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Spectra of join of quasi-corona $\mathcal{R}$ graph

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Abstract. In this paper, we have determined the adjacency, the Laplacian and the signless Laplacian spectra of *quasi-corona $\mathcal{R}$-vertex join*, represented by $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ and *quasi-corona $\mathcal{R}$-edge join*, represented by $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ and obtain several adjacency, Laplacian and signless Laplacian cospectral of non-regular graphs. Further, we have also determined the Kirchhoff's indices, Laplacian-energy-like-invariant (*LEL*) and the number of spanning trees from Laplacian spectra.

1. Introduction

Consider a simple graph having n vertices and m edges, denoted as $\mathcal{G} = (\mathcal{U}, \mathcal{E})$. Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be the vertex set and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ be the edge set of $\mathcal{G}$.
The adjacency matrix of the graph G is $n \times n$ square matrix and defined as $A(\mathcal{G}) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } u_i \sim u_j, \\ 0, & \text{otherwise.} \end{cases}$$

The incidence matrix of G is $n \times m$ matrix and defined as $B(\mathcal{G}) = [b_{ij}]$, where

$$b_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident on } u_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}(\mathcal{G})$ be the line graph and consider $B(\mathcal{G}) = B$. Then $B^T B = A(\mathcal{L}(\mathcal{G})) + 2I_m$ and $BB^T = A(\mathcal{G}) + rI_n$, where I_n and I_m are the identity matrices. The Laplacian matrix $L(\mathcal{G})$ and the signless Laplacian matrix $Q(\mathcal{G})$ is defined as $D(\mathcal{G}) - A(\mathcal{G})$ and $D(\mathcal{G}) + A(\mathcal{G})$ respectively, where $D(\mathcal{G})$ be the diagonal matrix. The characteristic polynomials of $A(\mathcal{G})$, $L(\mathcal{G})$ and $Q(\mathcal{G})$ are defined as $\Phi_{\mathcal{G}}(A; x) = |xI_n - A(\mathcal{G})|$, $\Phi_{\mathcal{G}}(L; x) = |xI_n - L(\mathcal{G})|$ and $\Phi_{\mathcal{G}}(Q; x) = |xI_n - Q(\mathcal{G})|$, respectively. The eigenvalues of $A(\mathcal{G})$ are the adjacency eigenvalues of $\mathcal{G}$ and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Similarly, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ denote respectively the eigenvalues of $L(\mathcal{G})$ and $Q(\mathcal{G})$. Also, the eigenvalues (with multiplicities) of A, L and Q-spectrum is denoted by $\{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_n^{m_n}\}$, $\{\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_n^{m_n}\}$ and $\{\nu_1^{m_1}, \nu_2^{m_2}, \ldots, \nu_n^{m_n}\}$ respectively, where $m_1, m_2, \ldots, m_n$ are its multiplicities. Moreover, if two graphs share the same spectrum, they are referred to as cospectral.

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For any connected graph $\mathcal{G}$, the sum of the resistance distances between all pairs of vertices of $\mathcal{G}$ is the Kirchhoff index, denoted by $Kf(\mathcal{G})$ and is defined as $Kf(\mathcal{G}) = n\sum_{i=2}^{n} \frac{1}{\mu_i(\mathcal{G})}$. The Laplacian spectrum based graph invariant, Laplacian energy-like-invariant LEL, is defined as $LEL(\mathcal{G}) = \sum_{i=2}^{n_2} \sqrt{\mu_i}$ and the spanning trees with n vertices is determined by $t(\mathcal{G}) = \frac{\mu_2(\mathcal{G})...\mu_n(\mathcal{G})}{n}$.

Several graph operations exist in the literature, such as the complement, union, join, corona operations and graph product. Their spectra are determined in [1, 3, 5, 8, 11, 13, 14, 17]. Borah, Singh and Prasad [2] defined four new graphs based on subdivision and central graph, and obtained their A, L, and Q spectra. As an application, the number of spanning trees and the Kirchhoff's indices are determined. Given a graph $\mathcal{G}$, the $\mathcal{R}$- graph [5] is the graph obtained from $\mathcal{G}$ by introducing a new vertex to each edge of $\mathcal{G}$ and then joining each new vertex to the end vertices of that edge. Lan and Zhou [12] determined A, L and Q-spectra of the resulting graphs based on $\mathcal{R}$- graph. Also, they used their results to obtain several pairs of non-regular A, L and Q-cospectral graphs. Das and Panigrahi [7] obtained A, L and Q- spectra of $\mathcal{R}$-vertex and edge join graphs and determined pairs of non-regular A, L and Q-cospectral graphs. Hou et al.[10] defined quasi-corona SG -vertex join and multiple SG- vertex join of graphs and obtained their adjacency spectra for regular graphs.

Consider two graphs $\mathcal{G}$ and $\mathcal{H}$ with n_1 and n_2 vertices, and m_1 and m_2 edges.

Definition 1.1. The *quasi-corona $\mathcal{R}$-vertex join* of $\mathcal{G}$ and $\mathcal{H}$, represented by $\mathcal{G}\lfloor u\rfloor\mathcal{H}$, is a graph constructed from $\mathcal{R}(\mathcal{G})$ and H by choosing a copy of $\mathcal{R}(\mathcal{G})$ and n_1 copies of $\mathcal{G}$ and then connecting each old vertex of $\mathcal{G}$ to every vertex of $\mathcal{H}$.

Definition 1.2. The *quasi-corona $\mathcal{R}$-edge join* of $\mathcal{G}$ and $\mathcal{H}$, represented by $\mathcal{G}\lfloor e\rfloor\mathcal{H}$, is a graph constructed from $\mathcal{R}(\mathcal{G})$ and $\mathcal{H}$ by choosing a copy of $\mathcal{R}(\mathcal{G})$ and n_1 copies of $\mathcal{H}$ and then connecting each new vertex of $\mathcal{G}$ to every vertex of $\mathcal{H}$.

We observe that $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ and $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ have the same number of vertices $n_1 + m_1 + n_1n_2$ and $\mathcal{G}\lfloor v\rfloor\mathcal{H}$ has $2n_1 + n_1m_2 + n_1^2n_2$ edges and $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ has $n_1 + m_1m_2 + m_1^2n_2$ edges.

Example 1.3. Let us take $\mathcal{G} = K_3$ and $\mathcal{H} = P_2$, then $K_3\lfloor u\rfloor P_2$ and $K_3\lfloor e\rfloor P_2$ are given by Figure 1 and Figure 2 respectively.

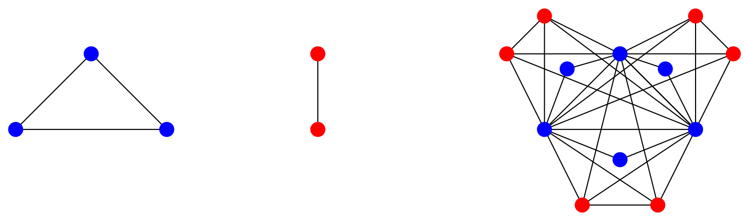

Figure 1: $K_3\lfloor u\rfloor P_2$

The M-Coronal, represented by $\Gamma_M(x)$[16], is defined as $\Gamma_M(x) = J_n^T(xI_n - M)^{-1}J_n$, where M is the square matrix of order n and J_n is the column matrix of order $n \times 1$ whose entries are 1 and $\Gamma_M(x) = \frac{n}{x-t}$ if row sum of n order square matrix is equal to a constant t. Further, for the Laplacian matrix $L(\mathcal{G})$, $\Gamma_L(x) = \frac{n}{x}$ [16] and for the signless Laplacian matrix $Q(x)$, $\Gamma_Q(x) = \frac{n}{x-2r}$ [6].

From [5, 7, 14], we get the following lemmas which will be used in our proof.

Lemma 1.4 (([14])). $\det(M + \gamma J_{n\times n}) = \det(M) + \gamma J_{n\times 1}^T adj(M)J_{n\times 1}$, *where $adj(M)$ is the adjoint of M and γ is a real number. Further,* $\det(xI_n - M - \gamma J_n) = \{1 - \gamma\Gamma_M(x)\}\det(xI_n - M)$.

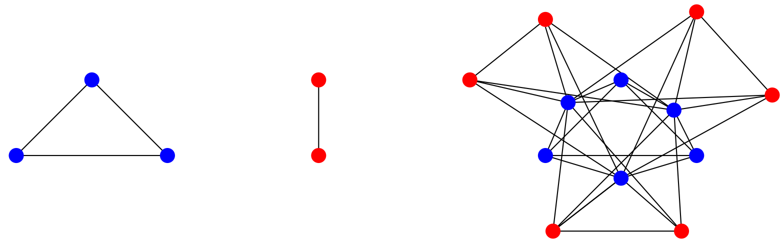

Figure 2: $K_3\lfloor e\rfloor P_2$

Lemma 1.5. ([5]) *Let B_1, B_2, B_3 and B_4 be four $b_1 \times b_1$, $b_1 \times b_2$, $b_2 \times b_1$ and $b_2 \times b_2$ matrices, where B_1 and B_4 are non-singular square matrices. Then,*

$$\det\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \det(B_4)\det(B_1 - B_2B_4^{-1}B_3) = \det(B_1)\det(B_4 - B_3B_1^{-1}B_2).$$

Lemma 1.6. ([7]) *For any real numbers $p, q > 0$, we get*

$$(pI_n - qJ_{n\times n})^{-1} = \frac{1}{p}I_n + \frac{q}{p(p-nq)}J_{n\times n}.$$

The Kronecker product of two matrices $A = (a_{ij})$ of order $a_1 \times a_2$ and $B = (b_{ij})$ of order $b_1 \times b_2$, denoted by $A \otimes B$, is defined as the matrix of order $a_1b_1 \times a_2b_2$ and is obtained by replacing each a_{ij} of A by $a_{ij}B$ [9]. Also, for any four matrices B_1, B_2, B_3 and B_4, we get $(B_1 \otimes B_2)(B_3 \otimes B_4) = B_1B_3 \otimes B_2B_4$. Further, for any two non-singular matrices B_1 and B_2 it follows that $(B_1 \otimes B_2)^{-1} = B_1^{-1} \otimes B_2^{-1}$ and $\det(B_1 \otimes B_2) = (\det B_1)^u(\det B_2)^v$, where u and v are respectively the order of the square matrices B_1 and B_2.

First, we determine A, L and Q spectra of quasi-corona $\mathcal{R}$-vertex and quasi-corona $\mathcal{R}$-edge join of graphs. Then, we have shown the existence of simultaneous pairs of cospectral graphs of these two graphs. Further, we obtain the Kirchhoff index, Laplacian-energy-like-invariant and the number of spanning trees.

2. Spectra of quasi-corona $\mathcal{R}$- vertex join

We start with the following result about adjacency spectra of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$.

Theorem 2.1. ([10]) *Let $\mathcal{G}$ be an r_1- regular and $\mathcal{H}$ be any graph, then*

$$\Phi_{\mathcal{G}\lfloor u\rfloor\mathcal{H}}(A;x) = x^{m_1-n_1}\prod_{i=2}^{n_2}\{x-\lambda_i(\mathcal{H})\}^{n_1}\prod_{i=2}^{n_1}\{x^2-\lambda_i(\mathcal{G})x-r_1-\lambda_i(\mathcal{G})\}\{x^2-r_1x-2r_1-n_1x\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\}.$$

Now, we have the following observations from the above Theorem 2.1.

Observations.

(1) If $\mathcal{H}$ is an r_2 regular graph, then A-spectrum of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ contains the following eigenvalues
  - (i) 0 with multiplicities $m_1 - n_1$.
  - (ii) $\lambda_j(\mathcal{H})$ with multiplicities n_1, $j = 2, 3, \cdots n_2$
  - (iii) the roots of the quadratic equation $x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda_i(\mathcal{G}) = 0$, $i = 2, 3, \cdots n_1$
  - (iv) the roots of the cubic equation $x^3 - (r_1 + r_2)x^2 + (r_1r_2 - 2r_1 - n_1^2n_2)x + 2r_1r_2 = 0$

(2) If $\mathcal{H} = K_{a,b}$, then the A-spectrum of $\mathcal{G}\lfloor u\rfloor K_{a,b}$ contains the following eigenvalues
  - (i) 0 with multiplicities $m_1 + n_1(a + b - 3)$

(*ii*) $\pm\sqrt{ab}$ with multiplicities n_1

(*iii*) the roots of the quadratic equation $x^2 - \lambda_i(\mathcal{G})x - \lambda_i(\mathcal{G}) - r_1 = 0$ and

(*iv*) the roots of the quadratic equation $x^2 - r_1 x - 2r_1 - n_1 x\Gamma_{A(K_{a,b})\otimes I_{n_1}}(x) = 0$.

Now, we determine the L- spectra of $G\lfloor v\rfloor H$.

Theorem 2.2. *Let $\mathcal{G}$ be an r_1-regular and $\mathcal{H}$ be any graph, then*

$$\Phi_{\mathcal{G}\lfloor u\rfloor\mathcal{H}}(L;x) = (x-2)^{m_1-n_1}x\{x^2-(2+r_1+n_1+n_1n_2)x+(2n_1+n_1r_1+2n_1n_2)\}\prod_{j=2}^{n_2}\{x-n_1-\mu_j(\mathcal{H})\}^{n_1}$$

$$\prod_{i=2}^{n_1}\{x^2-(r_1+n_1n_2+2+\mu_i(\mathcal{G}))x+2n_1n_2+3\mu_i(\mathcal{G})\}.$$

Proof. By proper labelling of the vertices, $L(\mathcal{G}\lfloor u\rfloor\mathcal{H})$ can be expressed as

$$L(\mathcal{G}\lfloor u\rfloor\mathcal{H}) = \begin{pmatrix} (r_1+n_1n_2)I_{n_1}+L(\mathcal{G}) & -B & -J_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & 2I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & n_1I_{n_2}+L(\mathcal{H})\otimes I_{n_1} \end{pmatrix}.$$

The characteristic polynomial of $L(\mathcal{G}\lfloor u\rfloor\mathcal{H})$ is $\Phi_{\mathcal{G}\lfloor u\rfloor\mathcal{H}}(L;x)$

$$\begin{aligned}
&= \det(xI_{n_1n_2+n_1+m_1} - L(\mathcal{G}\lfloor u\rfloor\mathcal{H})) \\
&= \det\begin{pmatrix} (x-r_1-n_1n_2)I_{n_1}-L(\mathcal{G}) & B & J_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & (x-2)I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & \{(x-n_1)I_{n_2}-L(\mathcal{H})\}\otimes I_{n_1} \end{pmatrix} \\
&= \det\{((x-n_1)I_{n_2}-L(\mathcal{H}))\otimes I_{n_1}\}\det S,
\end{aligned}$$

where,

$$\begin{aligned}
S &= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1}-L(\mathcal{G}) & B \\ B^T & (x-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} J_{n_1\times n_2}\otimes J_{n_1}^T \\ 0_{m_1\times n_2}\otimes J_{n_1}^T \end{pmatrix}\Big(\big((x-n_1)I_{n_2}-L(\mathcal{H})\big)^{-1}\otimes I_{n_1}\Big)\Big(-J_{n_2}\otimes J_{n_1}^T \quad 0_{n_2\times n_1}\otimes J_{n_1}^T\Big) \\
&= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1}-L(\mathcal{G}) & B \\ B^T & (x-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} \Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & 0_{n_1\times m_1} \\ 0_{m_1\times n_1} & 0_{m_1\times m_1} \end{pmatrix} \\
&= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1}-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1}-L(G) & B \\ B^T & (x-2)I_{m_1} \end{pmatrix}.
\end{aligned}$$

So,

$$\begin{aligned}
\det S &= \det\{(x-2)I_{m_1}\}\det\{(x-r_1-n_1n_2)I_{n_1}-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1}-L(\mathcal{G})-\frac{1}{x-2}BB^T\} \\
&= \det\{(x-2)I_{m_1}\}\{1-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-n_1)\Gamma_{L(G)+\frac{BB^T}{x-2}}(x-r_1-n_1n_2)\}\det\{(x-r_1-n_1n_2)I_{n_1}-\frac{1}{x-2}BB^T\}
\end{aligned}$$

Since,

$$\Gamma_{L(G)+\frac{BB^T}{x-2}}(x-r_1-n_1n_2) = \frac{n_1(x-2)}{x^2-(2+r_1+n_1n_2)x+2n_1n_2}$$

and

$$\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-n_1) = \frac{n_1n_2}{x-n_1},$$

We get,

$$\det S = \det\{(x-2)I_{m_1}\}\left\{1 - \frac{n_1 n_2}{x-n_1}\left(\frac{n_1(x-2)}{x^2-(2+r_1+n_1n_2)x+2n_2}\right)\right\}$$
$$\times \det\left\{(x-r_1-n_1n_2)I_{n_1} - \frac{1}{x-2}(A(\mathcal{G})+r_1I_{n_1})\right\}$$
$$= (x-2)^{m_1-n_1}\{x^3-(2+r_1+n_1+n_1n_2)x^2+(2n_1+n_1r_1+2n_1n_2)x\}$$
$$\prod_{i=1}^{n_1}\{x^2-(r_1+n_1n_2+2+\mu_i(\mathcal{G}))x+2n_1n_2+3\mu_i(\mathcal{G})\}$$

Applying the fact that $\lambda_i(\mathcal{G}) = r_1 - \mu_i(\mathcal{G})$, $\mu_1(\mathcal{G}) = 0$ and $\mu_1(H) = 0$, gives the desired L-spectrum of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$. □

From Theorem 2.2, we get the following observations.
Observations.

(1) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is a r_2 regular graphs, then L-spectrum of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ contains

- (i) 0
- (ii) 2 with multiplicities $m_1 - n_1$
- (iii) $n_1 + \mu_j(\mathcal{H})$ with multiplicities n_1
- (iv) the roots of the quadratic equation $x^2-(2+r_1+n_1n_2+\mu_i(\mathcal{G}))x+2n_1n_2+3\mu_i(\mathcal{G}) = 0, i = 2,3,4,\ldots,n_1$ and
- (v) the roots of the quadratic equation $x^2-(2+r_1+n_1+n_1n_2)x+(2n_1+n_1r_1+2n_1n_2) = 0$.

(2) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H} = K_{n_2}$, then L-spectrum of $\mathcal{G}\lfloor u\rfloor K_{n_2}$ contains

- (i) 0
- (ii) 2 with multiplicities $m_1 - n_1$
- (iii) n_1 with multiplicities n_1
- (iv) $n_1 + n_2$ with multiplicities $n_1n_2 - n_1$
- (v) the roots of the quadratic equation $x^2-(2+r_1+n_1n_2+\mu_i(\mathcal{G}))x+2n_1n_2+3\mu_i(\mathcal{G}) = 0, i = 2,3,4,\ldots,n_1$ and
- (vi) the roots of the quadratic equation $x^2-(2+r_1+n_1+n_1n_2)x+(2n_1+n_1r_1+2n_1n_2) = 0$.

Next, we determine the Q-spectrum of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$.

Theorem 2.3. *Let $\mathcal{G}$ be an r_1-regular and $\mathcal{H}$ be a r_2 regular graph, then*

$$\Phi_{\mathcal{G}\lfloor u\rfloor\mathcal{H}}(Q;x) = (x-2)^{m_1-n_1}(x^3-(3r_1+2+n_1n_2+3r_2)x^2+(2n_1n_2+4r_1+3n_1n_2r_2+9r_1r_2+6r_2-n_1^2n_2)x$$
$$-6n_1n_2r_2-12r_1r_2+2n_1^2n_2)\prod_{j=2}^{n_2}\{x-n_1-\nu_j(\mathcal{H})\}^{n_1}\prod_{i=2}^{n_1}\{x^2-(r_1+n_1n_2+2+\nu_i(\mathcal{G}))x+2n_1n_2+2r_1-\nu_i(\mathcal{G})\}.$$

Proof. The Q matrix of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ can be expressed as

$$L(\mathcal{G}\lfloor u\rfloor\mathcal{H}) = \begin{pmatrix} (r_1+n_1n_2)I_{n_1}+Q(\mathcal{G}) & -B & -J_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & 2I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & n_1I_{n_2}+Q(\mathcal{H})\otimes I_{n_1} \end{pmatrix}$$

The proof of the remaining part is similar to the proof of Theorem 2.2. □

From Theorem 2.3, we get the following observations.

Observations.

(1) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is a r_2 regular graph, then Q-spectrum of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ contains the following eigenvalues

- (i) 2 with multiplicities $m_1 - n_1$
- (ii) $n_1 + \nu_j(\mathcal{H})$ with multiplicities n_1
- (iv) the roots of the quadratic equation $x^2-(2+r_1+n_1n_2+\nu_i(\mathcal{G}))x+2n_1n_2+2r_1-\nu_i(\mathcal{G}) = 0, i = 2, 3, 4, \dots, n_1$ and
- (v) the roots of the cubic equation $x^3 - (3r_1 + 2 + n_1n_2 + 3r_2)x^2 + (2n_1n_2 + 4r_1 + 3n_1n_2r_2 + 9r_1r_2 + 6r_2 - n_1^2n_2)x - 6n_1n_2r_2 - 12r_1r_2 + 2n_1^2n_2 = 0.$

3. Spectra of the quasi-corona $\mathcal{R}$ edge join $\mathcal{G}\lfloor e\rfloor\mathcal{H}$

We begin with the adjacency spectra of $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ for regular graphs.

Theorem 3.1. *Let $\mathcal{G}$ be an r_1- regular and $\mathcal{H}$ be a r_2 regular graph, then*

$$\Phi_{\mathcal{G}\lfloor e\rfloor\mathcal{H}}(A;x) = x^{m_1-n_1}x\{x^3 - (r_1+r_2)x^2 - (2r_1 + m_1n_1n_2 - r_1r_2)x + 2r_1r_2 + r_1m_1n_1n_2\}\prod_{j=2}^{n_2}\{x - \lambda_j(\mathcal{H})\}^{n_1}$$
$$\prod_{i=2}^{n_1}\{x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda(\mathcal{G})\}.$$

Proof. By labelling the vertices appropriately, $A(\mathcal{G}\lfloor e\rfloor\mathcal{H})$ becomes

$$A(\mathcal{G}\lfloor e\rfloor\mathcal{H}) = \begin{pmatrix} A(\mathcal{G}) & B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & 0_{m_1\times m_1} & J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & J_{n_2\times m_1}\otimes J_{n_1} & A(\mathcal{H})\otimes I_{n_1}\end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned}\Phi_{\mathcal{G}\lfloor e\rfloor\mathcal{H}}(A;x) &= \det\begin{pmatrix} xI_{n_1} - A(\mathcal{G}) & -B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & xI_{m_1} & -J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & -J_{n_2\times m_1}\otimes J_{n_1} & (xI_{n_2} - A(\mathcal{H}))\otimes I_{n_1}\end{pmatrix}\\ &= \det\{(xI_{n_2} - A(\mathcal{H}))\otimes I_{n_1}\}\det S\\ &= \det(xI_{n_2} - A(\mathcal{H}))^{n_1}\det S,\end{aligned}$$

where,

$$\begin{aligned}S &= \begin{pmatrix} xI_{n_1} - A(\mathcal{G}) & -B \\ -B^T & xI_{m_1}\end{pmatrix} - \begin{pmatrix} 0_{n_1\times n_2}\otimes J_{n_1}^T \\ -J_{m_1\times n_2}\otimes J_{n_1}^T\end{pmatrix}\Big((xI_{n_2} - A(\mathcal{H}))^{-1}\otimes I_{n_1}\Big)\Big(0_{n_2\times n_1}\otimes J_{n_1} \quad -J_{n_2\times m_1}\otimes J_{n_1}\Big)\\ &= \begin{pmatrix} xI_{n_1} - A(\mathcal{G}) & -B \\ -B^T & xI_{m_1} - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)J_{m_1\times m_1}\end{pmatrix}\end{aligned}$$

So,

$$\begin{aligned}\det S =& \det\{xI_{m_1} - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)J_{m_1\times m_1}\}\det\{xI_{n_1} - A(\mathcal{G}) - B\Big(xI_{m_1} - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)J_{m_1\times m_1}\Big)^{-1}B^T\}\\
=& x_1^m\left\{1 - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\frac{m_1}{x}\right\}\det\{xI_{n_1} - A(\mathcal{G}) - B(xI_{m_1} - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)J_{m_1\times m_1})^{-1}B^T\}\\
=& x_1^m\left\{1 - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\frac{m_1}{x}\right\}\det\left\{xI_{n_1} - A(\mathcal{G}) - B\left(\frac{1}{x}I_{m_1} + \frac{\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)}{x(x - m_1\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x))}J_{m_1\times m_1}\right)B^T\right\}\\
=& x_1^m\left\{1 - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\frac{m_1}{x}\right\}\det\left\{xI_{n_1} - A(\mathcal{G}) - \frac{BB^T}{x} - \frac{\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)}{x(x - m_1\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x))}r_1^2J_{n_1\times n_1}\right\}\\
=& x_1^m\left\{1 - \Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\frac{m_1}{x}\right\}\det\left\{xI_{n_1} - A(\mathcal{G}) - \frac{BB^T}{x}\right\}\left\{1 - \frac{r_1^2\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x)\Gamma_{A(\mathcal{G})+\frac{BB^T}{x}}(x)}{x(x - m_1\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x))}\right\}.\end{aligned}$$

Since,
$\Gamma_{A(\mathcal{G})+\frac{BB^T}{x}}(x) = \frac{n_1}{x-(r_1+\frac{2r_1}{x})}$ and $\Gamma_{A(\mathcal{H})\otimes I_{n_1}}(x) = \frac{n_1n_2}{x-r_2}$
We get,

$$\det S = x^{m_1-n_1}\prod_{i=1}^{n_1}\{x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda_i(\mathcal{G})\}\{x^4 - (r_1 + r_2)x^3 - (2r_1 + m_1n_1n_2 - r_1r_2)x^2 + (2r_1r_2 + r_1m_1n_1n_2)x\}.$$

Thus, we have

$$\begin{aligned}\Phi_{\mathcal{G}\lfloor e\rfloor\mathcal{H}}(A;x) = x^{m_1-n_1}\prod_{j=2}^{n_2}\{x - \lambda_j(\mathcal{H})\}^{n_1}\prod_{i=2}^{n_1}\{x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda_i(\mathcal{G})\}x\{x^3 - (r_1 + r_2)x^2 - (2r_1 + m_1n_1n_2\\
- r_1r_2)x + (2r_1r_2 + r_1m_1n_1n_2)\}\end{aligned}$$

□

From Theorem 3.1, we get the following observations.

Observations.

(1) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is a r_2 regular graph, then A-spectrum of $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ contains

- (*i*) 0 with multiplicities $m_1 - n_1 + 1$
- (*ii*) $\lambda_j(\mathcal{H})$ with multiplicities n_1
- (*iii*) the roots of the quadratic equation $x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda_i(\mathcal{G}) = 0$ and
- (*iv*) the roots of the quadratic equation $x^3 - (r_1 + r_2)x^2 - (2r_1 + m_1n_1n_2 - r_1r_2)x + (2r_1r_2 + r_1m_1n_1n_2 = 0$.

(2) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H} = K_{n_2}$, then A-spectrum of $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ contains

- (*i*) 0 with multiplicities $m_1 - n_1 + 1$
- (*ii*) $n_2 - 1$ with multiplicities n_1
- (*iii*) -1 with multiplicities $n_1(n_2 - 1)$
- (*iv*) the roots of the equation $x^2 - \lambda_i(\mathcal{G})x - r_1 - \lambda_i(\mathcal{G}) = 0$ and
- (*v*) the roots of the equation $x^3 - (n_2 - 1)x^2 - (2r_1 + n_1m_1n_2)x + 2r_1(n_2 - 1) = 0$.

The next result gives the L-spectrum of $\mathcal{G}\lfloor e\rfloor\mathcal{H}$.

Theorem 3.2. *Let $\mathcal{G}$ be a r_1-regular and $\mathcal{H}$ be any graph, then*

$$\Phi_{\mathcal{G}\lfloor e\rfloor\mathcal{H}}(L;x) = x(x-2-n_1n_2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-m_1-\mu_j(\mathcal{H})\}^{n_1}\prod_{i=2}^{n_1}\{x^2-(r_1+n_1n_2+\mu_i(\mathcal{G})+2)x+3\mu_i(\mathcal{G})+ n_1n_2\mu_i(\mathcal{G})+r_1n_1n_2\}\{x^3-(r_1+2n_1n_2+m_1+4)x^2+(2r_1n_1n_2+4n_1n_2+2r_1+4+n_1^2n_2^2+m_1r_1+4m_1+ n_1n_2m_1)x-(2r_1n_1n_2+r_1n_1^2n_2^2+m_1r_1n_1n_2+2m_1r_1+4m_1+2m_1n_1n_2)\}.$$

Proof. : By labelling the vertices appropriately, $L(\mathcal{G}\lfloor e\rfloor\mathcal{H})$ becomes

$$L(\mathcal{G}\lfloor e\rfloor\mathcal{H}) = \begin{pmatrix} r_1I_{n_1}+L(\mathcal{G}) & -B & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -B^T & (2+n_1n_2)I_{m_1} & -J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times m_1}\otimes J_{n_1} & -J_{n_2\times m_1}\otimes J_{n_1} & m_1I_{n_2}+L(\mathcal{H})\otimes I_{n_1}\end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned}\Phi_{\mathcal{G}\lfloor e\rfloor\mathcal{H}}(L:x) &= \det(xI_{n_1n_2+n_1+m_1}-L(\mathcal{G}\lfloor e\rfloor\mathcal{H}))\\ &= \det\begin{pmatrix}(x-r_1)I_{n_1}-L(\mathcal{G}) & B & 0_{m_1\times n_2}\otimes J_{n_1}^T\\ B^T & (x-2-n_1n_2)I_{m_1} & J_{m_1\times n_2}\otimes J_{n_1}^T\\ 0_{n_2\times m_1}\otimes J_{n_1} & J_{n_2\times m_1}\otimes J_{n_1} & (x-m_1I_{n_2}-L(\mathcal{H}))\otimes I_{n_1}\end{pmatrix}\\ &= \det\{(x-m_1I_{n_2}-L(\mathcal{H}))\otimes I_{n_1}\}\det S\end{aligned}$$

where,

$$\begin{aligned}S &= \begin{pmatrix}x-r_1I_{n_1}-L(\mathcal{G}) & B\\ B^T & (x-2-n_1n_2)I_{m_1}\end{pmatrix} - \begin{pmatrix}0_{m_1\times n_2}\otimes J_{n_1}^T\\ J_{m_1\times n_2}\otimes J_{n_1}^T\end{pmatrix}\Big((x-m_1I_{n_2}-L(\mathcal{H}))^{-1}\otimes I_{n_1}\Big)\Big(0_{n_2\times n_1}\otimes J_{n_1} \quad J_{n_2\times m_1}\otimes J_{n_1}\Big)\\ &= \begin{pmatrix}x-r_1I_{n_1}-L(\mathcal{G}) & B\\ B^T & (x-2-n_1n_2)I_{m_1}-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)J_{m_1\times m_1}.\end{pmatrix}\end{aligned}$$

Therefore, we get

$$\begin{aligned}\det S &= \det\Big\{(x-2-n_1n_2)I_{m_1}-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)J_{m_1\times m_1}\Big\}\det\Big\{x-r_1I_{n_1}-L(\mathcal{G})-\\ &\quad B\Big((x-2-n_1n_2)I_{m_1}-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)J_{m_1\times m_1}\Big)^{-1}B^T\Big\}\\ &=(x-2-n_1n_2)^{m_1}\Big\{1-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)\frac{m_1}{x-2-n_1n_2}\Big\}\det\Big\{x-r_1I_{n_1}-L(G)-\\ &\quad B\Big(\frac{1}{x-2-n_1n_2}I_{m_1}+\frac{\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)}{(x-2-n_1n_2)(x-2-n_1n_2-m_1\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1))}J_{m_1\times m_1}\Big)B^T\Big\}\\ &=(x-2-n_1n_2)^{m_1}\Big\{1-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)\frac{m_1}{x-2-n_1n_2}\Big\}\det\Big\{(x-r_1)I_{n_1}-L(G)-\\ &\quad \frac{BB^T}{x-2-n_1n_2}-\frac{\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)}{(x-2-n_1n_2)\big(x-2-n_1n_2-m_1\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)\big)}r_1^2J_{n_1\times n_1}\Big\}\\ &=(x-2-n_1n_2)^{m_1}\Big\{1-\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)\frac{m_1}{x-2-n_1n_2}\Big\}\det\Big\{(x-r_1)I_{n_1}-L(G)-\\ &\quad \frac{BB^T}{x-2-n_1n_2}-\frac{\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)}{(x-2-n_1n_2)(x-2-n_1n_2-m_1\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1))}r_1^2J_{n_1\times n_1}\Big\}\\ &=(x-2-n_1n_2)^{m_1}\Big\{1-\frac{n_1n_2}{x-m_1}.\frac{m_1}{x-2-n_1n_2}\Big\}\det\Big\{(x-r_1)I_{n_1}-L(\mathcal{G})-\frac{BB^T}{x-2-n_1n_2}\Big\}\\ &\quad \Big\{(1-\frac{\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1)}{(x-2-n_1n_2)(x-2-n_1n_2-m_1\Gamma_{L(\mathcal{H})\otimes I_{n_1}}(x-m_1))}r_1^2\Gamma_{L(\mathcal{G})+\frac{BB^T}{x-2-n_1n_2}}(x-r_1)\Big\}.\end{aligned}$$

Thus, we have

$$\Phi_{\mathcal{G}\lfloor e\rceil\mathcal{H}}(L;x) = x(x-2-n_1n_2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-m_1-\mu_j(\mathcal{H})\}^{n_1}\prod_{i=2}^{n_1}\{x^2-(r_1+n_1n_2+\mu_i(\mathcal{G})+2)x+3\mu_i(\mathcal{G})+ n_1n_2\mu_i(\mathcal{G})+r_1n_1n_2\}\{x^3-(r_1+2n_1n_2+m_1+4)x^2+(2r_1n_1n_2+4n_1n_2+2r_1+4+n_1^2n_2^2+m_1r_1+4m_1+ n_1n_2m_1)x-(2r_1n_1n_2+r_1n_1^2n_2^2+m_1r_1n_1n_2+2m_1r_1+4m_1+2m_1n_1n_2)\}.$$

□

From Theorem 3.2, we get the following observations.

Observations. If $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is any graph, then L-spectra of $\mathcal{G}\lfloor e\rceil\mathcal{H}$ contains

(*i*) 0

(*ii*) $2+n_1n_2$ with multiplicities m_1-n_1

(*iii*) $m_1+\mu_i(\mathcal{H})$, where $i=2,3,4,...,n_2$ with multiplicities n_1

(*iv*) the roots of the quadratic equation $x^2-\{r_1+2+n_1n_2+\mu_i(\mathcal{G})\}x+r_1n_1n_2+3\mu_i(\mathcal{G})+n_1n_2\mu_i(\mathcal{G})=0$, $i=2,3,4,...,n_1$ and

(*v*) the roots of the cubic equation $x^3-(r_1+2n_1n_2+m_1+4)x^2+(2r_1n_1n_2+4n_1n_2+2r_1+4+n_1^2n_2^2+m_1r_1+4m_1+n_1n_2m_1)x-(2r_1n_1n_2+r_1n_1^2n_2^2+m_1r_1n_1n_2+2m_1r_1+4m_1+2m_1n_1n_2)=0$

Next, we get Q-spectrum of $\mathcal{G}\lfloor e\rceil\mathcal{H}$

Theorem 3.3. *Let $\mathcal{G}$ be an r_1-regular and $\mathcal{H}$ be a r_2 regular graph, then*

$$\Phi_{\mathcal{G}\lfloor e\rceil\mathcal{H}}(Q;x) = (x-2-n_2n_2)^{m_1-n_1}\prod_{i=2}^{n_1}\{x^2-(r_1+2+n_1n_2+\nu_j(\mathcal{G}))x+2r_1+n_1n_2r_1+\nu_j(\mathcal{G})+n_1n_2\nu_j(\mathcal{G})\}$$
$$\prod_{j=2}^{n_2} x-m_1-\nu_j(\mathcal{H})^{n_1}(x^4-(4+2n_1n_2+m_1+3r_1)x^3+(4m_1+8r_2+4+4n_1n_2+4n_1n_2r_2+n_1^2n_2^2+ n_1m_1n_2+10r_1+3r_1n_1n_2+3r_1m_1+6r_1r_2)x^2-(4m_1+2m_1n_1n_2+4n_1n_2r_2+4r_1n_1n_2+2n_1^2n_2^2r_2+ 6r_1m_1+12r_1r_2+8r_1+4r_1m_1-6r_1^2n_1n_2r_2^2)x+8r_1m_1+16r_1r_2-8r_1r_2n_1n_2-r_1^2n_1^2n_2)$$

Proof. The Q- matrix $(\mathcal{G}\lfloor v\rceil\mathcal{H})$ can be written as

$$L(\mathcal{G}\lfloor e\rceil\mathcal{H}) = \begin{pmatrix} r_1I_{n_1}+Q(\mathcal{G}) & -B & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -B^T & (2+n_1n_2)I_{m_1} & -J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times m_1}\otimes J_{n_1} & -J_{n_2\times m_1}\otimes J_{n_1} & m_1I_{n_2}+Q(\mathcal{H})\otimes I_{n_1} \end{pmatrix}$$

The proof of the remaining part of the theorem is similar to Theorem 3.2. □

From Theorem 3.3, we get the following observations.

Observations.

(1) If $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is any graph, then Q-spectra of $\mathcal{G}\lfloor e\rceil\mathcal{H}$ contains

- (*i*) $2+n_1n_2$ with multiplicities m_1-n_1
- (*ii*) $m_1+\nu_j(\mathcal{H})$, where $j=2,3,4,...,n_2$ with multiplicities n_1
- (*iii*) the roots of the quadratic equation $x^2-(r_1+2+n_1n_2+\nu_j(\mathcal{G}))x+2r_1+n_1n_2r_1+\nu_j(\mathcal{G})+n_1n_2\nu_j(\mathcal{G})=0$, $i=2,3,4,...,n_1$ and
- (*iv*) the roots of the bi-quadratic equation $x^4-(4+2n_1n_2+m_1+3r_1)x^3+(4m_1+8r_2+4+4n_1n_2+4n_1n_2r_2+n_1^2n_2^2+n_1m_1n_2+10r_1+3r_1n_1n_2+3r_1m_1+6r_1r_2)x^2-(4m_1+2m_1n_1n_2+4n_1n_2r_2+4r_1n_1n_2+2n_1^2n_2^2r_2+6r_1m_1+12r_1r_2+8r_1+4r_1m_1-6r_1^2n_1n_2r_2^2)x+8r_1m_1+16r_1r_2-8r_1r_2n_1n_2-r_1^2n_1^2n_2=0$

4. Pair of simultaneous cospectral graphs

From Theorems 2.1, 2.2, 2.3, 3.1, 3.2 and 3.3, it is observed that the A, L and Q-spectra of the join graphs $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ and $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ depend only on the number of vertices, edges, degree of regularities and the corresponding spectrum of $\mathcal{G}$ and $\mathcal{H}$. The following are the main observations.

Observations.

(1) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two regular graphs that are both A and L-cospectral, and let $\mathcal{F}$ be any graph. Then, $\mathcal{F}_1\lfloor u\rfloor\mathcal{F}$ (respectively, $\mathcal{F}_1\lfloor e\rfloor\mathcal{F}$) and $\mathcal{F}_2\lfloor u\rfloor\mathcal{F}$ (respectively, $\mathcal{F}_2\lfloor e\rfloor\mathcal{F}$) are also simultaneously A and L-cospectral.

(2) Let $\mathcal{F}$ is a regular graph, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ are any two graphs that are both A and L-cospectral, then $\mathcal{F}\lfloor u\rfloor\mathcal{F}_1$ (respectively, $\mathcal{F}\lfloor e\rfloor\mathcal{F}_1$) and $\mathcal{F}\lfloor u\rfloor\mathcal{F}_2$ (respectively, $\mathcal{F}\lfloor e\rfloor\mathcal{F}_2$) are also simultaneously A and L-cospectral.

(3) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are any two regular graphs that are L or Q-cospectral and let $\mathcal{H}_1$ and $\mathcal{H}_2$ are any two regular L or Q-cospectral, then, $\mathcal{F}_1\lfloor u\rfloor\mathcal{H}_1$ (respectively, $\mathcal{F}_1\lfloor e\rfloor\mathcal{H}_1$) and $\mathcal{F}_2\lfloor u\rfloor\mathcal{H}_2$ (respectively, $\mathcal{F}_2\lfloor e\rfloor\mathcal{H}_2$) are also L or Q-cospectral.

5. Applications

As an application of these two graph operations, we determine the following invariants from the Laplacian spectra of the join of graphs $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ and $\mathcal{G}\lfloor e\rfloor\mathcal{H}$, where $\mathcal{G}$ is an r_1 regular and $\mathcal{H}$ is any graph.

(1) $Kf(\mathcal{G}\lfloor u\rfloor H)$, $LEL(\mathcal{G}\lfloor u\rfloor\mathcal{H})$ and $t(\mathcal{G}\lfloor u\rfloor H)$ of $\mathcal{G}\lfloor u\rfloor\mathcal{H}$ are as follows

(i) $Kf(\mathcal{G}\lfloor u\rfloor H) = (m_1 + n_1 + n_1n_2)\left(\frac{m_1-n_1}{2} + \frac{2+r_1+n_1+n_1n_2}{2n_1+2n_1n_2+n_1r_1} + \sum_{j=2}^{n_2}\frac{n_1}{n_1+\mu_j(H)} + \sum_{i=2}^{n_1}\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})}{2n_1n_2+3\mu_i(\mathcal{G})}\right)$

(ii) $LEL(\mathcal{G}\lfloor u\rfloor\mathcal{H}) = (m_1 - n_1)2^{1/2} + n_1\{n_1 + \mu_j(\mathcal{H})\}^{1/2} + \left(\frac{r_1+2+n_1+n_1n_2+\sqrt{\Delta_1}}{2}\right)^{1/2} + \left(\frac{r_1+2+n_1+n_1n_2-\sqrt{\Delta_1}}{2}\right)^{1/2} + \left(\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})+\sqrt{\Delta_2}}{2}\right)^{1/2} + \left(\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})-\sqrt{\Delta_2}}{2}\right)^{1/2}$, where $\Delta_1 = (2+r_1+n_1+n_1n_2)^2 - 4(2n_1+n_1r_1+2n_1n_2)$ and $\Delta_2 = \{2 + r_1 + n_1n_2 + \mu_i(\mathcal{G})\}^2 - 4\{2n_1n_2 + 3\mu_i(\mathcal{G})\}$

(iii) $t(\mathcal{G}\lfloor u\rfloor H) = \frac{2^{m_1-n_1}(2n_1+n_1r_1+2n_1n_2)\prod_{j=2}^{n_2}(n_1+\mu_j(H))^{n_1}\prod_{i=2}^{n_1}(2n_1n_2+3\mu_i(\mathcal{G}))}{n_1+m_1+n_1n_2}$

(2) $Kf(\mathcal{G}\lfloor e\rfloor H)$, $LEL(\mathcal{G}\lfloor e\rfloor\mathcal{H})$ and $t(\mathcal{G}\lfloor e\rfloor H)$ of $\mathcal{G}\lfloor e\rfloor\mathcal{H}$ are as follows

(i) $Kf(\mathcal{G}\lfloor e\rfloor\mathcal{H}) = (m_1+n_1+n_1n_2)\left\{\frac{m_1-n_1}{2+n_1n_2} + \sum_{i=2}^{n_1}\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})}{r_1n_1n_2+3\mu_i(\mathcal{G})+n_1n_2\mu_i(\mathcal{G})} + \left(\frac{2r_1n_1n_2+m_1r_1+4m_1+m_1n_1n_2+2r_1+4+4n_1n_2+n_1^2n_2^2}{m_1r_1n_1n_2+2r_1n_1n_2+2m_1r_1+4m_1+2m_1n_1n_2+n_1^2n_2^2r_1}\right) + \sum_{j=2}^{n_2}\frac{n_1}{m_1+\mu_j(\mathcal{H})}\right\}$

(ii) $LEL(\mathcal{G}\lfloor e\rfloor\mathcal{H}) = (m_1-n_1)(2+n_1n_2)^{1/2} + n_1\{m_1+\mu_j(\mathcal{H})\}^{1/2} + \left(\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})+\sqrt{\Delta_3}}{2}\right)^{1/2} + \left(\frac{r_1+2+n_1n_2+\mu_i(\mathcal{G})-\sqrt{\Delta_3}}{2}\right)^{1/2} + {\omega_i}^{1/2}$, where $\Delta_3 = \{2 + r_1 + \mu_i(\mathcal{G}) + n_1n_2\}^2 - 4\{r_1n_1n_1 + n_1n_2\mu_i(\mathcal{G}) + 3\mu_i(\mathcal{G})\}$ and $\omega_i, i = 1, 2, 3$ are the roots of the cubic equation.

(iii) $t(\mathcal{G}\lfloor e\rfloor\mathcal{H}) = \frac{1}{n_1+m_1+n_1n_2}\{(2 + n_1n_2)^{m_1-n_1}(m_1r_1n_1n_2 + 2r_1n_1n_2 + 2m_1r_1 + 4m_1 + 2m_1n_1n_2 + n_1^2n_2^2r_1)$ $\prod_{i=2}^{n_1}(r_1n_1n_2 + n_1n_2\mu_i(\mathcal{G}) + 3\mu_i(\mathcal{G}))\prod_{j=2}^{n_2}(m_1 + \mu_j(\mathcal{H}))^{n_1}\}$

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