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Quotient spaces of strongly topological gyrogroups

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Abstract. The concept of gyrogroups was introduced during the researches of Einstein velocity addition that Einstein introduced in [16] which founded the special theory of relativity. In this paper, we mainly investigate the quotient spaces generated from a strongly topological gyrogroup with respect to closed strong subgyrogroups. It is shown that if G is a strongly topological gyrogroup and H is a closed first-countable and separable strong subgyrogroup of G, the quotient space G/H is an $\aleph_0$-space, then G is an $\aleph_0$-space; if the quotient space G/H is a cosmic space, then G is also a cosmic space; if the quotient space G/H has a star-countable *cs*-network or star-countable wcs^*-network, then G also has a star-countable *cs*-network or star-countable wcs^*-network, respectively.

1. Introduction

By the study of the c-ball of relativistically admissible velocities with Einstein velocity addition that Einstein introduced in [16] which founded the special theory of relativity, A.A. Ungar discovered that the seemingly structureless Einstein addition of relativistically admissible velocities possesses a rich grouplike structure and he introduced the concept of gyrogroups in [34]. Clearly, every group is a gyrogroup. Then the gyrogroup was equipped with a topology by W. Atiponrat [3] such that the binary operation $\oplus : G \times G \to G$ is jointly continuous and the inverse mapping $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous and she called it a topological gyrogroup. By further researches of the classical Möbius gyrogroups, Einstein gyrogroups, and Proper Velocity gyrogroups, Bao and Lin [6] found that each of them has an open neighborhood base at the identity element 0 such that all elements of the base are invariant under the groupoid automorphisms with standard topology. Therefore, they posed the concept of strongly topological gyrogroups. A series of results on topological gyrogroups and strongly topological gyrogroups have been obtained in [2, 4, 5, 7–11, 20–23, 29, 36–38].

In [32], T. Suksumran and K. Wiboonton introduced the notion of L-subgyrogroups and showed that if H is an L-subgyrogroup of a gyrogroup G, then the set $\{a \oplus H : a \in G\}$ forms a disjoint partition of G. Then, Bao and Xu [12] constructed a subgyrogroup H in a strongly topological gyrogroup G such that $gyr[x, y](H) = H$ for all $x, y \in G$, hence they introduced the concept of strong subgyrogroups in topological

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gyrogroup. Clearly, for a topological gyrogroup G, every strong subgyrogroup of G is an L-subgyrogroup. Therefore, it is natural to investigate the quotient spaces of a strongly topological gyrogroup with respect to strong subgyrogroups. In particular, since every topological group is a strongly topological gyrogroup, we would like to extend some important results of coset spaces of topological groups to quotient spaces of strongly topological gyrogroups.

As we all known, being separable or metrizable is a three space property on topological groups, and every separable metrizable space is a cosmic space, but being a cosmic space is not a three space property [35]. Moreover, it is proved in [26, Corollary 3.5] that if H is a closed second-countable subgroup of a topological group G, and the quotient space G/H is an $\aleph_0$-space (resp., cosmic space), then G is also an $\aleph_0$-space (resp., cosmic space). In this paper, we extend these important results to strongly topological gyrogroups. We show that if G is a strongly topological gyrogroup and H is a closed first-countable and separable strong subgyrogroup of G, the quotient space G/H is an $\aleph_0$-space, then G is an $\aleph_0$-space; if the quotient space G/H is a cosmic space, then G is also a cosmic space; if the quotient space G/H has a star-countable cs-network or star-countable wcs^*-network, then G also has a star-countable cs-network or star-countable wcs^*-network, respectively.

2. Preliminaries

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. The readers may consult [1, 17, 27, 33] for notation and terminology not explicitly given here. Next we recall some definitions and facts.

Definition 2.1. ([3]) Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a *groupoid* or a *magma*. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a *groupoid homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We write $\mathrm{Aut}(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.2. ([33]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions:

$(G1)$ There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$.

$(G2)$ For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.

$(G3)$ For all $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$.

$(G4)$ For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$.

Lemma 2.3. ([33]) *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the followings:*

1. $(\ominus x) \oplus (x \oplus y) = y$. (left cancellation law)
2. $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$. (right cancellation law)
3. $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$.
4. $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$.
5. $(x \oplus y) \oplus z = x \oplus (y \oplus gyr[y, x](z))$.

Proposition 2.4. ([32]) *Let G be a gyrogroup and let $X \subseteq G$. Then the followings are equivalent:*
(1) $gyr[a, b](X) \subseteq X$ for all $a, b \in G$;
(2) $gyr[a, b](X) = X$ for all $a, b \in G$.

Notice that a group is a gyrogroup $(G, \oplus)$ such that $\mathrm{gyr}[x, y]$ is the identity function for all $x, y \in G$. The definition of a subgyrogroup is as follows.

Definition 2.5. ([32]) Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a *subgyrogroup*, denoted by $H \leq G$, if H forms a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup*, denoted by $H \leq_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Definition 2.6. ([3]) A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup* if the following statements hold:

(1) (G, τ) is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

Obviously, every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group. In particular, it was proved that the Einstein gyrogroup with the standard topology is a topological gyrogroup but not a topological group, see Example 2 in [3].

Next, we introduce the definition of a strongly topological gyrogroup, it is very important in this paper.

Definition 2.7. ([6]) Let G be a topological gyrogroup. We say that G is a *strongly topological gyrogroup* if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\mathrm{gyr}[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0.

For each $U \in \mathscr{U}$, we can set $V = U \cup (\ominus U)$. Then,

$$\mathrm{gyr}[x, y](V) = \mathrm{gyr}[x, y](U \cup (\ominus U)) = \mathrm{gyr}[x, y](U) \cup (\ominus \mathrm{gyr}[x, y](U)) = U \cup (\ominus U) = V,$$

for all $x, y \in G$. Obviously, the family $\{U \cup (\ominus U) : U \in \mathscr{U}\}$ is also a neighborhood base of 0. Therefore, we may assume that U is symmetric for each $U \in \mathscr{U}$ in Definition 2.7.

In [6], the authors proved that there is a strongly topological gyrogroup which is not a topological group, see Example 3.1 in [6].

Definition 2.8. ([12]) A subgyrogroup H of a topological gyrogroup G is called *strong subgyrogroup* if for any $x, y \in G$, we have $gyr[x, y](H) = H$.

It is noted that every strongly topological gyrogroup G contains some strong subgyrogroups which are union-generated from open neighborhoods of the identity element by construction, see [12, Proposition 3.11].

We recall the following concept of the coset space of a topological gyrogroup.

Let $(G, \tau, \oplus)$ be a topological gyrogroup and H an L-subgyrogroup of G. It follows from [32, Theorem 20] that $G/H = \{a \oplus H : a \in G\}$ is a coset space which defines a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Clearly, for each $a \in G$, we have $\pi^{-1}(\pi(a)) = a \oplus H$. Indeed, for any $a \in G$ and $h \in H$,

$$\begin{aligned}
(a \oplus h) \oplus H &= a \oplus (h \oplus \mathrm{gyr}[h, a](H)) \\
&= a \oplus (h \oplus \mathrm{gyr}^{-1}[a, h](H)) \\
&= a \oplus (h \oplus H) \\
&= a \oplus H
\end{aligned}$$

Denote by $\tau(G)$ the topology of G, the quotient topology on G/H is as follows:

$$\tau(G/H) = \{O \subseteq G/H : \pi^{-1}(O) \in \tau(G)\}.$$

Throughout this paper, denote by π the natural homomorphism from a topological gyrogroup G to its quotient topology on G/H.

3. Quotient spaces with locally compact strong subgyrogroups

In [14], the authors proved the following result, which generalized the well-known result [1, Theorem 3.2.2], which is important in this section.

Theorem 3.1. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. Then there exists an open neighborhood U of the identity element 0 such that $\pi(\overline{U})$ is closed in G/H and the restriction of π to $\overline{U}$ is a perfect mapping from $\overline{U}$ onto the subspace $\pi(\overline{U})$.*

Then, we give some applications about Theorem 3.1 combining generalized metric properties. In particular, we assume that the strong subgyrogroup H of a strongly topological gyrogroup G is locally compact.

A topological gyrogroup is *feathered* [6] if it contains a non-empty compact set K of countable character in G. A space X is said to be a *p-space* [17] if it is Tychonoff and there exists a countable collection $\mathscr{E}=\{\gamma_n : n \in \mathbb{N}\}$ of families γ_n of open sets in the Čech-Stone compactification βX of X such that $\bigcap\{St(x,\gamma_n) : n \in \mathbb{N}\} \subseteq X$, for every $x \in X$.

Lemma 3.2. ([9]) *Let G be a strongly topological gyrogroup. Then the followings are equivalent:*

(1) *G is feathered,*

(2) *G is a p-space, and*

(3) *G is a paracompact p-space.*

Theorem 3.3. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. If the quotient space G/H is a p-space, then G is a paracompact p-space.*

Proof. By Theorem 3.1, there exists an open neighborhood U of the identity element 0 in G such that $\overline{U}$ is a preimage of a closed subset of G/H under a perfect mapping. Moreover, since the class of p-spaces is closed under taking closed subspaces, it follows from [1, Proposition 4.3.36] that $\overline{U}$ is a p-space, hence a feathered space. Therefore, U contains a non-empty compact subspace F with a countable base of neighborhoods in G, thus G is a paracompact p-space by Lemma 3.2. □

Definition 3.4. ([31]) Let X be a topological space. A space is called *strictly Fréchet-Urysohn at a point $x \in X$* if whenever $\{A_n\}_n$ is a sequence of subsets in X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists $x_n \in A_n$ for each $n \in \mathbb{N}$ such that the sequence $\{x_n\}_n$ converges to x. A space X is called *strictly Fréchet-Urysohn* if it is strictly Fréchet-Urysohn at every point $x \in X$.

Lemma 3.5. ([26]) *Suppose that X is a regular space, and that $f : X \to Y$ is a closed mapping. Suppose also that $b \in X$ is a G_δ-point in the space $F = f^{-1}(f(b))$ (i.e., the singleton $\{b\}$ is a G_δ-set in the space F) and F is countably compact and strictly Fréchet-Urysohn at b. If the space Y is strictly Fréchet-Urysohn at $f(b)$, then X is strictly Fréchet-Urysohn at b.*

Theorem 3.6. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H is strictly Fréchet-Urysohn, then G is also strictly Fréchet-Urysohn.*

Proof. By Theorem 3.1, there exists an open neighborhood U of the identity element 0 in G such that $\pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H.

Put $f = \pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$. Then $f(\overline{U}) = \pi(\overline{U})$ is strictly Fréchet-Urysohn. For each $b \in \overline{U}$, $f^{-1}(f(b)) = \pi^{-1}(\pi(b)) \cap \overline{U} = (b \oplus H) \cap \overline{U}$ is compact and metrizable. It follows from Lemma 3.5 that $\overline{U}$ is strictly Fréchet-Urysohn. Therefore, G is locally strictly Fréchet-Urysohn and G is strictly Fréchet-Urysohn. □

However, for the property of Fréchet-Urysohn, we do not know whether it has the similar result. Therefore, we pose the following question.

Question 3.7. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H is Fréchet-Urysohn, is G also Fréchet-Urysohn?*

Theorem 3.8. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H has property $\mathcal{P}$, where $\mathcal{P}$ is a topological property. Then the space G is locally in $\mathcal{P}$ if $\mathcal{P}$ satisfies the following:*

(1) $\mathcal{P}$ is closed hereditary;

(2) $\mathcal{P}$ contains point G_δ-property, and

(3) let $f : X \to Y$ be a perfect mapping, if X has G_δ-diagonal and Y is $\mathcal{P}$, then X is $\mathcal{P}$.

Proof. By the hypothesis, since G/H is in $\mathcal{P}$ and $\mathcal{P}$ contains point G_δ-property, $\{H\}$ is a G_δ-subset in G/H, that is, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open sets in G/H such that $\{H\} = \bigcap_{n\in\mathbb{N}} V_n$. Therefore, $H = \bigcap_{n\in\mathbb{N}} \pi^{-1}(V_n)$. Since H is a metrizable strong subgyrogroup of G, there is a family $\{W_n : n \in \mathbb{N}\}$ of open neighborhoods of the identity element 0 such that $\{W_n \cap H : n \in \mathbb{N}\}$ is an open countable neighborhood base in H. Hence,

$$\{0\} = \bigcap_{n\in\mathbb{N}}(W_n \cap H) = \bigcap_{n\in\mathbb{N}}(W_n \cap \pi^{-1}(V_n)).$$

Then G has point G_δ-property. It follows from [7] that every strongly topological gyrogroup with countable pseudocharacter is submetrizable. So G has G_δ-diagonal.

By Theorem 3.1, there is an open neighborhood U of the identity element 0 in G such that $\pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H. Then by (1) and (3), the subspace $\overline{U}$ is in $\mathcal{P}$. Therefore, G is locally in $\mathcal{P}$. □

It is well-known that all stratifiable spaces, semi-stratifiable spaces and σ-spaces satisfy the conditions in Theorem 3.8, respectively. Moreover, it was claimed in [10] and [25] that if a strongly topological gyrogroup G has point G_δ-property, then G has a KG-sequence and if $f : X \to Y$ is a perfect map and Y is a k-semistratifiable space, then X is a k-semistratifiable space if and only if X has a KG-sequence. Therefore, the following corollary is obtained.

Corollary 3.9. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H is a stratifiable space (semi-stratifiable space, k-semistratifiable, σ-space), then G is a locally stratifiable space (semi-stratifiable space, k-semistratifiable, σ-space).*

4. Quotient spaces with closed first-countable and separable strong subgyrogroups

In this section, we study the quotient space G/H with some generalized metric properties, where G is a strongly topological gyrogroup and H is a closed first-countable and separable strong subgyrogroup of G. In particular, we prove that if the quotient space G/H is an $\aleph_0$-space, then G is an $\aleph_0$-space; if the quotient space G/H is a cosmic space, then G is also a cosmic space; if the quotient space G/H has a star-countable cs-network or star-countable wcs^*-network, then G also has a star-countable cs-network or star-countable wcs^*-network, respectively.

Definition 4.1. ([19, 24]) Let $\mathcal{P}$ be a family of subsets of a topological space X.

1. $\mathcal{P}$ is called a *k-network* for X if whenever $K \subseteq U$ with K compact and U open in X, there exists a finite family $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \bigcup \mathcal{P}' \subseteq U$.

2. $\mathcal{P}$ is called a *wcs^*-network* for X if, given a sequence $\{x_n\}_n$ converging to a point x in X and a neighborhood U of x in X, there exists a subsequence $\{x_{n_i}\}_i$ of the sequence $\{x_n\}_n$ such that $\{x_{n_i} : i \in \mathbb{N}\} \subseteq P \subseteq U$ for some $P \in \mathcal{P}$.

Definition 4.2. ([30]) Let X be a topological space.

1. X is called *cosmic* if X is a regular space with a countable network.

2. X is called an $\aleph_0$-space if it is a regular space with a countable k-network.

It was claimed in [27] that every base is a k-network and a cs-network for a topological space, and every k-network or every cs-network is a wcs^*-network for a topological space, but the converse does not hold. Moreover, a space X has a countable cs-network if and only if X has a countable k-network if and only if X has a countable wcs^*-network, see [26]. Therefore, it is natural that a topological space is an $\aleph_0$-space if and only if it is a regular space with a countable cs-network. Moreover, every $\aleph_0$-space is a cosmic space and every cosmic space is a paracompact and separable space.

The following lemmas are necessary.

Lemma 4.3. ([6]) *Let G be a topological gyrogroup and H an L-subgyrogroup of G. Then the natural homomorphism π from a topological gyrogroup G to its quotient topology on G/H is an open and continuous mapping.*

Lemma 4.4. ([10]) *Suppose that G is a topological gyrogroup and H is a closed and separable L-subgyrogroup of G. If Y is a separable subset of G/H, $\pi^{-1}(Y)$ is also separable in G.*

Lemma 4.5. ([8]) *Every locally paracompact strongly topological gyrogroup is paracompact.*

A family $\mathscr{P}$ of subsets of a topological space X is called *star-countable* [17] if the collection $\{P \in \mathscr{P} : P \cap P_0 \neq \emptyset\}$ is countable for any $P_0 \in \mathscr{P}$.

Lemma 4.6. ([15]) *Every star-countable family $\mathcal{P}$ of subsets of a topological space X can be expressed as $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha \in \Lambda\}$, where each subfamily $\mathcal{P}_\alpha$ is countable and $(\bigcup \mathcal{P}_\alpha) \cap (\bigcup \mathcal{P}_\beta) = \emptyset$ whenever $\alpha \neq \beta$.*

Theorem 4.7. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H is a local $\aleph_0$-space, then G is a topological sum of $\aleph_0$-subspace.*

Proof. Let G be a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$ at 0. Since the quotient space G/H is a local $\aleph_0$-space, we can find an open neighborhood Y of H in G/H such that Y has a countable cs-network. Put $X = \pi^{-1}(Y)$. By Lemma 4.3, the natural homomorphism π from G onto G/H is an open and continuous mapping, so X is an open neighborhood of the identity element 0 in G. Since Y is an $\aleph_0$-space and each $\aleph_0$-space is separable, it follows from Lemma 4.4 that X is separable. Therefore, there is countable subset $B = \{b_m : m \in \mathbb{N}\}$ of X such that $\overline{B} = X$.

Since H is first-countable, there exists a countable family $\{V_n : n \in \mathbb{N}\} \subseteq \mathscr{U}$ of open symmetric neighborhoods of 0 in G such that $V_{n+1} \oplus (V_{n+1} \oplus V_{n+1}) \subseteq V_n \subseteq X$ for each $n \in \mathbb{N}$ and the family $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H. Since Y is an $\aleph_0$-space, there is a countable cs-network $\{P_k : k \in \mathbb{N}\}$ for Y.

Claim 1. X is an $\aleph_0$-space.

Put $\mathcal{F} = \{\pi^{-1}(P_k) \cap (b_m \oplus V_n) : k, m, n \in \mathbb{N}\}$. Then $\mathcal{F}$ is a countable family of subsets of X. Suppose that $\{x_i\}_i$ is a sequence converging to a point x in X and U be a neighborhood of x in X. Then U is also a neighborhood of x in G. Let V be an open neighborhood of 0 in G such that $x \oplus (V \oplus V) \subseteq U$. Since $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H, there is $n \in \mathbb{N}$ such that $V_n \cap H \subseteq V \cap H$. Moreover, $(x \oplus V_{n+1}) \cap X$ is a non-empty open subset of X and $\overline{B} = X$, whence $B \cap (x \oplus V_{n+1}) \neq \emptyset$. Therefore, there exists $b_m \in B$ such that $b_m \in x \oplus V_{n+1}$. Furthermore, $(x \oplus V_{n+1}) \cap (x \oplus V)$ is an open neighborhood of x and $\pi : G \to G/H$ is an open mapping, so $\pi((x \oplus V_{n+1}) \cap (x \oplus V))$ is an open neighborhood of $\pi(x)$ in the space Y and the sequence $\{\pi(x_i)\}_i$ converges to $\pi(x)$ in Y. It is obtained that

$$\{\pi(x)\} \cup \{\pi(x_i) : i \geq i_0\} \subseteq P_k \subseteq \pi((x \oplus V_{n+1}) \cap (x \oplus V)) \text{ for some } i_0, k \in \mathbb{N}.$$

By the left cancellation law of Lemma 2.3, it is easy to verify that $(x \oplus V_{n+1}) \cap (x \oplus V) = x \oplus (V_{n+1} \cap V)$. Therefore, for an arbitrary $z \in \pi^{-1}(P_k) \cap (b_m \oplus V_{n+1})$, $\pi(z) \in P_k \subseteq \pi(x \oplus (V_{n+1} \cap V))$. Since $z \in (x \oplus (V_{n+1} \cap V)) \oplus H$, and H is a strong subgyrogroup, then

$$z \in (x \oplus (V_{n+1} \cap V)) \oplus H = \bigcup_{t \in V_{n+1} \cap V} \{(x \oplus t) \oplus H\} = \bigcup_{t \in V_{n+1} \cap V} \{x \oplus (t \oplus \mathrm{gyr}[t, x](H))\} = x \oplus ((V_{n+1} \cap V) \oplus H).$$

Therefore, $\ominus x \oplus z \in (V_{n+1} \cap V) \oplus H$. Moreover, from $z \in b_m \oplus V_{n+1}$ and $b_m \in x \oplus V_{n+1}$, it follows that

$$\begin{aligned} z &\in (x \oplus V_{n+1}) \oplus V_{n+1} \\ &= \bigcup_{u,v \in V_{n+1}} \{(x \oplus u) \oplus v\} \\ &= \bigcup_{u,v \in V_{n+1}} \{x \oplus (u \oplus \mathrm{gyr}[u,x](v))\} \\ &= x \oplus (V_{n+1} \oplus V_{n+1}). \end{aligned}$$

So, $(\ominus x) \oplus z \in V_{n+1} \oplus V_{n+1}$. Hence, $(\ominus x) \oplus z \in ((V_{n+1} \cap V) \oplus H) \cap (V_{n+1} \oplus V_{n+1})$. There exist $a \in (V_{n+1} \cap V)$, $h \in H$ and $u_3, v_3 \in V_{n+1}$ such that $(\ominus x) \oplus z = a \oplus h = u_3 \oplus v_3$, whence $h = (\ominus a) \oplus (u_3 \oplus v_3) \in V_{n+1} \oplus (V_{n+1} \oplus V_{n+1}) \subseteq V_n$. Therefore, $(\ominus x) \oplus z \in (V_{n+1} \cap V) \oplus (V_n \cap H)$, and consequently, $z \in x \oplus ((V_{n+1} \cap V) \oplus (V_n \cap H)) \subseteq x \oplus (V \oplus V) \subseteq U$. Thus, we obtain that $\pi^{-1}(P_k) \cap (b_m \oplus V_{n+1}) \subseteq U$.

Since $b_m \in x \oplus V_{n+1}$, there is $u \in V_{n+1}$ such that $b_m = x \oplus u$, whence

$$\begin{aligned} x &= (x \oplus u) \oplus \mathrm{gyr}[x,u](\ominus u) \\ &= b_m \oplus \mathrm{gyr}[x,u](\ominus u) \\ &\in b_m \oplus \mathrm{gyr}[x,u](V_{n+1}) \\ &= b_m \oplus V_{n+1}. \end{aligned}$$

Therefore, there exists $i_1 \geq i_0$ such that $x_i \in b_m \oplus V_{n+1}$ when $i \geq i_1$, whence $\{x\} \cup \{x_i : i \geq i_1\} \subseteq \pi^{-1}(P_k) \cap (b_m \oplus V_{n+1})$. Thus $\mathcal{F}$ is a countable *cs*-network for X, and hence X is an $\aleph_0$-space.

Since G is homogeneous, it is clear that G is a local $\aleph_0$-space. Therefore, G is a locally paracompact space. Furthermore, every locally paracompact strongly topological gyrogroup is paracompact by Lemma 4.5, so G is paracompact. Let $\mathcal{A}$ be an open cover of G by $\aleph_0$-subspace. Because the property of being an $\aleph_0$-space is hereditary, we can assume that $\mathcal{A}$ is locally finite in G by the paracompactness of G. Moreover, as every point-countable family of open subsets in a separable space is countable, the family $\mathcal{A}$ is star-countable. Then $\mathcal{A} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \Lambda\}$ by Lemma 4.6, where each subfamily $\mathcal{B}_\alpha$ is countable and $(\bigcup \mathcal{B}_\alpha) \cap (\bigcup \mathcal{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Set $X_\alpha = \bigcup \mathcal{B}_\alpha$ for each $\alpha \in \Lambda$. Then $G = \bigoplus_{\alpha \in \Lambda} X_\alpha$.

Claim 2. X_α is an $\aleph_0$-subspace for each $\alpha \in \Lambda$.

Put $\mathcal{B}_\alpha = \{B_{\alpha,n} : n \in \mathbb{N}\}$, where each $B_{\alpha,n}$ is an open $\aleph_0$-subspace of G, and put $\mathcal{P}_\alpha = \bigcup_{n \in \mathbb{N}} \mathcal{P}_{\alpha,n}$, where $\mathcal{P}_{\alpha,n}$ is a countable *cs*-network for the $\aleph_0$-space $B_{\alpha,n}$ for each $n \in \mathbb{N}$. Then $\mathcal{P}_\alpha$ is a countable *cs*-network for X_α. Thus, X_α is an $\aleph_0$-space.

In conclusion, G is a topological sum of $\aleph_0$-subspace. □

Corollary 4.8. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H is an $\aleph_0$-space, G is also an $\aleph_0$-space.*

By the similar proof of Theorem 4.7, the following result is obvious.

Theorem 4.9. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H is a locally cosmic space, then G is a topological sum of cosmic subspaces.*

Corollary 4.10. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H is a cosmic space, G is also a cosmic space.*

Theorem 4.11. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H has a star-countable cs-network, G also has a star-countable cs-network.*

Proof. Let $\mathscr{U}$ be a symmetric neighborhood base at 0 such that $gyr[x,y](U) = U$ for any $x, y \in G$ and $U \in \mathscr{U}$. Since the subgyrogroup H of G is first-countable at the identity element 0 of G, there exists a countable family $\{V_n : n \in \mathbb{N}\} \subseteq \mathscr{U}$ such that $(V_{n+1} \oplus (V_{n+1} \oplus V_{n+1})) \subseteq V_n$ for each $n \in \mathbb{N}$ and the family $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H.

Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a star-countable *cs*-network for the space G/H. For each $\alpha \in \Lambda$, the family $\{P_\alpha \cap P_\beta : \beta \in \Lambda\}$ is a countable *wcs**-network for P_α. Therefore, P_α is a cosmic space, and P_α is separable. Then it follows from Lemma 4.4 that $\pi^{-1}(P_\alpha)$ is separable. We can find a countable subset $B_\alpha = \{b_{\alpha,m} : m \in \mathbb{N}\}$ of $\pi^{-1}(P_\alpha)$ such that $\overline{B_\alpha} = \pi^{-1}(P_\alpha)$.

Put

$$\mathcal{F} = \{\pi^{-1}(P_\alpha) \cap (b_{\alpha,m} \oplus V_n) : \alpha \in \Lambda, \text{ and } m, n \in \mathbb{N}\}.$$

Then $\mathcal{F}$ is a star-countable family of G.

Claim. $\mathcal{F}$ is a *cs*-network for G.

Let $\{x_i\}_i$ be a sequence converging to a point x in G and let U be a neighborhood of x in G. Choose an open neighborhood V of 0 in G such that $(x \oplus (V \oplus V)) \subseteq U$. Since $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H, there exists $n \in \mathbb{N}$ such that $V_n \cap H \subseteq V \cap H$. Since $\pi : G \to G/H$ is an open and continuous mapping, there are $i_0 \in \mathbb{N}$ and $\alpha \in \Lambda$ such that $\{\pi(x)\} \cup \{\pi(x_i) : i \geq i_0\} \subseteq P_\alpha \subseteq \pi((x \oplus V_{n+1}) \cap (x \oplus V))$. Since $x \in \pi^{-1}(P_\alpha)$, $(x \oplus V_{n+1}) \cap \pi^{-1}(P_\alpha)$ is non-empty and open in the subspace $\pi^{-1}(P_\alpha)$. Moreover, since $\overline{B_\alpha} = \pi^{-1}(P_\alpha)$, there exists $m \in \mathbb{N}$ such that $b_{\alpha,m} \in x \oplus V_{n+1}$.

For an arbitrary $z \in \pi^{-1}(P_\alpha) \cap (b_{\alpha,m} \oplus V_{n+1})$, $\pi(z) \in P_\alpha \subseteq \pi((x \oplus V_{n+1}) \cap (x \oplus V)) = \pi(x \oplus (V_{n+1} \cap V))$. Then, $z \in x \oplus ((V_{n+1} \cap V) \oplus H)$ since H is a strong subgyrogroup. Since $z \in b_{\alpha,m} \oplus V_{n+1}$ and $b_{\alpha,m} \in x \oplus V_{n+1}$, we have

$$\begin{aligned} z &\in (x \oplus V_{n+1}) \oplus V_{n+1} \\ &= \bigcup_{u,v \in V_{n+1}} \{(x \oplus u) \oplus v\} \\ &= \bigcup_{u,v \in V_{n+1}} \{x \oplus (u \oplus \text{gyr}[u, x](v))\} \\ &= x \oplus (V_{n+1} \oplus V_{n+1}). \end{aligned}$$

Then, $(\ominus x) \oplus z \in V_{n+1} \oplus V_{n+1}$. Hence, $(\ominus x) \oplus z \in ((V_{n+1} \cap V) \oplus H) \cap (V_{n+1} \oplus V_{n+1})$. Therefore, there exist $a \in (V_{n+1} \cap V)$, $h \in H$ and $u_1, u_2 \in V_{n+1}$ such that $(\ominus x) \oplus z = a \oplus h = u_1 \oplus u_2$, whence $h = (\ominus a) \oplus (u_1 \oplus u_2) \in V_{n+1} \oplus (V_{n+1} \oplus V_{n+1}) \subseteq V_n$. It follows that $(\ominus x) \oplus z \in (V_{n+1} \cap V) \oplus (V_n \cap H)$. Thus $z \in x \oplus ((V_{n+1} \cap V) \oplus (V_n \cap H)) \subseteq x \oplus (V \oplus V) \subseteq U$. Hence, $\pi^{-1}(P_\alpha) \cap (b_{\alpha,m} \oplus V_{n+1}) \subseteq U$.

Since $b_{\alpha,m} \in x \oplus V_{n+1}$, there is $u_3 \in V_{n+1}$ such that $b_{\alpha,m} = x \oplus u_3$. Thus,

$$\begin{aligned} x &= (x \oplus u_3) \oplus \text{gyr}[x, u_3](\ominus u_3) \\ &= b_{\alpha,m} \oplus \text{gyr}[x, u_3](\ominus u_3) \\ &\in b_{\alpha,m} \oplus \text{gyr}[x, u_3](V_{n+1}) \\ &= b_{\alpha,m} \oplus V_{n+1}. \end{aligned}$$

Therefore, there exists $i_1 \geq i_0$ such that $x_i \in b_{\alpha,m} \oplus V_{n+1}$ whenever $i \geq i_1$, whence $\{x\} \cup \{x_i : i \geq i_1\} \subseteq \pi^{-1}(P_\alpha) \cap (b_{\alpha,m} \oplus V_{n+1})$.

In conclusion, G has a star-countable *cs*-network. □

Theorem 4.12. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H has a star-countable wcs*-network, G also has a star-countable wcs*-network.*

Proof. Let $\mathscr{U}$ be a symmetric neighborhood base at 0 such that $gyr[x, y](U) = U$ for any $x, y \in G$ and $U \in \mathscr{U}$. Since the subgyrogroup H of G is first-countable at the identity element 0 of G, there exists a countable family $\{V_n : n \in \mathbb{N}\} \subseteq \mathscr{U}$ in G such that $(V_{n+1} \oplus (V_{n+1} \oplus V_{n+1})) \subseteq V_n$ for each $n \in \mathbb{N}$ and the family $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H.

We construct $\mathcal{P}$ and $\mathcal{F}$ by the same way in Theorem 4.11, and we show that $\mathcal{F}$ is a *wcs**-network for G.

Let $\{x_i\}_i$ be a sequence converging to a point x in G and U be a neighborhood of x in G. Choose an open neighborhood V of 0 in G such that $(x \oplus (V \oplus V)) \subseteq U$. Since $\{V_n \cap H : n \in \mathbb{N}\}$ is a local base at 0 for H, there exists $n \in \mathbb{N}$ such that $V_n \cap H \subseteq V \cap H$. Since $\mathcal{P}$ is a *wcs**-network for G/H, there exists a subsequence $\{\pi(x_{i_j})\}_j$ of the sequence $\{\pi(x_i)\}_i$ such that $\{\pi(x_{i_j}) : j \in \mathbb{N}\} \subseteq P_\alpha \subseteq \pi((x \oplus V_{n+1}) \cap (x \oplus V))$ for some $\alpha \in \Lambda$. As the sequence $\{x_i\}_i$ converges to x, we have some $x_{i_j} \in x \oplus V_{n+2}$ for each $j \in \mathbb{N}$. Furthermore, since $x_{i_1} \in \pi^{-1}(P_\alpha)$, $(x_{i_1} \oplus V_{n+2}) \cap \pi^{-1}(P_\alpha)$ is non-empty and open in $\pi^{-1}(P_\alpha)$. Then it follows from $\overline{B_\alpha} = \pi^{-1}(P_\alpha)$ that there exists $m \in \mathbb{N}$ such that $b_{\alpha,m} \in x_{i_1} \oplus V_{n+2}$. Then

$$\begin{aligned} b_{\alpha,m} &\in x_{i_1} \oplus V_{n+2} \\ &\subseteq (x \oplus V_{n+2}) \oplus V_{n+2} \\ &= \bigcup_{u,v \in V_{n+2}} \{(x \oplus u) \oplus v\} \\ &= \bigcup_{u,v \in V_{n+2}} \{x \oplus (u \oplus \mathrm{gyr}[u,x](v))\} \\ &= x \oplus (V_{n+2} \oplus V_{n+2}). \end{aligned}$$

Moreover, it is proved in Theorem 4.11 that $\pi^{-1}(P_\alpha) \cap (b_{\alpha,m} \oplus V_{n+1}) \subseteq U$.

In conclusion, G has a star-countable wcs^*-network. □

Lemma 4.13. ([28]) *Let $\mathscr{P}$ be a point-countable family of subsets of a space X. Then $\mathscr{P}$ is a k-network for X if and only if it is a wcs^*-network for X and each compact subset of X is first-countable (or sequential).*

Theorem 4.14. *Let G be a strongly topological gyrogroup and H a closed first-countable and separable strong subgyrogroup of G. If the quotient space G/H has a star-countable k-network, G also has a star-countable k-network.*

Proof. By Theorem 4.12, G has a star-countable wcs^*-network. It follows from Lemma 4.13 that each compact subset of G/H is first-countable. Then, since "every compact subset is first-countable" is a three-space property in topological gyrogroups by [13, Corollary 4.10], we know that every compact subset of G is first-countable, thus G has a star-countable k-network by Lemma 4.13. □

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