^2 z_{k,j} + \\
&+2H(z_{k,j}) \gamma_k^{2\alpha} sh z_{k,j} sin z_{k,j} + \gamma_k{}^2 2\alpha \sin^2 z_{k,j} + H(z_{k,j}) \gamma_k^{2\alpha} z_{k,j}{}^2 ch^2 z_{k,j} + \\
&+2 z_{k,j}{}^2 \gamma_k^{2\alpha} H(z_{k,j}) ch z_{k,j} cos z_{k,j} + \gamma_k{}^2 2\alpha z_{k,j}{}^2 cos^2 z_{k,j}.
\end{aligned} \tag{20}$$

Note that there is no need to compute the integral terms in (20) because the relationship established below allows us to use norms of eigenvectors in general form, without specifying.

3. On relation between characteristic determinant and norms of eigenvectors

Recalling that $\gamma_1 \leq \gamma_2 \leq \ldots$ are eigenvalues and $\{\varphi_k\}$, $k = \overline{1, \infty}$ are orthonormal eigenvectors of the operator A, in virtue of basicity of $\{\varphi_k\}$ in H any $y(t)$ from $L_2(H, (0,1))$ is expanded as

$$y(t) = \sum_{k=1}^{\infty} (y(t)\varphi_k, \varphi_k)\varphi_k$$

Denoting $(y(t)\varphi_k, \varphi_k) = y_k(t)$, in (1)-(4) we have the following spectral problem for the scalar functions $y_k(t)$:

$$l_k y_k(t) \equiv y_k^{IV}(t) + \gamma_k y_k(t) = \lambda y_k(t) \tag{21}$$

$$y_k(0) = y_k^{''}(0) = 0 \tag{22}$$

$$-y_k{}'''(1) = \lambda \gamma_k^{\alpha} y_k(1) \tag{23}$$

$$y_k^{''}(1) = \lambda \gamma_k^{\alpha} y_k^{'}(1) \tag{24}$$

For each fixed $k\,(k = 1, \infty)$ we denote the eigenvalues of that problem by $\lambda_{k,j}$, and the solutions of (21),(22) by $y_k(t, \lambda - \gamma_k)$. Obviously,

$$y_k(t, \lambda - \gamma_k) = c_{1k} sh \sqrt[4]{\lambda - \gamma_k}\, t + c_{2k} \sin \sqrt[4]{\lambda - \gamma_k}\, t$$

$\lambda_{k,j}, k = 1, \infty,\ j = 1, \infty$ are the eigenvalues of problem (1)-(4).

It is easy to see that the vectors

$$\left(y_{k,j}\left(t, \lambda_{k,j} - \gamma_k\right), \gamma_k^{\alpha} y_{k,j}\left(1, \lambda_{k,j} - \gamma_k\right), \gamma_k^{\alpha} y'_{k,j}\left(1, \lambda_{k,j} - \gamma_k\right)\right)$$

form a set of eigenvectors of the operator L_{0k} associated with the scalar problem (21)-(24) in space Λ and acting as

$$L_{0k}\left(y_k(t), \gamma_k^{\alpha} y_k(1), \gamma_k^{\alpha} y_k'(1)\right) = \left(l_{0k} y_k(t), -y_k'''(1), y_k''(1)\right)$$

Obviously, eigenvectors of L_{0k} are orthogonal due to self-adjointness of L_{0k} and coincide with $\Psi_{k,j}$:

$$\left(y_{k,j}\left(t, \lambda_{k,j} - \gamma_k\right), \gamma_k^{\alpha} y_{k,j}\left(1, \lambda_{k,j} - \gamma_k\right),\ \gamma_k^{\alpha} y_{k,j}'(1, \lambda_{k,j} - \gamma_k)\right) = \Psi_{k,j}.$$

$\Upsilon_{k,j} = c_{k,j}\Psi_{k,j}$ and $Y_{k,j} = c_{k,j}\Phi_{k,j}$ are orthonormal eigenvectors of problems (21)-(24) and (1)-(4), respectively.

Recall here the notations $\sqrt[4]{\lambda - \gamma_k} = z$, $\sqrt[4]{\lambda_{k,j} - \gamma_k} = z_{k,j}$ from previous sections.

Introduce the notations

$$\omega_1(z) \equiv y_k'''\left(1, \lambda - \gamma_k\right) + \lambda \gamma_k^{\alpha} y_k\left(1, \lambda - \gamma_k\right) =$$

$$= y_k'''\left(1, z^4\right) + \left(z^4 + \gamma_k\right)\gamma_k^{\alpha} y_k\left(1, z^4\right) \tag{25}$$

$$\omega_2(z) \equiv y_k''\left(1, \lambda - \gamma_k\right) - \lambda \gamma_k^{\alpha} y_k'\left(1, \lambda - \gamma_k\right) =$$

$$= y_k''\left(1, z^4\right) - \left(z^4 + \gamma_k\right)\gamma_k^{\alpha} y_k'\left(1, z^4\right) \tag{26}$$

The eigenvalues of (21)-(24) are defined from the system

$$\omega_1(z) = 0 \tag{27}$$

$$\omega_2(z) = 0 \tag{28}$$

Introduce the following function:

$$f_k(z) \equiv y_k(1, z^4)\overline{y_k(1, z^4)}\left[\frac{\omega_1(z)}{y_k(1, z^4)}\right] - y_k'(1, z^4)\overline{y_k'(1, z^4)}\left[\frac{\omega_2(z)}{y_k'(1, z^4)}\right]. \tag{29}$$

Define c_{1k} in terms of c_{2k} from (28). Now with that c_{1k} in (27) $\omega_2(z)$ becomes a characteristic determinant of L_{0k}. Then $f_k(z)$ will take the form

$$f_k(z) \equiv \overline{y_k(1, z^4)}\omega_1(z).$$

Prove the next theorem which has an important role in deriving the trace formula.

Theorem 3.1.

$$f_k'\left(z_{k,j}\right) = 4z_{k,j}^3\left\|\Psi_{k,j}\right\|_{\Lambda}^2 = 4z_{k,j}^3\frac{1}{c_{k,j}^2}$$

where $c_{k,j}^2$ are norming constants.

Proof. Let $y_k\left(t, \lambda - \gamma_k\right)$ and $y_k\left(t, \lambda_{k,j} - \gamma_k\right)$ be the solutions of equation (21) with λ and $\lambda_{k,j}$, respectively:

$$y_k^{IV}\left(t, \lambda - \gamma_k\right) + \gamma_k y_k\left(t, \lambda - \gamma_k\right) = \lambda y_k\left(t, \lambda - \gamma_k\right) \tag{30}$$

$$\overline{y_k^{IV}\left(t, \lambda_{k,j} - \gamma_k\right)} + \gamma_k \overline{y_k\left(t, \lambda_{k,j} - \gamma_k\right)} = \lambda_{k,j}\overline{y_k\left(t, \lambda_{k,j} - \gamma_k\right)} \tag{31}$$

Multiply (30) by $y_k\left(t,\lambda_{k,j}-\gamma_k\right)$, (31) by $y_k\left(t,\lambda-\gamma_k\right)$, then subtract the second one from the first, integrate both sides of the obtained relation along $(0,1)$, and to the obtained results add the term

$$\left(\lambda-\gamma_k-(\lambda_{k,j}-\gamma_k)\right)\gamma_k{}^{\alpha}y_k\left(1,\lambda-\gamma_k\right)\overline{y_k\left(1,\lambda_{k,j}-\gamma_k\right)}+$$
$$+\left(\lambda-\gamma_k-(\lambda_{k,j}-\gamma_k)\right)y_k'\left(1,\lambda-\gamma_k\right)\overline{y_k'\left(1,\lambda_{k,j}-\gamma_k\right)}\gamma_k{}^{\alpha}. \tag{32}$$

Note that addition of the last term is needed for finding the norm of the eigenvector of the operator L_{0k} in the direct sum space $\Lambda_1 = L_2\left((0,1)\right)\oplus C^2$ as it will become clear in the next derivations.

Thus,

$$\int_0^1 y_k{}^{IV}\left(t,z^4\right)\overline{y_k\left(t,z_{k,j}^4\right)}dt-\int_0^1\overline{y_k^{IV}\left(t,z_{k,j}^4\right)}y_k\left(t,z^4\right)dt+$$
$$+\left(z^4-z_{k,j}^4\right)\gamma_k^{\alpha}y_k\left(1,z^4\right)\overline{y_k\left(1,z_{k,j}^4\right)}+\left(z^4-z_{k,j}^4\right)y_k'\left(1,z^4\right)\overline{y_k'\left(1,z_{k,j}^4\right)}\gamma_k^{\alpha}=$$
$$=\left(z^4-z_{k,j}^4\right)\int_0^1 y_k\left(t,z^4\right)\overline{y_k\left(t,z_{k_j}^4\right)}dt+\left(z^4-z_{k,j}^4\right)\gamma_k^{\alpha}y_k\left(1,z^4\right)\overline{y_k\left(1,z_{k,j}^4\right)}+$$
$$+\left(z^4-z_{k,j}^4\right)y_k'\left(1,z^4\right)\overline{y_k'\left(1,z_{k,j}^4\right)}\gamma_k^{\alpha} \tag{33}$$

Denoting the expression on the left of (33) by N and integrating by parts there yields

$$N\equiv y_k'''\left(1,z^4\right)\overline{y_k(1,z_{k,j}^4)}-$$
$$\overline{y_k'''\left(1,z_{k,j}^4\right)}y_k\left(1,z^4\right)-y_k''\left(1,z^4\right)\overline{y_k'(1,z_{k,j}^4)}+\overline{y_k''\left(1,z_{k,j}^4\right)}y_k'\left(1,z^4\right)+$$
$$+\left(z^4-z_{k,j}^4\right)\gamma_k{}^{\alpha}y_k\left(1,z^4\right)\overline{y_k\left(1,z_{k,j}^4\right)}+\left(z^4-z_{k,j}^4\right)y_k'\left(1,z^4\right)\overline{y_k'\left(1,z_{k,j}^4\right)}\gamma_k{}^{\alpha}=$$
$$=\overline{y_k\left(1,z_{k,j}^4\right)}y_k\left(1,z^4\right)\left[\frac{y_k'''\left(1,z^4\right)}{y_k\left(1,z^4\right)}-\frac{\overline{y_k'''\left(1,z_{k,j}^4\right)}}{\overline{y_k\left(1,z_{k,j}^4\right)}}-\left(z^4-z_{k,j}^4\right)\gamma_k^{\alpha}\right]$$
$$-y_k'\left(1,z^4\right)\overline{y_k'\left(1,z_{k,j}^4\right)}\left[\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)}-\frac{\overline{y_k''\left(1,z_{k,j}^4\right)}}{\overline{y_k'\left(1,z_{k,j}^4\right)}}-\left(z^4-z_{k,j}^4\right)\gamma_k^{\alpha}\right] \tag{34}$$

Because of relations(25)-(28) for terms in brackets in right side of (34) we have

$$\frac{y_k'''\left(1,z_{k,j}^4\right)}{y_k\left(1,z_{k,j}^4\right)}=-\lambda_{k,j},$$

$$\frac{y_k''\left(1,z_{k,j}^4\right)}{y_k'\left(1,z_{k,j}^4\right)}=\lambda_{k,j}.$$

But $\gamma_{k,j} = z_{k,j}^4 + \gamma_k$, ($\gamma_k$ are real as eigenvalues of self-adjoint operator and $z_{k,j}$ are roots of (27), (28) and $z_{k,j}^4$ as we know might be only real).Hence

$$\frac{\overline{y_k'''\left(1, z_{k,j}^4\right)}}{\overline{y_k\left(1, z_{k,j}^4\right)}} = \frac{y_k'''\left(1, z_{k,j}^4\right)}{y_k\left(1, z_{k,j}^4\right)}$$

$$\frac{\overline{y_k''\left(1, z_{k,j}^4\right)}}{\overline{y_k'\left(1, z_{k,j}^4\right)}} = \frac{y_k''\left(1, z_{k,j}^4\right)}{y_k'\left(1, z_{k,j}^4\right)}$$

With that in mind substituting the expression for N from (34) into (33), dividing both sides of (33), by $z - z_{k,j}$ and passing to the limit as $z \to z_{k,j}$, also recalling definition of dot product in Λ we get

$$4z_{k,j}^3\left[\int_0^1 \left|y_k(t, z_{k,j}^4)\right|^2 dt + \gamma_k^\alpha \left|y_k(1, z_{k,j}^4)\right|^2 + \gamma_k^\alpha \left|y_k'\left(1, z_{k,j}^4\right)\right|^2\right] =$$

$$= \lim_{z\to z_{k,j}}\left(\frac{\overline{y_k\left(1, z_{k,j}^4\right)}y_k\left(1, z^4\right)\left[\frac{y_k'''\left(1,z^4\right)}{y_k\left(1,z^4\right)} - \frac{y_k'''\left(1,z_{k,j}^4\right)}{y_k\left(1,z_{k,j}^4\right)} + (z^4 - z_{k,j}^4)\gamma_k^\alpha\right]}{z - z_{k,j}} - \right.$$

$$\left. - \frac{\overline{y_k\left(1, z_{k,j}^4\right)}y_k'\left(1, z^4\right)\left[\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - \frac{y_k''\left(1,z_{k,j}^4\right)}{y_k'\left(1,z_{k,j}^4\right)} - (z^4 - z_{k,j}^4)\gamma_k^\alpha\right]}{z - z_{k,j}}\right) =$$

$$= \left|y_k(1, z_{k,j}^4)\right|^2 \left[\frac{y_k'''\left(1, z^4\right)}{y_k\left(1, z^4\right)}\right]'\Bigg|_{z=z_{k,j}} + 4z_{k,j}^3\gamma_k^\alpha \left|y_k(1, z_{k,j}^4)\right|^2 -$$

$$\left|y_k'(1, z_{k,j}^4)\right|^2 \left[\frac{y_k''\left(1, z^4\right)}{y_k'\left(1, z^4\right)}\right]'\Bigg|_{z=z_{k,j}} - 4z_{k,j}^3{\gamma_k}^\alpha \left|y_k'(1, z_{k,j}^4)\right|^2 =$$

$$= \left|y_k(1, z_{k,j}^4)\right|^2 \left[\frac{y_k'''\left(1, z^4\right)}{y_k\left(1, z^4\right)} + z^4\gamma_k^\alpha\right]'\Bigg|_{z=z_{k,j}} - \left|y_k'(1, z_{k,j}^4)\right|^2 \left[\frac{y_k''\left(1, z^4\right)}{y_k'\left(1, z^4\right)} - z^4\gamma_k^\alpha\right]'\Bigg|_{z=z_{k,j}} =$$

$$= \left|y_k(1, z_{k,j}^4)\right|^2 \left[\frac{\omega_1(z)}{y_k(1, z^4)}\right]'\Bigg|_{z=z_{k,j}} - \left|y_k'(1, z_{k,j}^4)\right|^2 \left[\frac{\omega_2(z)}{y_k'(1, z^4)}\right]'\Bigg|_{z=z_{k,j}} \tag{35}$$

(derivatives of expressions within square bracket in the last relation are taken with respect to z)

Note that $\int\limits_0^1 \left|y_k(t,z_{k,j}^4)\right|^2 dt+\gamma_k^\alpha \left|y_k(1,z_{k,j}^4)\right|^2+\gamma_k^\alpha \left|y_k^{'}(1,z_{k,j}^4)\right|^2$ standing on the left side of (35) is square of the norm of eigenvectors of the operator associated with problem (21)-(24) in Λ:

$$\int_0^1 \left|y_k(t,z_{k,j}^4)\right|^2 dt+\gamma_k^\alpha \left|y_k(t,z_{k,j}^4)\right|^2+\gamma_k^\alpha \left|y_k^{'}(1,z_{k,j}^4)\right|^2=\left\|\Psi_{k,j}\right\|_\Lambda^2 \tag{36}$$

Using (36) on the left side of (35) and notations (25), (26), we arrive at

$$4z_{k,j}^3\left\|\Psi_{k,j}\right\|_\Lambda^2=\left|y_k(1,z_{k,j}^4)\right|^2\left[\frac{\omega_1(z)}{y_k(1,z^4)}\right]'\Bigg|_{z=z_{k,j}}-\left|y_k'(1,z_{k,j}^4)\right|^2\left[\frac{\omega_2(z)}{y_k'(1,z^4)}\right]'\Bigg|_{z=z_{k,j}} \tag{37}$$

As it was stated, the solution of problem (21) satisfying (22) is

$$y_k\left(t,\lambda-\gamma_k\right)=c_{1k}sh\sqrt[4]{\lambda-\gamma_k}t+c_{2k}sin\sqrt[4]{\lambda-\gamma_k}t$$

or

$$y_k\left(t,z^4\right)=c_{1k}shzt+c_{2k}sinzt \tag{38}$$

For this function to be the first component of the eigenvector $\Psi_{k,j}$ of problem (21)-(24), it must satisfy also (23) and (24) or equivalently (27),(28). Substituting it into (27), (28), we get again (8), (9).

Writing $y_k\left(1,z^4\right)$ from (38) with c_{1k} defined from (13) into (24) yields (11) from which $z_{k,j}$ are found giving $\lambda_{k,j}=z_{k,j}^4+\gamma_k$

Evaluate the derivative of $f_k(z)$ at $z_{k,j}$

$$f_k'\left(z_{k,j}\right)=\left(\left|y_k(1,z_{k,j}^4)\right|^2\right)'\Bigg|_{z=z_{k,j}}\left[\frac{\omega_1\left(z_{k,j}\right)}{y_k\left(1,z_{k,j}^4\right)}\right]+\left|y_k(1,z_{k,j}^4)\right|^2\left[\frac{\omega_1(z)}{y_k\left(1,z^4\right)}\right]'\Bigg|_{z=z_{k,j}}- \tag{39}$$

$$-\left(\left|y_k'(1,z_{k,j}^4)\right|^2\right)'\Bigg|_{z=z_{k,j}}\left[\frac{\omega_2\left(z_{k,j}\right)}{y_k'\left(1,z_{k,j}^4\right)}\right]-\left|y_k'(1,z_{k,j}^4)\right|^2\left[\frac{\omega_2(z)}{y_k'\left(1,z^4\right)}\right]'\Bigg|_{z=z_{k,j}}$$

which in virtue of $\omega_1\left(z_{k,j}\right)=0,\ \omega_2\left(z_{k,j}\right)=0$ yields

$$f_k'\left(z_{k,j}\right)=\left|y_k(1,z_{k,j}^4)\right|^2\left[\frac{\omega_1(z)}{y_k(1,z^4)}\right]'\Bigg|_{z=z_{k,j}}-\left|y_k'(1,z_{k,j}^4)\right|^2\left[\frac{\omega_2(z)}{y_k'(1,z^4)}\right]'\Bigg|_{z=z_{k,j}} \tag{40}$$

From (40) with (36) in mind we get

$$f_k^{'}\left(z_{k,j}\right)=4z_{k,j}^3\left\|\Psi_{k,j}\right\|_\Lambda^2=4z_{k,j}^3\frac{1}{c_{k,j}^2} \tag{41}$$

where

$$c_{k,j}^2=\frac{1}{\left\|\Psi_{k,j}\right\|_\Lambda^2}=\frac{1}{\left\|\Phi_{k,j}\right\|_{\mathcal{H}}^2}$$

which completes the proof.

Thus, (41) relates the characteristic determinant $f_k(z)$ and norms of orthogonal eigenvectors. The function $f_k(z)$ has essential role in our derivation of trace formula described in the next section.

If c_{1k} is defined from (28), then $f_k(z)$ from (29) is simplified to the form

$$f_k(z) \equiv \overline{y_k(1, z^4)}\omega_1(z) \tag{42}$$

$\Upsilon_{k,j} = c_{k,j}\Psi_{k,j}$ are orthonormal eigenvectors of the operator L_{0k} associated with problem (21)-(24) in the space Λ, and

$$Y_{k,j} = c_{k,j}\Phi_{k,j} = c_{k,j}\Psi_{k,j}\varphi_k$$

are orthonormal eigenvectors of the operator L_0 associated with problem (1)-(4) in $\mathcal{H}$. □

4. Evaluation of regularized trace

Before passing to derivations, put on $q(t)$ the following condition:

$$\sum_{k=1}^{\infty} (q(t)\varphi_k, \varphi_k)_H < \infty. \tag{43}$$

Define operator Q in $\mathcal{H}$ as $Q(Y) = \{q(t)y(t), 0, 0\}, (Y = \{y(t), y_1, y_2\} \in \mathcal{H})$
Since Q is bounded in $\mathcal{H}$ and L_0 is operator with purely discrete spectrum, then $L_1 = L_0 + Q$ is also discrete. The eigenvalues of L_0 and L_1 are denoted by $\lambda_1 \leq \lambda_2 \leq \dots$ and $\mu_1 \leq \mu_2 \dots$, respectively. We found that roots $z_{k,j}$ of equation (11) and eigenvalues λ_n of L_0 have the next asymptotics:

$$z_{k,j} \sim \begin{cases} \pi j + \frac{\pi}{4} + O\left(\frac{1}{k}\right) \\ i\left(\pi j + \frac{\pi}{4} + O\left(\frac{1}{k}\right)\right) \end{cases}$$

$$\lambda_n \sim Cn^{\frac{4\beta}{4+\beta}}, n \to \infty,$$

where β by our assumption is order of eigenvalues γ_j of operator $A : \gamma_k \sim dj^{\beta}, d > 0, \beta > 0$.
In virtue of Theorem 3.1 from section 3 and Theorem 1 from [15] and above asymptotics of eigenvalues of L_0 holds relation

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} \left(\mu_n - \lambda_n - \left(QY_{k_n j_n}, Y_{k_n j_n}\right)_{\mathcal{H}}\right) = 0 \tag{44}$$

for some subsequence of natural numbers $\{n_m\}$.

In virtue of the asymptotic formula for $z_{k,j}$ and formula (16) for eigenvectors, the next lemma is valid (proof is similar to one , from our work [14]).

Lemma 4.1. *The series*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(QY_{k,j}, Y_{k,j}\right)_{\mathcal{H}}$$

is absolutely convergent.

From (44) and Lemma 4.1

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} (\mu_n - \lambda_n) = \lim_{m\to\infty} \sum_{n=1}^{n_m} \left(QY_{k_n j_n}, Y_{k_n j_n}\right)_{\mathcal{H}} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(QY_{k,j}, Y_{k,j}\right)_{\mathcal{H}} \tag{45}$$

Denote by $\sum\limits_{n=1}^{\infty'}(\mu_n-\lambda_n)$ the limit on the left side of (46) and call it a regularized trace of L_1.

$$\sum_{n=1}^{\infty'}(\mu_n-\lambda_n)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(QY_{k,j},Y_{k,j}\right)_{\mathcal{H}} \tag{46}$$

From Lemma 4.1 and from (, with (16) in mind, we get:

$$\sum_{n=1}^{\infty'}(\mu_n-\lambda_n)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(QY_{k,j},Y_{k,j}\right)_{\mathcal{H}}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}c_{k,j}^2\int_0^1 q_k(t)\ y_k^2(t,z_{k,j}^4)dt= \tag{47}$$

$$=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}c_{k,j}^2\int_0^1 q_k(t)\left[sin^2z_{k,j}t+2H\left(z_{j,k}\right)shz_{k,j}tsinz_{k,j}t+H\left(z_{j,k}\right)^2sh^2z_{k,j}t\right]dt,$$

where $q_k(t)=(q(t)\varphi_k,\varphi_k)$.

Without loss of generality, putting $\int_0^1 q_k(t)\,dt=0$ with Theorem 3.1 in mind we come to

$$\sum_{n=1}^{\infty'}(\mu_n-\lambda_n)=$$

$$=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}c_{k,j}^2\frac{\int_0^1 q_k(t)\ \left[-cos2z_{k,j}t+4H\left(z_{j,k}\right)shz_{k,j}tsinz_{k,j}t+H(z_{j,k})^2ch2z_{k,j}t\right]dt}{2} \tag{48}$$

Consider, the N-th partial sum of the inner series:

$$\sum_{k=1}^{N}c_{k,j}^2\frac{\int\limits_0^1 q_k(t)\left[-cos2z_{k,j}t+4H\left(z_{j,k}\right)shz_{k,j}t\sin z_{k,j}t+H(z_{j,k})^2ch2z_{k,j}\right]dt}{2} \tag{49}$$

or in a more compact form $\sum\limits_{k=1}^{N}c_{k,j}^2\int\limits_0^1 q_k(t)\,y_k^2(t,z_{k,j}{}^4)dt$

Our aim in that section is to find the sum of the series in (48). For that sake in our previous works , for example, for evaluating the value of the right side of (46) we use Cauchy's residue theorem, further tending contour of integration to infinity and using asymptotic formulas for the integrand. Namely, each time we have selected a function of a complex variable with poles at $z_{j,k}$ (zeros of characteristic determinant): they are the functions, the denominators of which are defined by the concrete form of characteristic determinants $\Delta(z)$ corresponding to the problem (whose equivalent in the present work determinant in (10) and numerators are suggested by numerators of series the sum of which to be evaluated (here the integrand of (49). Usually, the residues at poles of that function give terms of sum analog of which here is (49) which indicates on some relation between characteristic determinant of the associated operator and norming constants (normus of eigenvectors). Further, using asymptotic formulas found for $z_{k,j}$ on the integration contour, we get the desired formulas.

In this work, since the norming constants defined by (19) and (20) and $\Delta(z)$ from (10), (11) have too long expressions, and that is why to manipulate with them by using the above indicated methods is impossible.

For that reason , by (39) and (41) we establish the indicated above relation between $f_k(z)\,(\Delta(z))$ and norming constants existence of which was intuitively clear for us in all previous works. Remind that by determining, for example, c_{1k} from $\omega_1(z)=0$ and substituting in $\omega_2(z)=0$ yields an equation equivalent to $\Delta(z)=0$. Note that this is more general method and may be used in studying regularized traces in future.

Interchange the integrating and the sum in (49) and denote by $S_N(t)$ the following expression

$$S_N(t)=\sum_{j=1}^{N}c_{k,j}^2\ y_k^2(t,z_{k,j}^4) \tag{50}$$

Now using (41),(42), we see that the following functions of a complex variable z

$$F_k(z,t) = \frac{4z^3 y_k^2(t,z^4)}{f_k(z)} = \frac{4z^3 y_k^2(t,z^4)}{\overline{y_k(1,z^4)}\omega_1(z)} \tag{51}$$

have poles at common roots $z = z_{k,j}$ of system (27),(28) and the residues of $F_k(z,t)$ at these poles are the terms of the sum (49)

$$res_{z=z_{k,j}} F_k(z,t) = c_{k,j}^2\, y_k^2(t,z^4) \tag{52}$$

In virtue of (47), (52), we arrive at the next lemma

Lemma 4.2.

$$\sum_{n=1}^{\prime\infty} (\mu_n - \lambda_n) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=z_{k,j}} F_k(z,t)\, q_k(t)dt \tag{53}$$

where, $F_k(z,t)$ is defined by (52).

In our previous works, we write $c_{k,j}$ in the open form and write concrete form for the expressions analogous to $\overline{y_k(1,z^4)}\omega_1(z)$ in choice of $F_k(z,t)$.

The function $F_k(z,t)$ together with $z_{k,j}$ has poles also at zeros of $y_k\left(1,z^4\right)$. Denoting the zeros of $y_k\left(1,z^4\right)$ or $\overline{y_k(1,z^4)}$ by $z = \beta_{k,j}$ and since it doesn't matter for derivations writing $y_k(1,\cdot)$ instead of $\overline{y_k(1,\cdot)}$, we have

$$res_{z=\beta_{k,j}} F_k(z,t) = \frac{4{\beta_{k,j}}^3 y_k^2(t,{\beta_{k,j}}^4)}{\dot{y}_k\left(1,{\beta_{k,j}}^4\right)\omega_1\left(\beta_{k,j}\right)}$$

where the dot indicates a derivative with respect to z.

Taking into consideration $y_k\left(1,\ {\beta_{k,j}}^4\right) = 0$ in $\omega_1(z)$

$$res_{z=\beta_{k,j}} F_k(z,t) = \frac{4{\beta_{k,j}}^3 y_k^2(t,{\beta_{k,j}}^4)}{\dot{y}_k\left(1,{\beta_{k,j}}^4\right) y_k'''\left(1,\ {\beta_{k,j}}^4\right)}. \tag{54}$$

Note that ${\beta_{k,j}}^4 + \gamma_k$ are the eigenvalues of problem (21),(22) and (55),(56) for each fixed k

$$y_k(1) = 0 \tag{55}$$

$$y_k''(1) - \lambda\gamma_k^\alpha y_k'(1) = 0 \tag{56}$$

and the collection $\left\{{\beta_{k,j}}^4 + \gamma_k\right\}_{k,j=1}^{\infty}$ are the eigenvalues of problem (1), (2), and (57), (58)

$$y(1) = 0 \tag{57}$$

$$y''(1) - \lambda A^\alpha y'(1) = 0 \tag{58}$$

Selecting the rectangular contour l_N including inside it $z_{k,j}$ and $\beta_{k,j}$ for each fixed k and $j = \overline{1,N}$ (we can choose such a contour because of asymptotics of $z_{k,j}$, $\beta_{k,j}$), see, for example, [1, 2] and applying the Cauchy theorem about residues we have

$$\sum_{j=1}^{N} res_{z=z_{k,j}} F_k(z,t) = -\sum_{j=1}^{N} res_{z=\beta_{k,j}} F_k(z,t) + \int_{l_N} F_k(z,t)\,dz \tag{59}$$

Multiplying by $q_k(t)$, integrating along [0,1] and passing to the limit in (59) as $N\to\infty$ yields

$$\sum_{j=1}^{\infty}\int_0^1 res_{z=z_{k,j}} F_k(z,t)\, q_k(t)dt =$$

$$-\sum_{j=1}^{\infty}\int_0^1 res_{z=\beta_{k,j}}F_k(z,t)\,q_k(t)dt+\lim_{N\to\infty}\int_{l_N}\int_0^1 F_k(z,t)\,dz\,q_k(t)dtdz. \tag{60}$$

By using the asymptotics of $F_k(z,t)$ for large $|z|$ values it can be shown that as N tends to infinity, the integral along the extended contours approaches zero. So, with (52) and (54) in mind

$$\sum_{n=1}^{\infty\prime}(\mu_n-\lambda_n)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=z_{k,j}}F_k(z,t)\,q_k(t)dt=$$

$$=-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=\beta_{k,j}}F_k(z,t)\,q_k(t)dt=-\sum_{j=1}^{\infty}\frac{4\beta_{k,j}^3\int_0^1 y_k^2(t,\beta_{k,j}^4)q_k(t)dt}{2\dot{y}_k\left(1,{\beta_{k,j}}^4\right)y_k'''\left(1,{\beta_{k,j}}^4\right)} \tag{61}$$

Let $L_{11}=L_{01}+Q$, where L_{01} is an operator corresponding to (1),(2), (57), (58) which is defined in space $\mathcal{H}_2=L_2(H,(0,1))\oplus H$ of vectors $Y=(y(t),y_1), Z=(z(t),z_1)$ where $y_1,z_1\in H$, with a scalar product defined as, $(Y,Z)_{\mathcal{H}_2}=(y(t),z(t))_{L2(H,(01))}+(y_1,z_1)$,$D(L_{01})=\left\{Y\in D\left(L_0^*\right),y(1)=0,y_1=A^\alpha y'(1)\right\}$, $L_{01}Y=\{ly(t),\ y''(1)\}$ and Q this time is defined as $QY=\{q(t)\,y(t),\ 0\}$. Moreover, denote by L_{01k} the operator defined by $L_{01k}\left(y_k(t),\gamma_k^\alpha y_k'(1)\right)=\left\{l_k y_k(t),y_k''(1)\right\}$ in space $\Lambda_2=L_2(0,1)\oplus C$

Theorem 4.3.

$$res_{z=\beta_{k,j}}F_k(z,t)=-c_{k,j}^2\,y_k^2(t,\beta_{k,j}^4)$$

with $c_{k,j}^2=\frac{1}{\left\|\Phi_{k,j}\right\|_{\mathcal{H}_2}^2}=\frac{1}{\left\|\Psi_{k,j}\right\|_{\Lambda_2}^2}$*, where* $\left\{\Phi_{k,j}\right\}$ *this time are orthogonal eigenvectors of the operator* L_{01} *in* $\mathcal{H}_2$ *associated with problem* (1),(2), (57), (58) *and* $\Psi_{k,j}$ *are orthogonal eigenvectors of problem* (21), (22) (55), (56).

To prove it, from the right side of (61) *we can see that, it is enough to show*

$$c_{k,j}^2=-\frac{4\beta_{k,j}^3}{\dot{y}_k\left(1,{\beta_{k,j}}^4\right)y_k'''\left(1,{\beta_{k,j}}^4\right)}$$

Proof. Again multiplying (30) by $y_k\left(t,\lambda_{k,j}-\gamma_k\right)$, (31) $y_k(t,\lambda-\gamma_k)$, subtracting the second relation from the first, integrating the both sides of the obtained relation along (0,1), adding to the obtained results the term

$$\left(\lambda-\lambda_{k,j}\right)y_k'(1,\lambda-\gamma_k)\,y_k'\left(1,\lambda_{k,j}-\gamma_k\right){\gamma_k}^\alpha \tag{62}$$

keeping in mind,

$$y_k\left(1,\lambda_{k,j}-\gamma_k\right)=y_k\left(1,{\beta_{k,j}}^4\right)=0 \tag{63}$$

(we again keep notations $\sqrt[4]{\lambda-\gamma_k}=z,\ \sqrt[4]{\lambda_{k,j}-\gamma_k}=\beta_{k,j}$, where this time $\lambda_{k,j}={\beta_{k,j}}^4+\gamma_k$, $j=1,\infty$ are eigenvalues of the problem (21),(22),(55), (56) we get:

$$E\equiv\int_0^1 y_k^{IV}\left(t,z^4\right)y_k\left(t,\beta_{k,j}^4\right)dt-$$

$$-\int_0^1 y_k^{IV}\left(t,\beta_{k,j}^4\right)y_k\left(t,z^4\right)dt+\left(\lambda-\lambda_{k,j}\right)y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}^4\right)\gamma_k^\alpha=$$

$$=\left(z^4-{\beta_{k,j}}^4\right)\int_0^1 y_k\left(t,z^4\right)y_k\left(t,{\beta_{k,j}}^4\right)dt+\left(z^4-{\beta_{k,j}}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,{\beta_{k,j}}^4\right)\gamma_k^\alpha \tag{64}$$

Integration by parts gives

$$E\equiv y_k'''\left(1,z^4\right)y_k(1,{\beta_{k,j}}^4)-y_k'''\left(1,{\beta_{k,j}}^4\right)y_k\left(1,z^4\right)-y_k''\left(1,z^4\right)y_k'(1,{\beta_{k,j}}^4)+$$

$$+y_k''\left(1,{\beta_{k,j}}^4\right)y_k'\left(1,z^4\right)+\left(z^4-{\beta_{k,j}}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,{\beta_{k,j}}^4\right){\gamma_k}^\alpha=$$

$$= -y_k'''\left(1,\beta_{k,j}{}^4\right)y_k\left(1,z^4\right) - y_k''\left(1,z^4\right)y_k'(1,\beta_{k,j}{}^4) + y_k''\left(1,\beta_{k,j}{}^4\right)y_k'\left(1,z^4\right) +$$
$$+\left(z^4-\beta_{k,j}{}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\gamma_k{}^\alpha =$$
$$= -y_k'''\left(1,\beta_{k,j}{}^4\right)\left[y_k\left(1,z^4\right) - y_k\left(1,\beta_{k,j}{}^4\right)\right] -$$
$$-y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\left[\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - \frac{y_k''\left(1,\beta_{k,j}{}^4\right)}{y_k'\left(1,\beta_{k,j}{}^4\right)}\right] +$$
$$+\left(z^4-\beta_{k,j}{}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\gamma_k{}^\alpha.$$

Substituting it into (64),

$$\left(z^4-\beta_{k,j}{}^4\right)\int_0^1 y_k\left(t,z^4\right)y_k\left(t,\beta_{k,j}{}^4\right)dt + \left(z^4-\beta_{k,j}{}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\gamma_k{}^\alpha =$$
$$= -y_k'''\left(1,\beta_{k,j}{}^4\right)\left[y_k\left(1,z^4\right) - y_k\left(1,\beta_{k,j}{}^4\right)\right] - y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\left[\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - \frac{y_k''\left(1,\beta_{k,j}{}^4\right)}{y_k'\left(1,\beta_{k,j}{}^4\right)}\right] +$$
$$+\left(z^4-\beta_{k,j}{}^4\right)y_k'\left(1,z^4\right)y_k'\left(1,\beta_{k,j}{}^4\right)\gamma_k{}^\alpha \tag{65}$$

Recall that the term $y_k\left(1,\beta_{k,j}{}^4\right)$ can appear on the left side of (65) because of (63)

Dividing the both sides of (65) by $z-\beta_{k,j}$, passing to the limit as $z\to\beta_{k,j}$ and denoting orthogonal eigenvectors of the problem (21),(22),(55), (56) again by $\Psi_{k,j}$, we get

$$4\beta_{k,j}^3\left\|\Psi_{k,j}\right\|_{\Lambda_2}^2 = -y_k'''\left(1,\beta_{k,j}^4\right)\dot{y}_k\left(1,\beta_{k,j}^4\right) -$$
$$-y_k'\left(1,\beta_{k,j}^4\right)^2\lim_{z\to\beta_{k,j}}\left[\frac{\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - \frac{y_k''\left(1,\beta_{k,j}^4\right)}{y_k'\left(1,\beta_{k,j}{}^4\right)}}{z-\beta_{k,j}} - \frac{(z^4-\beta_{k,j}^4)\gamma_k^\alpha}{z-\beta_{k,j}}\right] =$$
$$= -y_k'''\left(1,\beta_{k,j}^4\right)y_k\left(1,z^4\right)'|_{z=\beta_{k,j}} - y_k'\left(1,\beta_{k,j}{}^4\right)^2\left[\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - z^4\gamma_k{}^\alpha\right]'|_{z=\beta_{k,j}} \tag{66}$$

If c_{2k} is defined from $\omega_2(z)=0$, then in the right side of (66) $\frac{y_k''\left(1,z^4\right)}{y_k'\left(1,z^4\right)} - z^4\gamma_k{}^\alpha \equiv 0$ and simplifies to

$$4\beta_{k,j}{}^3\left\|\Psi_{k,j}\right\|^2{}_{\Lambda_2} = -y_k'''\left(1,\beta_{k,j}{}^4\right)y_k\left(1,z^4\right)'|_{z=\beta_{k,j}} \tag{67}$$

or

$$\left\|\Phi_{k,j}\right\|_{\mathcal{H}_2}^2 = \left\|\Psi_{k,j}\right\|_{\Lambda_2}^2 = \frac{-y_k'''\left(1,\beta_{k,j}^4\right)y_k(1,z^4)'|_{z=\beta_{k,j}}}{4\beta_{k,j}^3} \tag{68}$$

We have (see (32), (63))

$$\omega_1\left(\beta_{k,j}\right) = -y_k'''\left(1,\beta_{k,j}^4\right) \tag{69}$$

So, for the norming constants of the problem (1), (2),(57), (58)

$$\begin{aligned}\frac{1}{c_{k,j}^2} &= \left\|\Phi_{k,j}\right\|_{\mathcal{H}_2}^2 = \left\|\Psi_{k,j}\right\|_{\Lambda_2}^2 = \frac{\omega_1(\beta_{k,j})y_k(1,z^4)'|_{z=\beta_{k,j}}}{4\beta_{k,j}^3} = \\ &= \frac{-y_k'''(1,4\beta_{k,j}^4)y_k(1,z^4)|_{z=\beta_{k,j}}}{\beta_{k,j}^3}\end{aligned} \tag{70}$$

From (54) and (70),

$$res_{z=\beta_{k,j}} F_k(z,t) = \frac{4{\beta_{k,j}}^3 y_k^2(t,\beta_{k,j}^4)}{\dot{y}_k\left(1,{\beta_{k,j}}^4\right)\omega_1\left(\beta_{k,j}\right)} = \frac{4{\beta_{k,j}}^3 y_k^2(t,\beta_{k,j}^4)}{\dot{y}_k\left(1,{\beta_{k,j}}^4\right)y_k'''\left(1,{\beta_{k,j}}^4\right)} =$$

$$= -\frac{y_k^2(t,\beta_{k,j}^4)}{\left\|\Psi_{k,j}\right\|_{\Lambda_2}^2} = -c_{k,j}^2 y_k^2(t,\beta_{k,j}^4) \tag{71}$$

$c_{k,j}$ (if k is fixed)are now the norming constants of the problem (31),(32),(55),(56) or for varying k of the problem (1),(2),(57),(58).

Denoting the eigenvalues of L_{01} and L_{11} by λ_{n1} , μ_{n1} respectively, we have for the regularized trace of L_{11} as in (47)

$$\sum_{n=1}^{\infty'}(\mu_{n1}-\lambda_{n1}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(QY_{k,j},Y_{k,j}\right)_{\mathcal{H}_2}, \tag{72}$$

where $\left\{Y_{k,j}\right\}$ are now orthonormal eigenvectors of the operator L_{01}.

In virtue of (72) and application of Theorem 4.3 to L_{01} yields

$$\sum_{n=1}^{\infty'}(\mu_{n1}-\lambda_{n1}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}c_{k,j}^2\int_0^1 q_k(t)y_k^2(t,\beta_{k,j}^4)dt = -\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 \underset{z=\beta_{k,j}}{res} F_k(z,t)q_k(t)dt \tag{73}$$

By comparing (73) and (61) we get □

Corollary 4.4.

$$\sum_{n=1}^{\infty'}(\mu_n-\lambda_n) = \sum_{n=1}^{\infty'}(\mu_{n1}-\lambda_{n1})$$

Thus, the problem is reduced to evaluating a regularized trace of the corresponding operator L_{11}.

From (73)

$$\sum_{n=1}^{\infty'}(\mu_{n1}-\lambda_{n1}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}c_{k,j}^2\int_0^1 q_k(t)y_k^2(t,\beta_{k,j}^4)dt. \tag{74}$$

To find the sum on the right side of (74), again apply the technique used above: select a function of a complex variable with the poles at $\beta_{k,j}$ and residues equal to the terms of the series on the left of side (74). Really, setting

$$K(z) \equiv -y_k(1,\lambda)\,y_k'''(1,\lambda) - y_k'(1,\lambda)\,[y_k''(1,\lambda) - \lambda\gamma_k^\alpha y_k'(1,\lambda)]$$

and in the solution of (21),(22) defining c_{2k} from condition (47) ($y_k''(1,\lambda) - \lambda\gamma_k^\alpha y_k'(1,\lambda) = 0$) we have

$$K'\left(\beta_{k,j}\right) = -\dot{y}_k\left(1,\beta_{k,j}^4\right)y_k'''\left(1,\beta_{k,j}^4\right) \tag{75}$$

So, if

$$F_{1k}(z,t) = \frac{4z^3 y_k^2(t,z^4)}{K(z)}$$

then in virtue of (75),(70) and Theorem 4.3 (or relation (71))

$$\begin{aligned} res_{z=\beta_{k,j}} F_{1k}(z,t) &= \frac{4{\beta_{k,j}}^3 y_k^2(t,\beta_{k,j}^4)}{K'\left(\beta_{k,j}\right)} = -\frac{4{\beta_{k,j}}^3 y_k^2(t,\beta_{k,j}^4)}{y_k'\left(1,\beta_{k,j}^4\right)y_k'''\left(1,\beta_{k,j}^4\right)} = \\ &= \frac{y_k^2(t,\beta_{k,j}^4)}{\left\|\Phi_{k,j}\right\|^2} = c_{k,j}^2 y_k^2(t,\beta_{k,j}^4) = res_{z=\beta_{k,j}} F_k(z,t) \end{aligned}$$

Now, if we define in the last relation c_{2k} from $y_k(1,\lambda)=0$, then $F_{1k}(z,t)$ will be simplified to the form

$$F_{1k}(z,t)=\frac{4z^3y_k^2(t,z^4)}{-2y_k'(1,z^4)\left[y_k''(1,z^4)-\lambda\gamma_k^{\alpha}y_k'(1,z^4)\right]} \tag{76}$$

Thus,

$$\sum_{n=1}^{\infty\prime}(\mu_{n1}-\lambda_{n1})=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 \underset{z=\beta_{k,j}}{res}F_{1k}(z,t)q_k(t)dt$$

But $F_{1k}(z)$ together with $\beta_{k,j}$ has poles also at the zeros of the function $y_k'\left(1,z^4\right)$. Denote them by $\delta_{k,j}$. Thus,

$$res_{z=\delta_{k,j}}F_{1k}(z,t)=\frac{4{\delta_{k,j}}^3y_k^2(t,z^4)}{-[y_k'(1,z^4)]'|_{z=\delta_{k,j}}y_k''\left(1,\delta_{k,j}^4\right)}. \tag{77}$$

Again taking the contour l_N $(j=\overline{1,N})$ including $\beta_{k,j}$ and $\delta_{k,j}$ and extending it to infinity, we will have

$$\sum_{j=1}^{\infty}\int_0^1 res_{z=\beta_{k,j}}F_{1k}(z,t)\,q_k(t)dt=-\sum_{j=1}^{\infty}\int_0^1 res_{z=\delta_{k,j}}F_{1k}(z,t)\,q_k(t)dt=$$

$$=\sum_{j=1}^{\infty}\frac{\int_0^1 4{\delta_{k,j}}^3y_k^2(t,z^4)\,q_k(t)dt}{[y_k'(1,\lambda)]'|_{z=\delta_{k,j}}y_k''\left(1,\delta_{k,j}^4\right)}, \tag{78}$$

where $\frac{-[y_k'(1,z^4)]'|_{z=\delta_{k,j}}y_k''\left(1,\delta_{k,j}^4\right)}{4\delta_{k,j}^3}$ is the norm of orthogonal eigenvectors of the operator corresponding to problem (21), (22) with the additional conditions

$$y_k(1)=0, \tag{79}$$

$$y_k'(1)=0 \tag{80}$$

or the norm of orthogonal eigenvectors of the operator L_{02} in $H_3\equiv L_2(H,(0,1))$ corresponding to problems (1),(2) and

$$y(1)=0, \tag{81}$$

$$y'(1)=0. \tag{82}$$

The perturbed operator corresponding to it is $L_{12}=L_{02}+q(t)$.

When justifying in (78) that

$$c_{k,j}^2=\frac{-4{\delta_{k,j}}^3}{[y_k'(1,z^4)]'|_{z=\delta_{k,j}}y_k''\left(1,\delta_{k,j}\right)}$$

really, are norming constants of the operator L_{02} corresponding to (1),(2),(81),(82) in $\mathcal{H}_3=L_2(H,(0,1))$ again use the above technique, but this time we will not add any additional terms like the term (62) in (64) or term (32) in (33) (there it was done for defining the norm in direct sum space, because of λ in the boundary conditions. The last boundary conditions don't depend on λ and those conditions define a selfadjoint operator in original space).

For not complicating notations denoting the eigenvectors again by technique $\Phi_{k,j}$ we have by illustrated in (33)-(41) or (62)-(68) and defining c_{2k} from (79):

$$\left(z^4-\delta_{k,j}^4\right)\int_0^1 y_k^2\left(t,z^4\right)dt=y_k'''\left(1,z^4\right)y_k\left(1,\delta_{k,j}^4\right)-y_k'''\left(1,\delta_{k,j}^4\right)y_k\left(1,z^4\right)-$$

$$-y_k''\left(1,z^4\right)y_k'\left(1,\delta_{k,j}^4\right)+y_k''\left(1,\delta_{k,j}^4\right)y_k'\left(1,z^4\right) \tag{83}$$

Dividing the both sides of this relation by $z-\delta_{k,j}$, letting $z\to\delta_{k,j}$, defining c_{2k} from (81) and taking into consideration $y_k'\left(1,\delta_{k,j}^4\right)=0$ yields

$$4\delta_{k,j}^3\left\|\Phi_{k,j}\right\|_{\mathcal{H}_3}^2=4\delta_{k,j}^3\left\|\Psi_{k,j}\right\|_{\Lambda_3}^2=[y_k'(1,z^4)]'|_{z=\delta_{k,j}}y_k''\left(1,\delta_{k,j}^4\right) \tag{84}$$

($\Lambda_3 = L_2(0,1)$)

Thus,

$$\sum_{n=1}^{\infty'} (\mu_{n1} - \lambda_{n1}) = -\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=\delta_{k,j}} F_{1k}(z,t)\, q_k(t)dt =$$

$$= \sum_{j=1}^{\infty} \frac{\int_0^1 4\delta_{k,j}^3 y_k^2(t,z^4) q_k(t)dt}{[y_k^{'}(1,\lambda)]'|_{z=\delta_{k,j}} y_k^{''}\left(1,\delta_{k,j}^4\right)} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(QY_{k,j}, Y_{k,j}\right)_3$$

where $Y_{k,j}$ are orthonormal eigenvectors of the operator L_{02} in H_3.

Denoting eigenvalues of L_{12}, L_{02} by μ_{n2}, λ_{n2}, respectively,

$$\sum_{n=1}^{\infty'} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty'} (\mu_{n1} - \lambda_{n1}) = \sum_{n=1}^{\infty'} (\mu_{n2} - \lambda_{n2})$$

Now we come to the evaluation of the sum of the series

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{\int_0^1 4\delta_{k,j}^3 y_k^2(t,z^4)\, q_k(t)dt}{[y_k^{'}(1,\lambda)]'|_{z=\delta_{k,j}} y_k^{''}\left(1,\delta_{k,j}^4\right)} \tag{85}$$

For that sake select the following function of a complex variable

$$F_{2k}(z,t) = \frac{4z^3 y_k^2(t,z^4)}{-y_k^{'''}(1,z^4)\, y_k(1,z^4) + y_k^{'}(1,z^4)\, y_k^{''}(1,z^4)]} \tag{86}$$

the residues at $\delta_{k,j}$ give terms of series (85). Selecting c_{2k}, the solution of boundary value problem from $y_k(1) = 0$, $F_{2k}(z,t)$ takes the form

$$F_{2k}(z,t) = \frac{4z^3 y_k^2(t,z^4)}{y_k^{'}(1,\lambda)\, y_k^{''}(1,\lambda)]} \tag{87}$$

and

$$res_{z=\delta_{k,j}} F_{2k}(z,t) = \frac{4{\delta_{k,j}}^3 y_k^2(t,{\delta_{k,j}}^4)}{[y_k^{'}\left(1,\delta_{k,j}^4\right)]'\ y_k^{''}(1,z)'\,|_{z=\delta_{k,j}}}. \tag{88}$$

Obviously $F_{2k}(z,t)$ will have poles also at the roots of the equation $y_k^{''}(1,\lambda) = 0$. Denote these roots by $\rho_{k,j}$. Thus, $\rho_{k,j}$ are common roots of the equations

$$y_k(0,\lambda) = 0, y_k^{''}(0,\lambda) = 0,$$

$$y_k(1,\lambda) = 0, \tag{89}$$

$$y_k^{''}(1,\lambda) = 0 \tag{90}$$

moreover,

$$res_{z=\rho_{k,j}} F_{2k}(z,t) = \frac{4\rho_{k,j}^3 y_k^2(t,\rho_{k,j}^4)}{\left[y_k^{'}(1,\rho_{k,j}^4)\right]' y_k^{''}(1,z)'|_{z=\rho_{k,j}}} \tag{91}$$

But

$$\frac{-[y_k^{'}\left(1,\rho_{k,j}\right)]'\, y_k^{''}(1,z)^{'}\,|_{z=\rho_{k,j}}}{4\rho_{k,j}^3} = \left\|\Phi_{k,j}\right\|_3^2$$

where $\Phi_{k,j}$ are the eigenvectors of problem (1), (2) with additional boundary conditions

$$y(1) = 0, y^{''}(1) = 0 \tag{92}$$

Really,

$$\left(z^4-\rho^4{}_{k,j}\right)\int_0^1 y_k\left(t,z^4\right)^2 dt = y_k'''\left(1,z^4\right)y_k\left(1,\rho^4_{k,j}\right)-y_k'''\left(1,\rho^4_{k,j}\right)y_k\left(1,z^4\right)-$$

$$-y_k''\left(1,z^4\right)y_k'\left(1,\rho^4_{k,j}\right)+y_k''\left(1,\rho^4_{k,j}\right)y_k'\left(1,z^4\right)=-y_k'''\left(1,\rho^4_{k,j}\right)\left[y_k\left(1,z^4\right)-y_k\left(1,\rho^4_{k,j}\right)\right]-$$

$$-y_k'\left(1,\rho^4_{k,j}\right)\left[y_k''\left(1,z^4\right)-y_k''\left(1,\rho^4_{k,j}\right)\right] \tag{93}$$

If c_{2k} is defined from (89), then from (93) as $z\to\rho_{k,j}$,

$$4\rho^3{}_{k,j}\left\|\Psi_{k,j}\right\|^2_{\Lambda_3}=-y_k'''\left(1,\rho^4_{k,j}\right)y_k\left(1,z^4\right)'|_{z=\rho_{k,j}}-y_k'\left(1,\rho^4_{k,j}\right)y_k''\left(1,z^4\right)'|_{z=\rho_{k,j}} \tag{94}$$

$$c^2_{k,j}=\left\|\Psi_{k,j}\right\|^2_{\Lambda_3}=\frac{-y_k'\left(1,\rho^4_{k,j}\right)y_k''\left(1,z^4\right)'|_{z=\rho_{k,j}}}{4\rho_{k,j}{}^3}$$

Denoting the eigenvalues of L_{03} and $L_{03}+q(t)$ in $L_2(H,(0,1))$ by $\lambda_{n3},\ \mu_{n3}$, we come to the next theorem

Theorem 4.5. $\sum\limits_{n=1}^{\infty\,\prime}(\mu_n-\lambda_n)=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n1}-\lambda_{n1})=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n2}-\lambda_{n2})=\sum_{n=1}^{\infty\,\prime}(\mu_{n3}-\lambda_{n3})$.

Hence,

$$\sum_{n=1}^{\infty\,\prime}(\mu_{n2}-\lambda_{n2})=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=\delta_{k,j}}F_{2k}(z,t)\,q_k(t)\,dt=$$

$$=-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 res_{z=\rho_{k,j}}F_{2k}(z,t)\,q_k(t)\,dt=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(QY_{k,j},Y_{k,j}\right)_{\mathcal{H}_3}$$

where $Y_{k,j}$ are now the set of orthonormal eigenvectors of the operator L_{03} .

But on the other hand, since the solution satisfying conditions (22) is given by (38), then from (89), (90) we have

$$c_{1k}sinz+c_{2k}shz=0$$

$$-z^2c_{1k}sinz+c_{2k}z^2shz=0$$

from which $c_{2k}=0$ and orthogonal eigenvectors are $c_{1k}sinzt$

From boundary conditions (89), (90) follows $sinz=0$ or $z=\pi j$, and eigenvalues are $\lambda_{k,j}=(\pi j)^4+\gamma_k$ and orthonormal eigenvectors of L_{03} are $Y_{k,j}=\sqrt{2}sin\pi jt\varphi_k,\ k,j=\overline{1,\infty}$

Thus, taking into consideration also the requirement (44)

$$\sum_{n=1}^{\infty\,\prime}(\mu_{n3}-\lambda_{n3})=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(QY_{k,j},Y_{k,j})_{\mathcal{H}_3}=-\sum_{k=1}^{\infty}\frac{q_k(1)+q_k(0)}{4}$$

Theorem 4.6. $\sum\limits_{n=1}^{\infty\,\prime}(\mu_n-\lambda_n)=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n1}-\lambda_{n1})=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n2}-\lambda_{n2})$

$$=\sum_{n=1}^{\infty}{}'(\mu_{n3}-\lambda_{n3})=-\sum_{k=1}^{\infty}{}'\frac{q_k(1)+q_k(0)}{4}. \tag{95}$$

If we put on $q(t)$ stronger condition than (44)*, namely would $q(t)$ belong to the trace class σ_1, then from (95)we get*

Corollary 4.7. $\sum\limits_{n=1}^{\infty\,\prime}(\mu_n-\lambda_n)=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n1}-\lambda_{n1})=\sum\limits_{n=1}^{\infty\,\prime}(\mu_{n2}-\lambda_{n2})=$

$$=\sum_{n=1}^{\infty\,\prime}(\mu_{n3}-\lambda_{n3})=-\frac{trq(1)+trq(0)}{4}.$$

Example 4.8. Consider in $\Omega \times [0,T]$, $\Omega= [0,1]\times[0,1]$ the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + Q(x,y)u \tag{96}$$

subject to

$$u(t,0,y) = 0 \tag{97}$$

$$u(t,x,0) = u(t,x,1) = 0 \tag{98}$$

$$u(t,0,y) = u_{xx}(t,0,y) = 0 \tag{99}$$

$$-u_{xxx}(t,1,y) = u_{ty}(t,1,y) \tag{100}$$

$$u_{xx}(t,1,y) = u_{tyx}(t,1,y), t \in [0,T], \; x \in [0,1], \; y \in [0,1]. \tag{101}$$

Look for the solution in the form $u(t,x,y) = U(t)\,V(x,y)$, substituting of which in equation yields

$$U'(t)\,V(x,y) = \frac{\partial^4 V(x,y)}{\partial x^4} U(t) + \frac{\partial^4 V}{\partial y^4} U(t) + Q(x,y)V(x,y)U(t). \tag{102}$$

Dividing both sides by of (102) $U(t)V(x,y)$ we get $\frac{U'(t)}{U(t)} = \frac{\partial^4 V}{V\partial x^4} + \frac{\partial^4 V}{V\partial y^4}$, denoting $\frac{U'(t)}{U(t)}$ by λ we come to

$$\frac{\partial^4 V}{\partial x^4} + \frac{\partial^4 V}{\partial y^4} + Q(x,y)V = \lambda V, \tag{103}$$

$$V(0,y) = V_{xx}(0,y) = 0 \tag{104}$$

$$-V_{xxx}(1,y) = \lambda V_y(1,y) \tag{105}$$

$$V_{xx}(1,y) = \lambda V_{yx}(1,y) \tag{106}$$

Define in $L_2(0,1)$ the operator A by

$$Av(y) = -v''(y), V(.,y) \equiv v(y) \tag{107}$$

$$D(A) = \left\{ v'(y) \text{ is absolutely continous in } L_2(0,1) \text{ and } v''(y) \in L_2(0,1), \; v(0) = v(1) = 0 \right\}$$

Obviously, the eigenvalues of the operator A are $\gamma_k = \pi^2 k^2$, orthonormal eigenfunctions are $\varphi_k = \sqrt{2} sin\pi ky$. For each fixed x the function $V(x,y)$ is from $L_2(H, \; (0,1))$. Let $Q_1 = Q_2 = A^{\frac{1}{2}}$, thus $Q_i V(1,y) = V_y(1,y)$, $i = 1,2$. For each y $q(x,y)$ acts in $L_2(0,1)$, thus denoting it by $q(x)$ and for each y denoting $V(x,y)$ by $u(x)$ we arrive at the following operator theoretical formulation of problem (103)-(106)

$$u^{iV}(x) + Au(x) + q(x)u(x) = \lambda u,$$

$$u(0) = u''(0) = 0$$

$$-u'''(1) = \lambda Q_1 u(1)$$

$$u''(1) = \lambda Q_2 u'(1)$$

By Lemma 4.2 the eigenvalues of the above theorem asymptotically behave as $\lambda_n \sim C_2 n^{\frac{3}{4}}$. Let $Q(x,y)$ be continuous in Ω and have second order partial derivatives with respect to x, moreover its expansion in Foirier series having only cosine terms at points $y = 0,1$ converges to its values at that points. Moreover, $\int_0^1 Q(0,y)\,dy = \int_0^1 Q(1,y)\,dy = 0$. Then by Theorem 4.6

$$\sum_{n=1}^{\infty'} (\mu_n - \lambda_n) = -\frac{\sum_{k=1}^{\infty} (q(0)\varphi_k, \varphi_k) + \sum_{k=1}^{\infty} (q(1)\varphi_k, \varphi_k)}{4}$$

But

$$\sum_{k=1}^{\infty}(q(0)\varphi_k,\varphi_k)=2\sum_{k=1}^{\infty}\int_0^1 Q(0,y)sin^2\pi kydy=$$

$$=\frac{2}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi}Q(0,\frac{y}{\pi})sin^2kydy=\frac{2}{\pi}\int_Q^{\pi}\left(0,\frac{y}{\pi}\right)\frac{1-cos2ky}{2}dy=-\frac{1}{\pi}\int_0^{\pi}Q\left(0,\frac{y}{\pi}\right)cos2kydy=$$

$$=-\frac{1}{4}\left[\sum_{k=0}^{\infty}\frac{2}{\pi}cosk\cdot 0\int_0^{\pi}Q(0,\frac{y}{\pi})coskydy\sum_{k=0}^{\infty}\frac{2}{\pi}cosk\cdot\pi\int_0^{\pi}Q(0,\frac{y}{\pi})coskydy\right]=$$

$$=-\frac{1}{4}\left[Q(0,0)+Q(0,1)\right]$$

In similar way one can show that

$$\sum_{k=1}^{\infty}(q(1)\varphi_k,\varphi_k)=-\frac{1}{4}\left[Q(1,0)+Q(1,1)\right].$$

Thus,

$$\sum_{n=1}^{\infty'}(\mu_n-\lambda_n)=\frac{1}{16}\left[Q(0,0)+Q(0,1)+Q(1,0)+Q(1,1)\right].$$

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