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On some properties of $\mathcal{P}\mathbb{R}$ ($\mathcal{P}m$)-factorizable semitopological groups

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Abstract. In this article, we introduce the notions of $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizability and $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizability. In the second part of this article, we study the relationship between $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$-factorizability and $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizability. In the third part of this article, we discuss some properties of certain continuous homomorphic images of $(\overline{\mathbb{R}}, \mathbb{SCSG})$- and $(\mathbb{MS}, \mathbb{SCSG})$-factorizable semitopological groups. In the fourth part of this article, $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizability and $(\mathbb{MS}, \mathbb{SCSG})$-factorizability are discussed. We prove the following results.

Let G and H be semitopological groups. If $G \to H$ is a continuous surjective closed and open homomorphism such that G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizable semitopological group with a q-point and H satisfies $Sm(H) \leq \omega$, then H is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizable. If $G \to H$ is a continuous open surjective homomorphism such that G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable semitopological group, then H is $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable. If G is a regular semitopological group with a q-point, property $(^{***})$ and satisfying $Sm(G) \leq \omega$, then G is topologically isomorphic to a subgroup of the product of a family of separable metrizable semitopological groups.

1. Introduction

Recall that a *paratopological group* [1] is a group with a topology such that multiplication on the group is jointly continuous. A *topological group* G [1] is a paratopological group such that the inverse mapping of G is continuous. A *semitopological group* is a group with a topology in which the left and the right translations are continuous [1]. A *quasitopological group* G is a semitopological group such that the inverse mapping of G is continuous [1]. Given a semitopological group G, the symbol $\mathcal{N}(e)$ denotes the family of open neighborhoods of the identity e in G.

Let $\mathbb{R}$ be the set of real numbers with the usual topology. Recall that a Hausdorff topological group G is $\mathbb{R}$-*factorizable* [24, 26] if, every continuous function $f : G \to \mathbb{R}$, one can find a continuous homomorphism $p : G \to H$ onto a second-countable Hausdorff topological group H and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ p$. A Hausdorff topological group G is m-*factorizable* if for every continuous mapping $f : G \to M$ to a metrizable space M, there exist a continuous homomorphism $\pi : G \to K$ onto a second-countable Hausdorff topological group K and a continuous mapping $g : K \to M$ such that $f = g \circ \pi$ ([1], p. 539). Every m-factorizable topological group is $\mathbb{R}$-factorizable.

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When factorizing a continuous mapping $f : G \to M$ of a given topological (paratopological, quasitopological, semitopological) group $G \in \mathbb{G}$ from a class $\mathbb{G}$ to a space $M \in \mathbb{M}$ (say, M is Hausdorff and second countable or metrizable), one chooses a class $\mathbb{H}$ of objects of Topological Algebra and a class $\mathbb{P}$ of continuous homomorphisms. Once this choice is done, one says that:

A group G is $(\mathbb{M}, \mathbb{H}, \mathbb{P})$-*factorizable* if for every continuous mapping $f : G \to M$ to a space $M \in \mathbb{M}$, there exist a group $H \in \mathbb{H}$ and a surjective continuous homomorphism $p : G \to H$, with $p \in \mathbb{P}$, such that $f = h \circ p$, for some continuous mapping $h : H \to M$.

Recall that a family $\{A_s : s \in S\}$ of subsets of a topological space is *closure-preserving* if $\overline{\bigcup_{s\in S_0} A_s} = \bigcup_{s\in S_0} \overline{A_s}$ for every $S_0 \subset S$. A family of sets is σ-*closure-preserving* if it can be represented as a countable union of closure-preserving families. Recall that an M_1-*space* is a regular space having a σ-closure preserving base [4].

Let us introduce the following notation, where $i \in \{0, 1, 2, 3, r, 3.5\}$ (r stands for regular):

- $\mathbb{TG}$, $\mathbb{PG}$, $\mathbb{SG}$ and $\mathbb{QG}$ are the classes of topological, paratopological, semitopological and quasitopological groups, respectively;
- if $\mathbb{O}$ is a class of spaces or objects of topological algebra, then $\mathbb{O}_i$ is the subclass of $\mathbb{O}$ that consists of all those $X \in \mathbb{O}$ that satisfy the T_i separation axiom (for example, $\mathbb{TG}_2$ is the class of Hausdorff topological groups);
- $\overline{\mathbb{R}}$ is the class that contains only one space, the real line $\mathbb{R}$;
- $\mathbb{SC}$ is the class of second-countable spaces (no separation restrictions are imposed);
- $\mathbb{SCTG}$, $\mathbb{SCPG}$, $\mathbb{SCSG}$ and $\mathbb{SCQG}$ are the classes of second-countable topological, paratopological, semitopological and quasitopological groups, respectively;
- $\mathbb{MS}$ is the class of metrizable spaces;
- $\mathbb{MTG}$, $\mathbb{MPG}$ and $\mathbb{MSG}$ are the classes of metrizable topological, paratopological and semitopological groups, respectively;
- $\mathbb{F}$ is the class of first-countable spaces;
- $\mathbb{FTG}$, $\mathbb{FPG}$ and $\mathbb{FSG}$ are the classes of first-countable topological, paratopological and semitopological groups, respectively;
- $\mathbb{M}_1$ is the class of M_1-spaces;
- $\mathbb{M}_1\mathbb{TG}$, $\mathbb{M}_1\mathbb{PG}$ and $\mathbb{M}_1\mathbb{SG}$ are the classes of topological, paratopological and semitopological groups, respectively, that are M_1-spaces;
- $\mathbb{FM}_1$ is the class of first-countable M_1-spaces;
- $\mathbb{FM}_1\mathbb{TG}$, $\mathbb{FM}_1\mathbb{PG}$ and $\mathbb{FM}_1\mathbb{SG}$ are the classes of first-countable topological, paratopological and semitopological groups, respectively, that are M_1-spaces;
- $\mathcal{P}\mathbb{TG}$, $\mathcal{P}\mathbb{PG}$, $\mathcal{P}\mathbb{SG}$ and $\mathcal{P}\mathbb{QG}$ are the classes of topological, paratopological, semitopological and quasitopological groups with property $\mathcal{P}$, respectively;
- $\mathbb{CH}$ is the class of continuous homomorphisms;
- $\mathbb{PH}$ is the class of perfect homomorphisms;
- $\mathbb{OH}$ is the class of open continuous homomorphisms.

In what follows, $(\mathbb{M}, \mathbb{H}, \mathbb{CH})$- factorizability shortens to $(\mathbb{M}, \mathbb{H})$-factorizability.

Making use of the above definitions, we conclude that a topological group G is $\mathbb{R}$-factorizable if and only if it is $(\overline{\mathbb{R}}, \mathbb{SCTG}_2)$-factorizable. Let $\mathcal{P}$ be a topological property. An $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$-factorizable semitopological group is called $\mathcal{P}\mathbb{R}$-*factorizable* in [12]. It is also called $\mathcal{P}$-factorizable in ([1], p. 562). Thus, an $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizable semitopological group is also called $\mathcal{P}m$-*factorizable*.

$(\overline{\mathbb{R}}, \mathbb{FSG})$-factorizability and $(\mathbb{MS}, \mathbb{FSG})$-factorizability are discussed in [12], where they are called $\mathcal{F}\mathbb{R}$-*factorizability* and $\mathcal{F}m$-*factorizability*, respectively. $\mathcal{M}$-factorizable topological groups are discussed in [33]. A Hausdorff topological group G is $\mathcal{M}$-*factorizable* if it is $(\overline{\mathbb{R}}, \mathbb{MTG})$-factorizable.

A point x of a space X is called a q-*point* of X if there exists a sequence $\{U_n\}_{n\in\omega}$ of open neighborhoods of x in X such that any sequence $\{x_n\}_{n\in\omega}$ with $x_n \in U_n$ for every $n \in \omega$ has a point of accumulation in X ([1], p. 389). A space X is called a q-*space* if every point of X is a q-point. The *symmetry number* of a

T_1 semitopological group G, denoted by $Sm(G)$, is the minimum cardinal number κ such that for every neighborhood U of the identity e in G, there exists a family $\mathcal{V} \subset \mathcal{N}(e)$ such that $\bigcap_{V\in\mathcal{V}} V^{-1} \subset U$ and $|\mathcal{V}| \leq \kappa$.

In this article, we introduce the notions of $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizability and $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizability. In the second part of this article, we study the relationship between $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$-factorizability and $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizability. In the third part of this article, we discuss some properties of certain continuous homomorphic images of $(\overline{\mathbb{R}}, \mathbb{SCSG})$- and $(\mathbb{MS}, \mathbb{SCSG})$-factorizable semitopological groups. In the fourth part of this article, $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizability and $(\mathbb{MS}, \mathbb{SCSG})$-factorizability are discussed.

In Theorem 2.10, we show the following result. Let $\mathcal{P}$ be a topological property such that every space with $\mathcal{P}$ is first-countable and $\mathcal{P}$ is inherited by subspaces. Let also G be a Tychonoff semitopological group such that $G \times \mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$-factorizable, where $\mathbb{Z}(2) = \{0, 1\}$ is the discrete group. Then G is $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizable.

In Theorems 3.14 and 3.17, we show the following result. Let G and H be semitopological groups. If $G \to H$ is a continuous surjective closed and open homomorphism such that G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizable semitopological group with a q-point and H satisfies $Sm(H) \leq \omega$, then H is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizable.

In Theorems 4.10 and 4.11 we show that if $G \to H$ is a continuous open surjective homomorphism such that G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable semitopological group, then H is $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable. In Theorem 4.2 we show that if G is a topological (paratopological, semitopological, quasitopological) group, then G is $(\overline{\mathbb{R}}, \mathbb{SCTG}_r)$ $((\overline{\mathbb{R}}, \mathbb{SCPG}_r)$, $(\overline{\mathbb{R}}, \mathbb{SCSG}_r)$, $(\overline{\mathbb{R}}, \mathbb{SCQG}_r))$-factorizable if and only if G is $(\overline{\mathbb{R}}, \mathbb{SCTG})$ $((\overline{\mathbb{R}}, \mathbb{SCPG})$, $(\overline{\mathbb{R}}, \mathbb{SCSG})$, $(\overline{\mathbb{R}}, \mathbb{SCQG}))$-factorizable. In Theorem 4.19 we show that if G is a regular semitopological group with a q-point, property (***) and satisfying $Sm(G) \leq \omega$, then G is topologically isomorphic to a subgroup of the product of a family of separable metrizable semitopological groups.

The set of all positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. In notation and terminology we will follow [1] and [5].

2. On $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$- and $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizable semitopological groups

A collection $\mathcal{B}$ of nonempty subsets of a space X is a *quasi-base* for X if, whenever $x \in X$ and U is a neighborhood of x, then there exists a $B \in \mathcal{B}$ such that $x \in B^\circ \subset B \subset U$, where B° denotes the interior of B in X. An M_2-*space* is a regular space with a σ-closure preserving quasi-base [4]. A space X is *stratifiable* if and only if X is an M_2-space [6, 8]. Let $\mathcal{P}$ be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of a space X with $P_1 \subset P_2$ for all $P \in \mathcal{P}$. Then $\mathcal{P}$ is called a *pair-base* for X if P_1 is open for all $P \in \mathcal{P}$ and if, for any $x \in X$ and any neighborhood U of x, there exists a $P \in \mathcal{P}$ such that $x \in P_1 \subset P_2 \subset U$ [4]. Moreover, $\mathcal{P}$ is called *cushioned* if for every $\mathcal{P}' \subset \mathcal{P}$, $\overline{\bigcup\{P_1 : P \in \mathcal{P}'\}} \subset \bigcup\{P_2 : P \in \mathcal{P}'\}$. $\mathcal{P}$ is called σ-*cushioned* if it is the union of countably many cushioned subcollections [4]. An M_3-*space* is a T_1-space with a σ-cushioned pair-base [4]. The classes of M_3- and M_2-spaces are equivalent ([10], Theorem VI, 28). Every first-countable M_3-space is a M_1-space [7]. A first-countable M_3-space is sometimes called a *Nagata space* ([10], p. 354). A first-countable M_1-semitopological (paratopological) group is called a Nagata semitopological (paratopological) group [11]. An internal characterization of subgroups of products of first-countable M_1-semitopological groups is given in [11].

In ([1], p. 539), it is pointed out that every m-factorizable topological group is $\mathbb{R}$-factorizable. Similarly, the following two statements are immediate from the fact $\overline{\mathbb{R}} \subset \mathbb{MS}$.

Proposition 2.1. *If G is a $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizable semitopological group, then G is $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$-factorizable.*

Corollary 2.2. *If G is a $(\mathbb{MS}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable semitopological group, then G is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\overline{\mathbb{R}}, \mathbb{SCSG}))$-factorizable.*

Recall that a space X is said to be *pseudo-$\aleph_1$-compact* if every discrete family of open sets in X is countable ([1], p. 538). A Hausdorff topological group G is m-factorizable if and only if G is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact ([1], Theorem 8.5.2).

A class C of spaces is a *CPS-class* if it contains countable products of its elements, is hereditary with respect to taking subspaces, and contains a one-point space [12]. By an argument similar to the one used in the proof of Lemma 8.1.2 in [1], we have the following result.

Lemma 2.3. *Let $\mathcal{P}$ be a CPS-*class *of semitopological (paratopological) groups. If $f : G \to X$ is a continuous mapping of an* $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$ $((\overline{\mathbb{R}}, \mathcal{P}\mathbb{PG}))$*-factorizable semitopological (paratopological) group G to a Tychonoff space X with $w(X) \leq \omega$, then one can find a continuous homomorphism $\pi : G \to K$ onto a semitopological (paratopological) group $K \in \mathcal{P}$ and a continuous mapping $g : K \to X$ such that $f = g \circ \pi$.*

Proof. We just prove the lemma in the case of semitopological groups. The proof in the other case is similar. According to ([5], Theorem 2.3.23), we can identify X with a subspace of $\mathbb{R}^\omega$. For every $n < \omega$, denote by p_n the projection of $\mathbb{R}^\omega$ to the nth factor. Then $p_n \circ f : G \to \mathbb{R}$ is a continuous mapping, so we can find a continuous homomorphism $\pi_n : G \to K_n$ onto a semitopological group $K_n \in \mathcal{P}$ and a continuous real-valued function g_n on K_n such that $p_n \circ f = g_n \circ \pi_n$. Denote by π the diagonal product of the homomorphisms π_n. Then $\pi : G \to \prod_{n<\omega} K_n$ is a continuous homomorphism and the image $K = \pi(G)$ is a subgroups of the semitopological group $\Pi = \prod_{n<\omega} K_n$. Since $K_n \in \mathcal{P}$ for every $n \in \omega$ and $\mathcal{P}$ is a *CPS*-class, we have $K \in \mathcal{P}$. For every $n < \omega$, let $q_n : \Pi \to K_n$ be the projection. Then $\pi_n = q_n \circ \pi$ for each $n < \omega$. Finally, denote by g the Cartesian product of the functions g_n, $n < \omega$. Then the mapping $g : \Pi \to \mathbb{R}^\omega$ is continuous.

From the definitions of g and π, we have $p_n \circ g \circ \pi = g_n \circ q_n \circ \pi = g_n \circ \pi_n = p_n \circ f$ for every $n \in \omega$. Thus $p_n \circ f = p_n \circ g \circ \pi$ for every $n \in \omega$. Then $f = g \circ \pi$, where π is a continuous homomorphism and the mapping g is continuous. Therefore, the homomorphism $\pi : G \to K$ and the mapping $h = g \upharpoonright K$ satisfy the equality $f = h \circ \pi$. $\square$

Proposition 2.4. *Let $\mathcal{P}$ be a CPS-class of semitopological groups. If G is an* $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$*-factorizable pseudo-$\aleph_1$-compact semitopological group, then G is* $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$*-factorizable.*

Proof. Let $f : G \to M$ be a continuous mapping of G onto a metrizable space M. Since G is pseudo-$\aleph_1$-compact and the mapping f is continuous, M is pseudo-$\aleph_1$-compact. Then M is a pseudo-$\aleph_1$-compact metrizable space. Thus M is a second-countable space. By Lemma 2.3, there exist a continuous homomorphism $\pi : G \to K$ onto a semitopological group $K \in \mathcal{P}$ and a continuous mapping $g : K \to X$ such that $f = g \circ \pi$. Thus G is $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$-factorizable. $\square$

By results of [3], $\mathbb{FM}_1$ is a *CPS*-class. Then by Proposition 2.4 we have the following result.

Corollary 2.5. *If G is an* $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\overline{\mathbb{R}}, \mathbb{SCSG}))$*-factorizable pseudo-$\aleph_1$-compact semitopological group, then G is* $(\mathbb{MS}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{SCSG}))$*-factorizable.*

Some properties of $(\overline{\mathbb{R}}, \mathbb{FSG})$ $((\mathbb{MS}, \mathbb{FSG}))$-factorizable semitopological groups are discussed in [12]. It is obvious that $\mathbb{F}$ is a *CPS*-class. Again, by Proposition 2.4, we have the following result.

Corollary 2.6. *If G is an* $(\overline{\mathbb{R}}, \mathbb{FSG})$*-factorizable pseudo-$\aleph_1$-compact semitopological group, then G is* $(\mathbb{MS}, \mathbb{FSG})$*-factorizable.*

By Lemma 8.5.4 in [1], we have the following result.

Lemma 2.7. *Let G be a subgroup of a product $\Pi = \prod_{i\in I} G_i$ of semitopological groups and $\pi : G \to H$ be a continuous homomorphism to a semitopological group H satisfying $\chi(H) \leq \kappa$. Then one can find a set $J \subset I$ with $|J| \leq \kappa$ and a continuous homomorphism $\varphi : p_J(G) \to H$ such that $\pi = \varphi \circ p_J|G$, where p_J is the projection of Π to $\Pi_J = \prod_{i\in J} G_i$. If, in particular, the space H is first-countable, then the set J can be chosen to be countable.*

If G is a Tychonoff semitopological group such that $G\times\mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}},\mathbb{FSG})$-factorizable, where $\mathbb{Z}(2)=\{0,1\}$ is the discrete group, then G is $(\mathbb{MS},\mathbb{FSG})$-factorizable ([12], Theorem 3.5). In what follows, we give a generalization of this result. First, we need some basic notions and results.

Recall that a subspace Y of a space X is said to be *C-embedded* in X if every continuous real-valued function on Y can be extended to a continuous real-valued function on X. Recall that a continuous mapping $f: X\to X$ is called a *retraction* of X, if $f\circ f=f$; the set of all values of a retraction of X is called a *retract* of X ([5], Exercise 1.5C). A subspace M of a topological space X is a retract of X if and only if every continuous mapping defined on M extendable over X ([5], Exercise 2.1D). Thus a retract of a space X is C-embedded in X.

Proposition 2.8. *Let $\mathcal{P}$ be a topological property which is hereditary with respect to taking subspaces. If a subgroup H of an $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$ $((\mathbb{MS},\mathcal{P}\mathbb{SG}))$-factorizable semitopological group G is C-embedded in G, then H is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$ $((\mathbb{MS},\mathcal{P}\mathbb{SG}))$-factorizable.*

Proof. We just consider the case of $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizability. The proof in the other case is similar. Let $f: H\to\mathbb{R}$ be any continuous real-valued function. Since H is C-embedded in G, there exists a continuous function $g: G\to\mathbb{R}$ such that $g|H=f$. Since G is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable, there exist a continuous homomorphism $\pi: G\to K$ onto a semitopological group $K\in\mathcal{P}$ and a continuous function $h: K\to\mathbb{R}$ such that $g=h\circ\pi$. If $h_1=h|\pi(H):\pi(H)\to\mathbb{R}$, then the mapping h_1 is continuous. If $\pi_1=\pi|H: H\to\pi(H)$, then the mapping π_1 is a continuous homomorphism onto the semitopological group $\pi(H)$ such that $f=h_1\circ\pi_1$. Since $K\in\mathcal{P}$ and $\mathcal{P}$ is hereditary with respect to taking subgroups, $\pi(H)\in\mathcal{P}$. Then G is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable. □

By a proof similar to the one used in b) in Theorem 8.5.5 in [1], we have the following result. To assist the reader, we give the proof of it.

Proposition 2.9. *Let G be a Tychonoff semitopological group. Let $\mathbb{Z}(2)=\{0,1\}$ be the discrete group. If $G\times\mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}},\mathbb{FSG})$-factorizable, then G is pseudo-$\aleph_1$-compact.*

Proof. Suppose that the semitopological group $G\times K$ is $(\overline{\mathbb{R}},\mathbb{FSG})$-factorizable, where $K=\mathbb{Z}(2)^{\omega_1}$. Denote by e_K the identity of the group K. It is clear that G is topologically isomorphic to $G\times\{e_K\}$ and G is a retract of $G\times K$. So Proposition 2.8 implies that the semitopological group G is $(\overline{\mathbb{R}},\mathbb{FSG})$-factorizable. Let us show that G is pseudo-$\aleph_1$-compact.

Assume the contrary. Then G contains a discrete family $\{U_\alpha:\alpha<\omega_1\}$ of nonempty open sets. Since G is a Tychonoff space, for every $\alpha<\omega_1$, one can choose a point $x_\alpha\in U_\alpha$ and define a continuous function $f_\alpha: G\to[0,1]$ such that $f_\alpha(x_\alpha)=1$ and $f_\alpha(x)=0$ if $x\in G\backslash U_\alpha$. We also consider the function $h:\mathbb{Z}(2)\to[0,1]$ such that $h(0)=0$ and $h(1)=1$. Let $\pi: G\times K\to G$ be the projection. If $y\in\mathbb{Z}(2)^{\omega_1}$ and $\alpha<\omega_1$, denote by y_α the α-th coordinate of y. For every $\alpha<\omega_1$, we define a function $g_\alpha: G\times K\to[0,1]$ as follows: If $(x,y)\in G\times K$, then $g_\alpha(x,y)=f_\alpha(x)\cdot h(y_\alpha)$. Clearly, g_α is continuous. Since the family $\{U_\alpha:\alpha<\omega_1\}$ is discrete, the function $g=\sum_{\alpha<\omega_1}g_\alpha$ is also continuous on $G\times K$. By $(\overline{\mathbb{R}},\mathbb{FSG})$-factorizability of $G\times K$, we can find a continuous homomorphism $\varphi: G\times K\to L$ to a first-countable semitopological group L and a continuous function $\widetilde{g}: L\to\mathbb{R}$ such that $g=\widetilde{g}\circ\varphi$. From Lemma 2.7 it follows that there exists a countable subset J of ω_1 and a continuous homomorphism $\psi: G\times\mathbb{Z}(2)^J\to L$ such that $\varphi=\psi\circ(id_G\times p_J)$, where id_G is the identity automorphism of G and $p_J:\mathbb{Z}(2)^{\omega_1}\to\mathbb{Z}(2)^J$ is the projection.

Since $g=\widetilde{g}\circ\varphi$ and $\varphi=\psi\circ(id_G\times p_J)$, we conclude that if $x\in G$ and $y,y'\in\mathbb{Z}(2)^{\omega_1}$ satisfy $p_J(y)=p_J(y')$ then $g(x,y)=g(x,y')$. Choose an ordinal $\alpha\in\omega_1\backslash J$. Now we define two points $y,y'\in\mathbb{Z}(2)^{\omega_1}$ by $y_\beta=y'_\beta=0$ if $\beta\neq\alpha$ and $y_\alpha=0, y'_\alpha=1$. Evidently, $p_J(y)=p_J(y')$. A simple calculation shows that $g(x_\alpha,y)=h(y_\alpha)=0$ and $g(x_\alpha,y')=h(y'_\alpha)=1$; it follows that $g(x_\alpha,y)\neq g(x_\alpha,y')$. This contradiction shows that G is pseudo-$\aleph_1$-compact. □

Theorem 2.10. *Let $\mathcal{P}$ be a topological property such that every space with $\mathcal{P}$ is first-countable and $\mathcal{P}$ is inherited to subspaces. Let also G be a Tychonoff semitopological group such that $G\times\mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable, where $\mathbb{Z}(2)=\{0,1\}$ is the discrete group. Then G is $(\mathbb{MS},\mathcal{P}\mathbb{SG})$-factorizable.*

Proof. Since G is a Tychonoff semitopological group such that $G\times\mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable and $\mathcal{P}\subset\mathbb{F}$, the group G is pseudo-$\aleph_1$-compact, by Proposition 2.9. Denote $K=\mathbb{Z}(2)^{\omega_1}$. Let e_K be the identity of the group K. Since G is topologically isomorphic to $G\times\{e_K\}$ and $G\times\{e_K\}$ is a retract of $G\times K$, by Proposition 2.8 G is $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable. Hence G is a $(\overline{\mathbb{R}},\mathcal{P}\mathbb{SG})$-factorizable pseudo-$\aleph_1$-compact semitopological group. By Proposition 2.4, G is $(\mathbb{MS},\mathcal{P}\mathbb{SG})$-factorizable. □

Corollary 2.11. *If G is a Tychonoff semitopological group such that $G\times\mathbb{Z}(2)^{\omega_1}$ is $(\overline{\mathbb{R}},\mathbb{FM}_1\mathbb{SG})$ $((\overline{\mathbb{R}},\mathbb{SCSG}))$-factorizable, then G is $(\mathbb{MS},\mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS},\mathbb{SCSG}))$-factorizable.*

3. On certain continuous homomorphic images of $(\overline{\mathbb{R}},\mathbb{FM}_1\mathbb{SG})$- and $(\mathbb{MS},\mathbb{FM}_1\mathbb{SG})$-factorizable semitopological groups

A family $\mathcal{U}$ of nonempty subsets of a semitopological group G is *dominated by a family* $\gamma\subset\mathcal{N}(e)$ if for every $U\in\mathcal{U}$ and $x\in U$ there exists $V\in\gamma$ such that $xV\subset U$ [17]. A family $\mathcal{V}$ of subsets of a set X is a *weak refinement* of a cover $\mathcal{U}$ of X if $\mathcal{V}$ contains a subfamily which is a cover of X and a refinement of $\mathcal{U}$ [17].

To give an internal characterization of subgroups of products of first-countable M_1-semitopological groups, the following notions are introduced in [11].

Let G be a semitopological group and let $A\subset G$ and $\gamma\subset\mathcal{N}(e)$. The closure of A with respect to γ is denoted by $\overline{A}^{(\gamma)}=\{x\in G: \text{for every } V\in\gamma,\ xV\cap A\neq\emptyset\}$.

A semitopological group G is said to have *property* $(c*)$ (resp., (M_3*)) ([11], Definition 2.1 (3.4)) if for every open neighborhood U of the identity e of G, the family $\{Ux: x\in G\}$ has a weak refinement $\mathcal{V}$ of open subsets of G such that the following properties hold:

(1) $\mathcal{V}$ is dominated by a countable family $\gamma\subset\mathcal{N}(e)$;
(2) For every $x\in G$, there exists $F_x\in\mathcal{V}$ such that $x\in F_x\subset Ux$ and there exists a cover $\{G_n: n\in\omega\}$ of G such that for any $n\in\omega$ and any $A_n\subset G_n$, $\overline{\bigcup\{F_x: x\in A_n\}}\subset\bigcup\{Ux: x\in A_n\}$ (resp., $\overline{\bigcup\{F_x: x\in A_n\}}^{(\gamma)}\subset\bigcup\{Ux: x\in A_n\}$).

Lemma 3.1. ([11], Proposition 3.5) *If G is a semitopological group with property (M_3*), then G has property $(c*)$.*

Lemma 3.2. ([11], Proposition 3.9) *Every Nagata semitopological group has property (M_3*).*

By Lemmas 3.1 and 3.2, every Nagata semitopological group has $(c*)$.

Lemma 3.3. ([11], Theorem 3.14) *Let G be a semitopological (paratopological) group. Then G is topologically isomorphic to a subgroup of the product of a family of Nagata semitopological (paratopological) groups if and only if G satisfies the T_0-separation axiom and has property (M_3*).*

Every open continuous homomorphic image of an $\mathbb{R}$-factorizable topological group is $\mathbb{R}$-factorizable ([26], Theorem 3.10). Peng and Zhang proved that every open continuous homomorphic image of an $\mathbb{R}$-factorizable paratopological group is $\mathbb{R}$-factorizable ([15], Theorem 1.7). In ([33], Corollary 3.8), it is proved that every quotient group of an $\mathcal{M}$-factorizable topological group is $\mathcal{M}$-factorizable. In [12], it is proved that if G is a Tychonoff $(\overline{\mathbb{R}},\mathbb{FSG})$ $((\mathbb{MS},\mathbb{FSG}))$-factorizable semitopological group, then every continuous open homomorphic image of G is $(\overline{\mathbb{R}},\mathbb{FSG})$ $((\mathbb{MS},\mathbb{FSG}))$-factorizable.

Now we discuss certain continuous homomorphic images of $(\overline{\mathbb{R}},\mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS},\mathbb{FM}_1\mathbb{SG}))$-factorizable semitopological groups.

Lemma 3.4. *If $f: X\to Y$ is an open continuous mapping and x is a q-point of X, then $f(x)$ is a q-point of Y.*

Proof. Since x is a q-point of X, there exists a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of x in X such that any sequence $\{x_n\}_{n\in\omega}$ with $x_n \in U_n$ for every $n \in \omega$ has an accumulation point in X. Since the mapping f is open, $\{f(U_n) : n \in \omega\}$ is a sequence of open neighborhoods of $f(x)$ in Y. If $\{y_n\}_{n\in\omega}$ is any sequence of points of Y such that $y_n \in f(U_n)$ for every $n \in \omega$, then there exists $x_n \in U_n$ such that $f(x_n) = y_n$ for every $n \in \omega$. Then the sequence $\{x_n\}_{n\in\omega}$ has an accumulation point a in X. Since the mapping f is continuous, $f(a)$ is an accumulation point of the sequence $\{y_n\}_{n\in\omega}$ in Y. □

By Lemma 3.4, we have the following result.

Corollary 3.5. *Let G and H be semitopological groups. If $f : G \to H$ is an open continuous homomorphism and G has a q-point, then H is a q-space.*

Lemma 3.6. *Let G and H be semitopological groups. If $f : G \to H$ is a continuous closed and open surjective homomorphism and G has $(c*)$, then H has property $(c*)$.*

Proof. Let e_H be the identity of H and let U be any element of $\mathcal{N}(e_H)$. Since the mapping f is a continuous homomorphism, the set $V = f^{-1}(U)$ is an open neighborhood of the identity e of G. Since G has property $(c*)$, there exists a family $\mathcal{V}_V$ of open subsets of G such that the following properties hold:

(1) $\mathcal{V}_V$ is dominated by a countable family $\gamma \subset \mathcal{N}(e)$;
(2) For every $x \in G$, there exists $F_x \in \mathcal{V}_V$ such that $x \in F_x \subset Vx$ and there exists a cover $\{G_m : m \in \omega\}$ of G such that for any $m \in \omega$ and any $A_m \subset G_m$, $\overline{\bigcup\{F_x : x \in A_m\}} \subset \bigcup\{Vx : x \in A_m\}$.

For any $y \in H$, there exists $x_y \in G$ such that $f(x_y) = y$. Thus $y \in f(F_{x_y}) \subset f(V)f(x_y) = Uy$ for every $y \in H$. Denote $\gamma_U = \{f(W) : W \in \gamma\}$. Since $\gamma \subset \mathcal{N}(e)$ is countable and the mapping f is open, γ_U is a countable subfamily of $\mathcal{N}(e_H)$. Since $\mathcal{V}_V$ is a family of open subsets in G and $\mathcal{V}_V$ is dominated by γ, the family $\mathcal{V}_U = \{f(B) : B \in \mathcal{V}_V\}$ is a family of open subsets in H and $\mathcal{V}_U$ is dominated by γ_U. For every $m \in \omega$, let $H_m = \{y \in H : x_y \in G_m\}$. Then $H = \bigcup\{H_m : m \in \omega\}$.

For any $m \in \omega$ and any $C_m \subset H_m$, we have $\{x_y : y \in C_m\} \subset G_m$. Thus $\overline{\bigcup\{F_{x_y} : y \in C_m\}} \subset \bigcup\{Vx_y : y \in C_m\}$.

Then $f(\overline{\bigcup\{F_{x_y} : y \in C_m\}}) \subset \bigcup\{f(Vx_y) : y \in C_m\} = \bigcup\{Uy : y \in C_m\}$. Since the mapping f is closed, the set $f(\overline{\bigcup\{F_{x_y} : y \in C_m\}})$ is a closed subset of H. Then $\overline{\bigcup\{f(F_{x_y}) : y \in C_m\}} \subset f(\overline{\bigcup\{F_{x_y} : y \in C_m\}}) \subset \bigcup\{Uy : y \in C_m\}$.

Thus the semitopological group H has property $(c*)$. □

Proposition 3.7. *Every Tychonoff $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group has property (M_3*).*

Proof. Let $\{f_i : i \in I\}$ be the family of continuous real-valued functions on an $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group G. For every $i \in I$, there exist a continuous homomorphism $\pi_i : G \to H_i$ onto a Nagata semitopological group H_i and a continuous mapping $g_i : H_i \to \mathbb{R}$ such that $f_i = g_i \circ \pi_i$. Since G is Tychonoff, the family $\{f_i : i \in I\}$ separates points and closed sets in G. Then the family $\{\pi_i : i \in I\}$ separates points and closed sets in G. If $\pi : G \to \prod_{i\in I} H_i$ is the diagonal product of the family $\{\pi_i : i \in I\}$, then the mapping π is a topological monomorphism of G onto the subgroup $H = \pi(G)$ of $\prod_{i\in I} H_i$. By Lemma 3.3, G has property (M_3*). □

By Lemma 3.1 and Proposition 3.7, we have the following result.

Corollary 3.8. *Every Tychonoff $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group has property $(c*)$.*

Recall that a real-valued function f on a paratopological group G is *left (resp., right) ω-quasi-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U} \subset \mathcal{N}(e)$ such that for every $x \in G$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$ (resp., $yx^{-1} \in U$) ([30], Definition 4.1). A real-valued function f on a paratopological group G is *ω-quasi-uniformly continuous* if f is both left and right ω-quasi-uniformly continuous ([30], Definition 4.2). A paratopological group G has *property ω-QU* if each continuous real-valued function on G is ω-quasi-uniformly continuous ([30], Definition 4.7). If we replace 'paratopological group' in the above definitions with 'semitopological group', then we get the corresponding definitions in the class of semitopological groups [14].

Let G be a semitopological group. A mapping $f : G \to M$ to a metric space (M, d) is *left (resp., right) $d\omega$-quasi-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U} \subset \mathcal{N}(e)$ such that for every $x \in G$, there exists $U \in \mathcal{U}$ such that $d(f(x), f(y)) < \varepsilon$ whenever $x^{-1}y \in U$ (resp., $yx^{-1} \in U$) [12]. A mapping $f : G \to M$ to a metric space (M, d) is *$d\omega$-quasi-uniformly continuous* if f is both left and right $d\omega$-quasi-uniformly continuous [12]. A semitopological group G has property *$d\omega$-QU* if each continuous mapping $f : G \to M$ to a metric space (M, d) is $d\omega$-quasi-uniformly continuous [12].

Lemma 3.9. ([12], Proposition 2.20) *Every $(\overline{\mathbb{R}}, \mathbb{FSG})$-factorizable semitopological group has property ω-QU and every $(\mathbb{MS}, \mathbb{FSG})$-factorizable semitopological group has property $d\omega$-QU.*

The next fact is immediate from the definition of the class $\mathbb{FM}_1$.

Proposition 3.10. *Every $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$ $((\mathbb{MS}, \mathbb{FM}_1\mathbb{SG}))$-factorizable semitopological group is $(\overline{\mathbb{R}}, \mathbb{FSG})$ $((\mathbb{MS}, \mathbb{FSG}))$-factorizable.*

By Lemma 3.9 and Proposition 3.10, we have the following result.

Corollary 3.11. *Every $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group has property ω-QU and every $(\mathbb{MS}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group has property $d\omega$-QU.*

Given a topological property $\mathcal{P}$, we say that a semitopological group G is *projectively $\mathcal{P}$* if for every neighborhood U of the identity in G there exists a continuous homomorphism $p : G \to H$ onto a semitopological group H with property $\mathcal{P}$ such that $p^{-1}(V) \subset U$, for some neighborhood V of the identity in H [23]. If $\mathcal{P}$ is any class of semitopological groups, then a projectively $\mathcal{P}$ semitopological group is also called *range-$\mathcal{P}$* ([1], p. 168). The only difference between the two definitions is that the mapping p in the definition of range-$\mathcal{P}$ is not required to be surjective.

Lemma 3.12. ([14], Lemma 3.7) *Let G be a semitopological (paratopological) group and $\mathcal{P}$ be a CPS-class of semitopological (paratopological) groups. If G has property ω-QU and is projectively $\mathcal{P}$, then G is $(\overline{\mathbb{R}}, \mathcal{P}\mathbb{SG})$ $((\overline{\mathbb{R}}, \mathcal{P}\mathbb{PG}))$-factorizable.*

Lemma 3.13. ([32], Proposition 2.4) *Let G and H be semitopological groups. If $G \to H$ is a continuous open surjective homomorphism and G has property ω-QU, then H has property ω-QU.*

Theorem 3.14. *Let G and H be semitopological groups. If $f : G \to H$ is a closed and open continuous surjective homomorphism such that G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group with a q-point and H satisfies $Sm(H) \leq \omega$, then H is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable.*

Proof. By Corollary 3.8, G has property $(c*)$. Since $f : G \to H$ is a closed and open continuous surjective homomorphism and G has property $(c*)$, by Lemma 3.6 H has property $(c*)$. Since G satisfies the T_1-separation axiom and f is a closed surjective mapping, H satisfies the T_1-separation axiom. Since G has a q-point and the mapping f is open, it follows from Lemma 3.4 that H has a q-point. Then H is a T_1 semitopological group which has property $(c*)$, contains a q-point and satisfies $Sm(H) \leq \omega$. It follows from Lemma 2.8 in [11] that H is projectively $\mathcal{P}$, where $\mathcal{P}$ is the class of Nagata semitopological groups. Since G is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable, it follows from Corollary 3.11 that G has property ω-QU. Then by Lemma 3.13 H has property ω-QU. The product of a countable family of stratifiable spaces is stratifiable [3]. Thus the class $\mathbb{FM}_1$ is a CPS-class. Thus by Lemma 3.12, H is $(\overline{\mathbb{R}}, \mathbb{FM}_1\mathbb{SG})$-factorizable. □

Lemma 3.15. ([12], Proposition 3.22) *Let G be a semitopological (paratopological) group and let $\mathcal{P}$ be a CPS-class of semitopological (paratopological) groups. If G has property $d\omega$-QU and is projectively $\mathcal{P}$, then G is $(\mathbb{MS}, \mathcal{P}\mathbb{SG})$ $((\mathbb{MS}, \mathcal{P}\mathbb{PG}))$-factorizable.*

Lemma 3.16. ([12], Lemma 3.28) *Let G and H be semitopological groups. If $G \to H$ is a continuous open surjective homomorphism and G has property $d\omega$-QU, then H has property $d\omega$-QU.*

By Lemmas 3.15 and 3.16, Corollary 3.11 and an argument which is similar to the one used in the proof of Theorem 3.14, we have the following result.

Theorem 3.17. *Let G and H be semitopological groups. If $f : G \to H$ is a continuous closed and open surjective homomorphism such that G is a Tychonoff $(\mathbb{MS}, \mathbb{FM}_1\mathbb{SG})$-factorizable semitopological group with a q-point and H satisfies $Sm(H) \leq \omega$, then H is $(\mathbb{MS}, \mathbb{FM}_1\mathbb{SG})$-factorizable.*

4. On $(\overline{\mathbb{R}}, \mathbb{SCSG})$- and $(\mathbb{MS}, \mathbb{SCSG})$-factorizable semitopological groups

The notions of $\mathbb{R}_i$-factorizablility paratopological groups are introduced in [19] for $i = 1, 2, 3, 3.5$. In [19], $\mathbb{R}_i$-factorizable paratopological groups are assumed to satisfy the T_i-separation axiom. In [31], $\mathbb{R}_i$-factorizable paratopological groups are defined as follows. A paratopological group G is $\mathbb{R}_0$-*factorizable* ($\mathbb{R}_i$-*factorizable*, for $i = 1, 2, 3, 3.5$) if for every continuous real-valued function f on G, one can find a continuous homomorphism $\pi : G \to H$ onto a second-countable paratopological group H satisfying the T_0 (resp., $T_i + T_1$) separation axiom and a continuous real-valued function h on H such that $f = h \circ \pi$ [31]. If we do not impose any separation restriction on H, we obtain the concept of $\mathbb{R}$-factorizability [31]. Thus the $\mathbb{R}$-factorizability of paratopological groups defined in [31] is just the $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizability defined in this article.

In ([31], Theorem 3.8), it is pointed out that the concepts of $\mathbb{R}$-, $\mathbb{R}_0$-, $\mathbb{R}_1$-, $\mathbb{R}_2$- and $\mathbb{R}_3$-factorizability coincide in the class of paratopological groups. Each regular paratopological group is completely regular [2].

Thus, by Theorem 3.8 in [31], we have the following result.

Proposition 4.1. *Let G be a paratopological group. Then the following statements are equivalent:*

(1) G is $\mathbb{R}_i$-factorizable, where $i \in \{0, 1, 2, 3, r, 3.5\}$;
(2) G is $\mathbb{R}$-factorizable;
(3) G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable;
(4) G is $(\overline{\mathbb{R}}, \mathbb{SCPG}_r)$-factorizable.

A class C of spaces is a *PS-class* if it contains arbitrary products of its elements, is hereditary with respect to taking subspaces, and contains a one-point space [27]. Let C be a *PS*-class of spaces and $\varphi_G^C : G \to H$ a continuous surjective homomorphism of semitopological groups. The pair (H, φ_G^C) is called a *C-reflection* of G if $H \in C$ and for every continuous mapping $f : G \to X$ to a space $X \in C$, there exists a continuous mapping $h : H \to X$ such that $f = h \circ \varphi_G^C$ ([27], Definition 1.1). For every semitopological group G and every $i \in \{0, 1, 2, 3, 3.5\}$, there exists the T_i-reflection $(T_i(G), \varphi_{G,i})$ of G. Similarly, there exist the regular reflection $(Reg(G), \varphi_{G,r})$ and the Tychonoff reflection $(Tych(G), \varphi_{G,t})$ of G. The homomorphism $\varphi_{G,i}$ is open for each $i = 0, 1, 2$ ([27], Proposition 2.5).

A subset U of a space X is called *regular open* if $U = \overline{U}^{\circ}$. Given a space (X, τ), denote by τ' the topology on X whose base consists of regular open subsets of (X, τ). The space (X, τ') is said to be the *semiregularization* of (X, τ) and is denoted by X_{sr}. It is easy to see that $\tau' \subset \tau$ and the spaces (X, τ) and (X, τ') have the same regular open subsets. The operation of semiregularization was defined by M.H. Stone in [21] and studied by M. Katetov [9]. A space whose regular open subsets form a base for its topology is called *semiregular* ([18], p. 44). If G is an arbitrary paratopological group, then so is G_{sr} ([28], Theorem 2.2).

Theorem 4.2. *Let G be a topological (semitopological, quasitopological) group. Then G is $(\overline{\mathbb{R}}, \mathbb{SCTG}_r)$ ($(\overline{\mathbb{R}}, \mathbb{SCSG}_r)$, $(\overline{\mathbb{R}}, \mathbb{SCQG}_r)$)-factorizable if and only if G is $(\overline{\mathbb{R}}, \mathbb{SCTG})$ ($(\overline{\mathbb{R}}, \mathbb{SCSG})$, $(\overline{\mathbb{R}}, \mathbb{SCQG})$)-factorizable.*

Proof. The necessity is obvious. We just need to prove the sufficiency.

Assume that G is an $(\overline{\mathbb{R}}, \mathbb{SCTG})$-factorizable topological group. Let f be any continuous real-valued function on G. Since G is an $(\overline{\mathbb{R}}, \mathbb{SCTG})$-factorizable topological group, there exists a continuous homomorphism φ of G onto a second-countable topological group H and a continuous real-valued function g on H such that $f = g \circ \varphi$. Let K be the smallest closed subgroup of H. It follow from Theorem 3.4 in [27] that K is invariant in G, the quotient mapping $\pi_H : H \to H/K$ is a continuous open homomorphism, and $H/K \cong T_1(H)$. Since H is a second-countable topological group and K is closed in H, the quotient group H/K is a second-countable T_1 topological group. Hence $H/K \cong T_1(H)$ is separable and metrazible. Then there exists a continuous real-valued function g_1 on H/K such that $g = g_1 \circ \pi_H$. Thus $f = g_1 \circ (\pi_H \circ \varphi)$ and $\pi_H \circ \varphi : G \to H/K$ is a continuous surjective homomorphism. Then G is $(\overline{\mathbb{R}}, \mathbb{SCTG}_r)$-factorizable.

Now we prove the theorem in the case of semitopological groups. We apply an argument similar to the one used in the above paragraph and the following results. The regular reflection $Reg(H)$ of a semitopological group H satisfies $Reg(H) = T_1(T_3(H))$ ([27], Theorem 3.8). $T_3(H)$ is the semiregularization of H, i.e., $T_3(H) \cong H_{sr}$, for every semitopological group H ([29], Theorem 3.4). If X is a second-countable space, then so is X_{sr} ([31], Lemma 3.6). Thus, $Reg(H)$ is a regular second-countable semitopological group if H is a second-countable semitopological group.

In the case of quasitopological groups, we use the following facts. The regular reflection $Reg(H)$ of a quasitopological group H satisfies $Reg(H) = T_1(T_3(H))$ ([22], Corollary 2.6). If H is a quasitopological group, then $T_3(H) \cong H_{sr}$ ([22], Theorem 2.4) and H_{sr} is a quasitopological group ([22], Lemma 2.3). If X is a second-countable space, then so is X_{sr} ([31] Lemma 3.6). Thus, $Reg(H)$ is a regular second-countable quasitopological group if H is a second-countable quasitopological group. □

Similarly, we have the following result.

Proposition 4.3. *If G is a topological (paratopological, semitopological, quasitopological) group, then G is $(\mathbb{MS}, \mathbb{SCTG}_r)$ $((\mathbb{MS}, \mathbb{SCPG}_r), (\mathbb{MS}, \mathbb{SCSG}_r), (\mathbb{MS}, \mathbb{SCQG}_r))$-factorizable if and only if G is $(\mathbb{MS}, \mathbb{SCTG})$ $((\mathbb{MS}, \mathbb{SCPG}), (\mathbb{MS}, \mathbb{SCSG}), (\mathbb{MS}, \mathbb{SCQG}))$-factorizable.*

An internal characterization of projectively T_i second-countable semitopological groups, for $i = 0, 1, 2$, is given in [13].

Now let us recall some notions. The *invariance number* $inv(G)$ of a semitopological group G is countable if for every open neighborhood U of the identity e in G, there exists a countable family $\mathcal{V}$ of open neighborhoods of e such that for each $x \in G$ there exists some $V \in \mathcal{V}$ such that $xVx^{-1} \subset U$ [1]. Any such a family $\mathcal{V}$ is called *subordinated* to U [1]. A topological (paratopological, semitopological) group G with $inv(G) \leq \omega$ is also called *ω-balanced* ([1], [23], [16]). Let $\mathcal{U}$ be a family of subsets of a space. A refinement $\mathcal{V}$ of $\mathcal{U}$ is called *basic* if for every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \mathcal{V}$ such that $x \in V \subset U$ [17]. A subset V of a semitopological group G is called *ω-good* if there exists a countable family $\mathcal{V} \subset \mathcal{N}(e)$ such that for every $x \in V$, there exists $W \in \mathcal{V}$ with $xW \subset V$ ([16], [20]). A semitopological group G is *locally ω-good* if the family $\mathcal{N}^*(e)$ of ω-good sets containing the identity e is a local base at e [16].

A semitopological group G is said to have *property* (***) [13] if for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic countable refinement which is dominated by a countable family γ. For a Hausdorff semitopological group G with the identity e, the *Hausdorff number* of G, denoted by $Hs(G)$, is the mininum cardinal number κ such that for every neighborhood U of e in G, there exists a family γ of open neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subset U$ and $|\gamma| \leq \kappa$ [23].

Lemma 4.4. ([13], Theorem 30) *Let G be a Hausdorff semitopological group. Then G admits a homeomorphic embedding as a subgroup into a product of Hausdorff second-countable semitopological groups if and only if G has property* (***) *and $Hs(G) \leq \omega$.*

Proposition 4.5. ([13], Proposition 27) *Every second-countable semitopological group has property* (***).

Proposition 4.6. *If G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable semitopological group, then G has property* (***).

Proof. By an argument similar to the one used in the proof of Proposition 8.1.3 in [1] (or Proposition 3.7), we deduce that G admits a homeomorphic embedding as a subgroup into a product of second-countable semitopological groups. By Lemma 4.4, G has property (***). □

Lemma 4.7. *Let G and H be semitopological groups. If $f : G \to H$ is an open continuous surjective homomorphism and G has property (***), then H has property (***).*

Proof. Let e_H be the identity of H and let U be any open neighborhood of e_H in H. Since f is a continuous homomorphism, $f^{-1}(U)$ is an open neighborhood of the identity e in G. Since G has property (***), the family $\{f^{-1}(U)x : x \in G\}$ has an open basic countable refinement $\mathcal{W}$ which is dominated by a countable family $\gamma \subset \mathcal{N}(e)$. For any $y \in H$, there exists $x_y \in G$ such that $f(x_y) = y$. Let $y \in H$. Since $\mathcal{W}$ is a basic refinement of $\{f^{-1}(U)z : z \in G\}$, there exists $W_y \in \mathcal{W}$ such that $x_y \in W_y \subset f^{-1}(U)x_y$. Then $y = f(x_y) \in f(W_y) \subset Uy$. This shows that $\{f(W) : W \in \mathcal{W}\}$ is a basic countable refinement of $\{Uh : h \in H\}$. For any $W \in \mathcal{W}$ and any $x \in W$, there exists $V \in \gamma$ such that $xV \subset W$. Since the mapping f is open, we have $\{f(V) : V \in \gamma\} \subset \mathcal{N}(e_H)$. Thus $\{f(W) : W \in \mathcal{W}\}$ is dominated by the countable family $\{f(V) : V \in \gamma\} \subset \mathcal{N}(e_H)$. Thus H has property (***). □

The following fact is evident.

Proposition 4.8. *Every $(\overline{\mathbb{R}}, \mathbb{SCSG})$ $((\mathbb{MS}, \mathbb{SCSG}))$-factorizable semitopological group is $(\overline{\mathbb{R}}, \mathbb{FSG})$ $((\mathbb{MS}, \mathbb{FSG}))$-factorizable.*

Arguing as in the proof of Theorem 29 in [13], we have the following result.

Lemma 4.9. *If G is a semitopological group with property (***), then G is projectively second-countable.*

Theorem 4.10. *Let G and H be semitopological groups. If $f : G \to H$ is an open continuous surjective homomorphism and G is Tychonoff $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable, then H is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable.*

Proof. Since G is a Tychonoff $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable semitopological group, by Proposition 4.6 G has property (***). Then it follows from Lemma 4.7 that H has property (***). By Proposition 4.8 and Lemmas 3.9 and 3.13, H has property ω-QU. Since $\mathbb{SC}$ is a CPS-class, it follows from Lemmas 3.12 and 4.9 that H is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable. □

By Lemma 3.15 and an argument similar to the one used in the proof of Theorem 4.10, we have the following result.

Theorem 4.11. *Let G and H be semitopological groups. If G is a Tychonoff $(\mathbb{MS}, \mathbb{SCSG})$-factorizable semitopological group and $f : G \to H$ is an open continuous surjective homomorphism, then H is $(\mathbb{MS}, \mathbb{SCSG})$-factorizable.*

In what follows, we discuss properties of a regular semitopological group G with property (***), a q-point and satisfying $Sm(G) \le \omega$.

Lemma 4.12. *If G is a first-countable semitopological group with property (***), then G is second-countable.*

Proof. Let $\{U_n : n \in \omega\}$ be a base of open neighborhoods for G at the identity e. Since G has property (***), for every $n \in \omega$ the family $\{U_n x : x \in G\}$ has an open basic countable refinement $\mathcal{V}_n$ which is dominated by a countable family $\gamma_n \subset \mathcal{N}(e)$.

For every $W \in \mathcal{V}_n$ and any $x \in W$, there exists $V \in \gamma_n$ such that $xV \subset W$. Thus W is open in G. If $\mathcal{B} = \bigcup\{\mathcal{V}_n : n \in \omega\}$, then $\mathcal{B}$ is a countable family of open subsets of G. For any open subset O of G and any $x \in O$, there exists $n \in \omega$ such that $x \in U_n x \subset O$. Then there exists $W \in \mathcal{V}_n$ such that $x \in W \subset U_n x$. Hence $x \in W \subset O$. Thus $\mathcal{B}$ is a countable base for G. Then G is second-countable. □

Lemma 4.13. ([13], Corollary 28) *Every semitopological group with property (***) is ω-balanced and locally ω-good.*

Lemma 4.14. ([16], Lemma 2.7) *Let G be a semitopological group with identity e. Suppose that a family $\gamma \subset \mathcal{N}(e)$ satisfies the following conditions:*

(a) for every $U \in \gamma$ and $x \in U$, there exists $V \in \gamma$ such that $xV \subset U$;
(b) γ is subordinated to U, for each $U \in \gamma$.

Then the set $N = \bigcap\{U \cap U^{-1} : U \in \gamma\}$ is an invariant subgroup of G. Further, $UN = NU = U$ for each $U \in \gamma$.

Lemma 4.15. *Let G be a semitopological group. If $\{V_n : n \in \omega\}$ is a sequence of open neighborhoods of the identity e in G such that $\overline{V_{n+1}} \subset V_n$ for every $n \in \omega$ and $N = \bigcap\{V_n : n \in \omega\}$ is a closed invariant subgroup of G with the property that $V_nN = NV_n$ for every $n \in \omega$ and $\{V_n : n \in \omega\}$ is a base for G at N, then the canonical quotient homomorphism $\pi : G \to G/N$ is closed and open and G/N is a first-countable regular semitopological group.*

Proof. Since N is a closed invariant subgroup of G, the quotient space G/N is a T_1 semitopological group. Since $\{V_n : n \in \omega\}$ is a base for G at N and π is an open continuous mapping, $\{\pi(V_n) : n \in \omega\}$ is a countable base of open neighborhoods for G/N at the identity e^* in G/N. By the homogeneity of G/N, G/N is first-countable. Now we show that the mapping $\pi : G \to G/N$ is closed.

Take an arbitrary closed subset F of G. For any $z \in G/N \setminus \pi(F)$, there exists $x \in G$ such that $\pi(x) = z$. Then $xN \cap F = \emptyset$. Thus $N \cap x^{-1}F = \emptyset$. Since $\{V_n : n \in \omega\}$ is a countable base of open neighborhoods for G at N, there exists $m \in \omega$ such that $V_m \cap x^{-1}F = \emptyset$. Thus $xV_m \cap F = \emptyset$. Since $V_m = V_mN$, we have $\pi(xV_m) \cap \pi(F) = \emptyset$. Then $(z\pi(V_m)) \cap F = \emptyset$. Thus $\pi(F)$ is closed in G/N. Then the mapping π is closed. Since $\overline{V_{n+1}} \subset V_n$ for every $n \in \omega$ and the mapping π is closed, $\pi(V_{n+1}) \subset \overline{\pi(V_{n+1})} \subset \pi(\overline{V_{n+1}}) \subset \pi(V_n)$. Since $\{\pi(V_n) : n \in \omega\}$ is a countable base for G/N at e^* and $\overline{\pi(V_{n+1})} \subset \pi(V_n)$ for every $n \in \omega$, the quotient space G/N is regular. □

Lemma 4.16. *Let G be a regular semitopological group with a q-point, property (***) and satisfying $Sm(G) \leq \omega$. Then for every open neighborhood U of the identity e in G there exists a closed countably compact invariant subgroup N with a countable base of open neighborhoods $\{V_n : n \in \omega\}$ such that the following properties hold:*

(1) $V_0 \subset U$;
(2) For every $n \in \omega$, $\overline{V_{n+1}} \subset V_n$;
(3) $N = \bigcap\{V_n : n \in \omega\} = \bigcap\{\overline{V_n} : n \in \omega\}$;
(4) $V_nN = NV_n = V_n$ for every $n \in \omega$;
(5) If $\{x_n\}_{n\in\omega}$ is a sequence of points of G such that $x_n \in V_n$ for every $n \in \omega$, then the sequence $\{x_n\}_{n\in\omega}$ has a point of accumulation in N;
(6) The quotient space G/N is a separable metrizable semitopological group and the the canonical quotient homomorphism $\pi : G \to G/N$ is closed and open such that $\pi^{-1}(\pi(V_0)) \subset U$.

Proof. By Lemma 4.13, G is ω-balanced and locally ω-good. Since G has a q-point and is homogeneous, the identity e of G is a q-point. Then there exists a decreasing sequence $\{W_n : n \in \omega\}$ of open neighborhoods of e in G such that any sequence $\{x_n\}_{n\in\omega}$ of points in G with $x_n \in W_n$ for every $n \in \omega$ has a point of accumulation in G.

Let U be an arbitrary open neighborhood of e in G. Thus $U \cap W_0$ is an open neighborhood of e in G. Since G is locally ω-good, there exists $V_0 \in \mathcal{N}^*(e)$ such that $V_0 \subset W_0 \cap U$.

Put $\mathcal{P}_0 = \{V_0\}$ and let $\mathcal{P}_0^* = \{V_{0,i} : i \in \omega\}$ such that $V_{0,i} = V_0$ for every $i \in \omega$. Suppose that for some $n \in \omega$, we have defined countable families $\mathcal{P}_0^*, \ldots, \mathcal{P}_n^*, \mathcal{P}_0, \ldots, \mathcal{P}_n$ satisfying the following conditions for every $k \leq n$:

(s1) $\mathcal{P}_k^* \subset \mathcal{N}^*(e)$ and $\mathcal{P}_k^* = \{V_{k,i} : k \leq i \text{ and } i \in \omega\}$;
(s2) $\mathcal{P}_k^*$ is closed under the finite intersections;
(s3) $\mathcal{P}_{k+1} = \{\bigcap \mathcal{F} : \mathcal{F} \subset \bigcup\{\mathcal{P}_i^* : i \leq k\}, 1 \leq |\mathcal{F}| < \omega\}$ if $k + 1 \leq n$;
(s4) If $1 \leq k \leq n$, then $\mathcal{P}_k^*$ is subordinated to W for each $W \in \mathcal{P}_k$;
(s5) If $1 \leq k \leq n$, then for every $W \in \mathcal{P}_k$ the inclusion $\bigcap_{O\in\mathcal{P}_k^*} O^{-1} \subset W$ holds;
(s6) For each $O \in \mathcal{P}_k^*$, the set O is a subset of W_k;

(s7) If $1 \le k \le n$, then for every $W \in \mathcal{P}_k$ and every $x \in W$ there exists $O \in \mathcal{P}_k^*$ such that $xO \subset W$;

(s8) If $1 \le k \le n$, then for every $W \in \mathcal{P}_k$ there exists $O \in \mathcal{P}_k^*$ such that $\overline{O} \subset W$;

(s9) If $1 \le k \le n$, then for every $W \in \mathcal{P}_k$ the family $\{Wx : x \in G\}$ has an open basic countable refinement $\mathcal{V}_W$ which is dominated by a countable family $\gamma_W \subset \mathcal{N}(e)$.

If $\mathcal{P}_{n+1} = \{\bigcap \mathcal{F} : \mathcal{F} \subset \bigcup\{\mathcal{P}_i^* : i \le n\}, 1 \le |\mathcal{F}| < \omega\}$, then $|\mathcal{P}_{n+1}| \le \omega$ and $\mathcal{P}_{n+1} \subset \mathcal{N}^*(e)$. Since $\mathcal{P}_{n+1} \subset \mathcal{N}^*(e)$ is countable and G is locally ω-good, there exists a countable family $\lambda_{n+1,1} \subset \mathcal{N}^*(e)$ such that $\bigcup \lambda_{n+1,1} \subset W_{n+1}$ and for every $W \in \mathcal{P}_{n+1}$ and any $x \in W$, there exists $O \in \lambda_{n+1,1}$ such that $xO \subset W$. As G is ω-balanced and $|\mathcal{P}_{n+1}| \le \omega$, there exists a countable family $\lambda_{n+1,2} \subset \mathcal{N}^*(e)$ such that $\bigcup \lambda_{n+1,2} \subset W_{n+1}$, and $\lambda_{n+1,2}$ is subordinated to every $W \in \mathcal{P}_{n+1}$. As G is regular and $|\mathcal{P}_{n+1}| \le \omega$, there exists a countable family $\lambda_{n+1,3} \subset \mathcal{N}^*(e)$ such that $\bigcup \lambda_{n+1,3} \subset W_{n+1}$ and for every $W \in \mathcal{P}_{n+1}$ there exists $O \in \lambda_{n+1,3}$ such that $\overline{O} \subset W$. As $Sm(G) \le \omega$ and G is locally ω-good, there exists a countable family $\lambda_{n+1,4} \subset \mathcal{N}^*(e)$ such that $\bigcup \lambda_{n+1,4} \subset W_{n+1}$ and for every $W \in \mathcal{P}_{n+1}$ the inclusion $\bigcap_{O \in \lambda_{n+1,4}} O^{-1} \subset W$ holds.

Since G has property (***), for every $W \in \mathcal{P}_{n+1}$ the family $\{Wx : x \in G\}$ has an open basic countable refinement which is dominated by a countable family $\gamma_W \subset \mathcal{N}^*(e)$ and $\bigcup \gamma_W \subset W_{n+1}$.

Let $\lambda_{n+1,5} = \bigcup\{\gamma_W : W \in \mathcal{P}_{n+1}\}$. Then $\lambda_{n+1,5} \subset \mathcal{N}^*(e)$, $\bigcup \lambda_{n+1,5} \subset W_{n+1}$ and $|\lambda_{n+1,5}| \le \omega$. Denote $\mathcal{P}_{n+1}^* = \{\bigcap \mathcal{F} : \mathcal{F} \subset \bigcup_{i=1}^{5} \lambda_{n+1,i} \text{ and } 1 \le |\mathcal{F}| < \omega\} = \{V_{n+1,i} : i \in \omega \text{ and } n+1 \le i\}$.

In this way, we get two sequences $\{\mathcal{P}_n^* : n \in \omega\}$ and $\{\mathcal{P}_n : n \in \omega\}$ of countable families of ω-good sets with properties (s1)-(s9).

If $\mathcal{P} = \bigcup\{\mathcal{P}_n^* : n \in \omega\}$, then $\mathcal{P} \subset \mathcal{N}^*(e)$ is countable and satisfies the following three conditions:

(a) For every $W \in \mathcal{P}$ and every $x \in W$, there exists $O \in \mathcal{P}$ such that $xO \subset W$;

(b) $\mathcal{P}$ is subordinated to W for every $W \in \mathcal{P}$;

(c) $\bigcap_{O \in \mathcal{P}} O^{-1} \subset W$ for every $W \in \mathcal{P}$.

By Lemma 4.14, the set $N = \bigcap\{U \cap U^{-1} : U \in \mathcal{P}\}$ is an invariant subgroup of G and $WN = NW = W$ for every $W \in \mathcal{P}$. By (c), $N = \bigcap \mathcal{P}$.

We enumerate the family $\mathcal{P}$ as $\{P_n : n \in \omega\}$. There exists $i_0 \in W$ such that $V_0 = P_{i_0}$. Then $P_{i_0} = V_0 \subset W_0$. Let $k_0 = i_0$ and $P_{k_0} = P_{i_0} = V_0$. Assume that for each $n \in \omega$, we have defined a strictly increasing finite sequence $\{k_i : i \le n\} \subset \mathbb{N}$ such that $P_{k_{i+1}} \subset \overline{P_{k_{i+1}}} \subset P_{k_i}$ for every $i < n$ and $V_i = P_{k_i} \subset W_i, \overline{P_{k_i}} \subset \bigcap\{P_j : 0 \le j \le k_{i-1}\}$ for every $1 \le i \le n$. Since $\{P_i : 0 \le i \le k_n\}$ is a finite subfamily of $\mathcal{P}$, there exists $m \in \omega$ such that $\bigcap\{P_i : 0 \le i \le k_n\} \subset \mathcal{P}_k$ for each $k \ge m$. Then for every $k \ge m$, there exists $O_k \in \mathcal{P}_k^*$ such that $\overline{O_k} \subset \bigcap\{P_i : 0 \le i \le k_n\}$. Then exists $k_{n+1} \in \mathbb{N}$ such that $k_{n+1} > k_n$, $\overline{P_{k_{n+1}}} \subset \bigcap\{P_i : 0 \le i \le k_n\}$ and $P_{k_{n+1}} \in \mathcal{P}_j^*$ for some $j \ge n+1$. If $V_{n+1} = P_{k_{n+1}}$, then $\overline{V_{n+1}}$ is a subset of $W_{n+1} \cap V_n$.

Thus we can get a strictly increasing sequence $\{k_n\}_{n \in \omega}$ and a strictly decreasing sequence $\{P_{k_n}\}_{n \in \omega}$ with $\{P_{k_n} : n \in \omega\} \subset \mathcal{P}$ satisfying the following conditions:

(a) $P_{k_0} = P_{i_0} = V_0 \subset W_0$;

(b) If $n \ge 1$, the $\overline{P_{k_n}} \subset \bigcap\{P_i : i \le k_{n-1}\}$;

(c) $V_n = P_{k_n} \subset W_n$ for each $n \in \omega$.

Thus $N = \bigcap \mathcal{P} = \bigcap\{P_{k_n} : n \in \omega\} = \bigcap\{V_n : n \in \omega\} = \bigcap\{\overline{V_n} : n \in \omega\}$ is a closed invariant subgroup of G. Since $V_n = P_{k_n} \subset W_n$ for every $n \in \omega$, any sequence $\{x_n\}_{n \in \omega}$ of points of G satisfying $x_n \in V_n$ for every $n \in \omega$ has a point of accumulation in G. Thus the family $\{V_n : n \in \omega\}$ is a countable base of open neighborhoods for G at N and N is a countably compact closed invariant subgroup of G. Then items (1)-(5) of the lemma hold.

Let $\pi : G \to G/N$ be the canonical quotient homomorphism. By Lemma 4.15, the mapping π is closed and open and G/N is a first-countable regular semitopological group. Since G has property (***), by Lemma 4.7 G/N has property (***). Then by Lemma 4.12 G/N is second-countable. Thus G/N is separable and metrizable. Since $V_0 = V_0N$ holds, we have $\pi^{-1}(\pi(V_0)) = V_0N = V_0 \subset U$. $\square$

By Lemma 4.16, we have the following result.

Corollary 4.17. *If G is a regular semitopological group with a q-point, property (***) and satisfying $Sm(G) \leq \omega$, then G is projectively separable and metrizable.*

Lemma 4.18. ([[1], Theorem 3.4.21]) *Let $\mathcal{P}$ be a class of topological groups (paratopological groups, semitopological groups), τ an infinite cardinal number, and G a T_1 topological group (paratopological group, semitopological group), which is range-$\mathcal{P}$ and has a local base $\mathcal{B}$ of open neighborhoods of the identity such that $|\mathcal{B}| \leq \tau$. Then G is topologically isomorphic to a subgroup of the product of a family $\{H_a : a \in A\}$ of groups such that $H_a \in \mathcal{P}$, for each $a \in A$, and $|A| \leq \tau$.*

Theorem 4.19. *If G be a regular semitopological group with a q-point, property (***) and satisfying $Sm(G) \leq \omega$, then G is topologically isomorphic to a subgroup of the product of a family of separable metrizable semitopological groups.*

Proof. The required conclusion follows from Corollary 4.17 and Lemma 4.18. □

In ([12], Theorem 3.12), it is proved that if G is a paratopological group then G is $(\overline{\mathbb{R}}, \mathbb{FPG}_r)$ $((\mathbb{MS}, \mathbb{FPG}_r))$-factorizable if and only if G is $((\overline{\mathbb{R}}, \mathbb{FPG})$ $((\mathbb{MS}, \mathbb{FPG}))$-factorizable.

By the an argument similar to the one used in the proof of Theorem 4.2, we have the following two results.

Proposition 4.20. *Let G be a topological (semitopological, quasitopological) group. Then G is $(\overline{\mathbb{R}}, \mathbb{FTG}_r)$ $((\overline{\mathbb{R}}, \mathbb{FSG}_r)$, $(\overline{\mathbb{R}}, \mathbb{FQG}_r))$-factorizable if and only if G is $(\overline{\mathbb{R}}, \mathbb{FTG})$, $((\overline{\mathbb{R}}, \mathbb{FSG})$, $(\overline{\mathbb{R}}, \mathbb{FQG}))$-factorizable.*

Proposition 4.21. *If G is a topological (semitopological, quasitopological) group, then G is $(\mathbb{MS}, \mathbb{FTG}_r)$ $((\mathbb{MS}, \mathbb{FSG}_r)$, $(\mathbb{MS}, \mathbb{FQG}_r))$-factorizable if and only if G is $(\mathbb{MS}, \mathbb{FTG})$, $(\mathbb{MS}, \mathbb{FSG})$, $(\mathbb{MS}, \mathbb{FQG}))$-factorizable.*

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