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Global asymptotic stability of a predator-prey model with general functional response and including recruitment and capture in both species

Manh Tuan Hoang[a], **Jose C. Valverde**[b,*]

[a]*Department of Mathematics, FPT University, Hoa Lac Hi-Tech Park, Km29 Thang Long Blvd, Hanoi, Viet Nam*
[b]*SIDIS Research Group, Department of Mathematics, University of Castilla-La Mancha,*
Avda. de Espana, s/n, 02071-Albacete, Spain

Abstract. In this paper, we complete the study of the (global) dynamics of predator-prey models introduced in a previous work, which considers the possibility of any (generic) functional response involved, and includes recruitment and capture in both species. More specifically, we provide demonstrations for the global asymptotic stability of all the non-trivial steady states. To do that, we ingeniously divide the invariant set of the system by means of an adequate partition and construct appropriate Lyapunov functions for the subsets of such partition, what leads us to the global stability of each non-trivial equilibria. Thus, the work not only extends the precedent studies but also complete the study of them, providing technical ideas which can be useful in other contexts. Finally, numerical simulations associated to several specific examples are shown to illustrate their coherence with the theoretical findings.

1. Introduction

Mathematics has become a fundamental tool for modeling and analyzing the behaviour of phenomena coming from other sciences like biology, ecology, or other natural sciences [3, 6, 7, 23]. Among the mathematical models, the predator-prey type continue being one of the most studied in the last decades (see, for instance, [11–14, 17–21, 25–27, 35, 37, 39, 40, 42, 47, 49–51]). The variety of the predator-prey models which have appeared in the last two decades of the present century mainly correspond to the consideration of different functional responses [12, 19, 30–34, 42, 48, 50], or to the variation of other factors affecting the evolution of the populations like, stage structure [16, 19, 34, 44], harvesting [21, 49], diseases [5, 9], environmental time variations [41, 45], etc.

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* Corresponding author: Jose C. Valverde
Email addresses: tuanhm16@fe.edu.vn; hmtuan01121990@gmail.com (Manh Tuan Hoang), jose.valverde@uclm.es (Jose C. Valverde)
ORCID iDs: https://orcid.org/0000-0001-6089-3451 (Manh Tuan Hoang), https://orcid.org/0000-0001-6089-3451 (Jose C. Valverde)

In particular, in [27], a more general model was introduced where the authors considered a general form of predator functional response, a general form of recruitment, and capture on both predators and prey at a rate proportional to their populations. Thus, the results there obtained cover the casuistry originated by generic types of functional responses and recruitment forms. For this generic model, in [27], it is proved that the changes in the number and local stability of the equilibria are associated with a two-dimensional threshold parameter $\mathcal{R} = (m_1, m_2)$ with $m_1, m_2 > 0$, instead of with a one-dimensional one, as usually occurs in predator-prey models.

For mathematical models, the analysis of global asymptotic stability (GAS) of equilibria is one of the central questions to get to know the (complete) dynamics [46], with several useful applications in real-world situations [3, 7, 20, 23, 25, 39, 42, 49]. In particular, for predator-prey models, GAS of an equilibrium with a component equal to zero means that the corresponding population will go extinct while, if an equilibrium having both components positive is GAS, this announces a future ecological balance between both species. In [27], although the existence, coexistence and local stability of the equilibria of its more general model were determined, the GAS was only partially solved. More specifically, the authors only proved the GAS of the trivial (extinction) equilibrium $(0, 0)$, while for the non-trivial ones only their local asymptotic stability was confirmed. In spite of this, posterior numerical studies suggested that the equilibria may be not only locally but also globally asymptotically stable [10, 27].

Motivated by the above reasons, the main purpose of this work is to provide demonstrations for the global asymptotic stability of all the non-trivial steady states of the general model in [27]. To do that, we ingeniously divide the invariant set of the system by means of an adequate partition and construct appropriate Lyapunov functions for the subsets of such partition, what leads us to the global stability of each non-trivial equilibria. Thus, the work not only improves the precedent studies but also complete them, providing technical ideas which can be useful in other contexts. In addition, numerical simulations associated to several specific examples are shown to illustrate their coherence with the theoretical findings.

The paper is structured in the following parts. The mathematical model and the previous results on its dynamics are recalled in Section 2. In Section 3, we fully complete the GAS of the predator-prey model with general functional response and including recruitment and capture in both species. Section 4 is devoted to illustrating the theoretical results by means of numerical simulations in two different specific exameple. The last section provides the main conclusions of this work and future research directions.

2. Mathematical model: existence, coexistence and local stability of equilibria

We will deal with the following continuous predator-prey model (1) with general functional response and including (general) recruitment and capture in both species, given by the following nonlinear system of ordinary differential equations, which was firstly stated and studied in [27].

$$\begin{cases} \dot{x} &= xf(x,y) = x\Big[r(x) - y\psi(x) - m_1\Big], \\ \dot{y} &= yg(x,y) = y\Big[s(y) + cx\psi(x) - m_2\Big]. \end{cases} \tag{1}$$

In such a model (1),

- x and y stand for prey and predator populations, respectively.
- The functions $r(x)$ and $s(y)$ represent the *per capita recruitment rates* of prey and predators, respectively, and they posses the following (similar) properties:

$$\begin{aligned} &\forall x \geq 0, \quad 0 < r(x), \quad r'(x) < 0, \quad 0 \leq [xr(x)]', \quad \text{and} \quad \lim_{x\to\infty} r(x) = 0, \\ &\forall y \geq 0, \quad 0 < s(y), \quad s'(y) < 0, \quad 0 \leq [ys(y)]', \quad \text{and} \quad \lim_{y\to\infty} s(y) = 0, \end{aligned} \tag{2}$$

- The function $\psi(x)$ has the following characteristics:

$$\forall x \geq 0, \quad 0 < \psi(x), \quad \psi'(x) \leq 0, \quad 0 \leq [x\psi(x)]'. \tag{3}$$

- m_1 and m_2 represent the *total mortality rates* of prey and predators respectively and, due to that, it is assumed that they both are positive.

- c is another positive parameter known as *conversion efficiency of prey into predators*, which verifies $c < 1$.

For this system (1), $\Lambda = \mathbb{R}^2_+$ is a positively invariant set. From [27, Proposition 1], we know the existence and coexistence of equilibria of system (1) in Λ. Specifically, it presents four types of (possible) equilibria, namely: a *trivial (extinction) equilibrium* $P_0 = (0, 0)$, for all the values of the parameter; a *predator extinction equilibrium* $P_1 = (K, 0)$, if and only if $m_1 < r(0)$, where K is the solution of $r(x) = m_1$; a *prey extinction equilibrium* $P_2 = (0, M)$, if and only if $m_2 < s(0)$, where M is the solution of $s(y) = m_2$; and an *ecological stability equilibrium* $P_3 = (x_3, y_3)$, if and only if $m_1 < r(0) - M\psi(0)$ and $m_2 < s(0)$ or $m_1 < r(0)$ and $s(0) < m_2 < s(0) + cK\psi(K)$, where x_3 is the solution of the equation

$$cx\psi(x) + s\Big(\frac{r(x) - m_1}{\psi(x)}\Big) - m_2 = 0,$$

and y_3 can be obtained from the equality,

$$y_3 = \frac{r(x_3) - m_1}{\psi(x_3)},$$

once the value of x_3 is known.

Likewise, from the local stability analysis of system (1) in [27], we have that: the trivial (extinction) equilibrium is locally asymptotically stable, if $r(0) < m_1$ and $s(0) < m_2$, and unstable otherwise; the predator extinction equilibrium is locally asymptotically stable, if $r(0) > m_1$ and $s(0) + cK\psi(K) < m_2$, and unstable otherwise; the prey extinction equilibrium is locally asymptotically stable, if $r(0) - M\psi(0) < m_1$ and $s(0) > m_2$, and unstable otherwise; and the ecological stability equilibrium is locally asymptotically stable, if it belongs to Λ.

Nevertheless, in relation to the GAS of the equilibria, in [27], the authors only prove the case of the trivial (extinction) equilibrium, when $m_1 \geq r(0)$ and $m_2 \geq s(0)$.

In spite of this, posterior numerical simulations suggested that the rest of the equilibria may be also globally asymptotically stable [10, 27]. Thus, motivated by this suggestions, in the next section, we will analyze the GAS of the non-trivial equilibria.

3. GAS analysis of the non-trivial equilibria

Here, we will demonstrate the GAS of the non-trivial equilibria of system (1). To do that, we will use the result in [43, Proposition B.7] for an initial value problem of autonomous system of ordinary differential equations

$$\dot{w}(t) = f(w(t)), \quad w(0) = w_0 \geq 0, (t \geq 0) \tag{4}$$

where $f, w : \mathbb{R}^n \to \mathbb{R}^n$ and $w \geq 0$ means that $w \in \mathbb{R}^n_+$. Such a result establishes that, in conditions of existence and uniqueness of solution of initial value problems, if for all $i = 1, \ldots, n$, $f_i(w) \geq 0$ for any $w \geq 0$ with $w_i = 0$, then every solution $w(t)$ with initial condition $w(0) = w_0 \in \mathbb{R}^n_+$ remains in $\mathbb{R}^n_+$ for all $t \geq 0$.

Also, we shall denote by $\mathrm{Fix}^*(P_i)$ the set of existing equilibria minus P_i.

Theorem 3.1. *The predator extinction equilibrium $P_1 = (K, 0)$ of system* (1) *is globally asymptotically stable with respect to the set $\Lambda - \mathrm{Fix}^*(P_1)$, if $r(0) > m_1$ and $s(0) + cK\psi(K) < m_2$.*

Proof. In order to prove this theorem, we will consider a partition of Λ into two different sets,

$$\Lambda_K = \Big\{(x, y) \in \Lambda \Big| x \leq K\Big\}$$

and

$$\Lambda^K = \Big\{(x, y) \in \Lambda \Big| x \geq K\Big\}. \tag{5}$$

Then, we will prove that the $P_1 = (K, 0)$ is globally asymptotically stable with respect to $\Lambda_K - \text{Fix}^*(P_1)$, and, also, with respect to $\Lambda^K - \text{Fix}^*(P_1)$. Since, $\Lambda = \Lambda^K \bigcup \Lambda_k$, from both issues, we will obtain the complete GAS of P_1.

Firstly, we will demonstrate that the set

$$\Lambda_K = \Big\{(x, y) \in \Lambda \Big| x \leq K\Big\}$$

is a positively invariant set for the system (1). To do that, we will prove that $\Big(x(t), y(t)\Big) \in \Lambda_K$ for all $t > 0$, if $\Big(x(0), y(0)\Big) \in \Lambda_K$, i.e. $x(t) \leq K$ and $y(t) \geq 0$ for all $t > 0$. Since Λ is positively invariant for system (1), we have that $y(t) \geq 0$ for all $t > 0$.

Now, consider the change of variables $z = K - x$. Then, system (1) becomes

$$\begin{cases} \dot{z} &= -(K - z)\Big[r(K - z) - y\psi(K - z) - m_1\Big], \\ \dot{y} &= y\Big[s(y) + c(K - z)\psi(K - z) - m_2\Big]. \end{cases} \tag{6}$$

Thus,

$$\begin{aligned} &\dot{z}|_{z=0} = -K[r(K) - y\psi(K) - m_1] = Ky\psi(K) + K[m_1 - r(K)] = Ky\psi(K) \geq 0, \\ &\dot{y}|_{y=0} = 0. \end{aligned}$$

At this point, using the mentioned result in [43], we have that every solution $(z(t), y(t))$ such that $(z(0), y(0)) \geq 0$ remains in $\mathbb{R}^2_+$ for all $t \geq 0$. In particular, $z(t) \geq 0$ for all $t \geq 0$, what implies that $x(t) \leq K$ for all $t \geq 0$.

Secondly, we will demonstrate that, if $s(0) + cK\psi(K) < m_2$, then $P_1 = (K, 0)$ is globally asymptotically stable with respect to the set $\Lambda_K - \text{Fix}^*(P_1)$. Observe that $s(0) + cK\psi(K) < m_2$ implies that $s(y) \leq s(0) < m_2$ for all $y \geq 0$. As $0 \leq [x\psi(x)]'$, for all $(x, y) \in \Lambda_K$, it follows

$$s(y) + cx\psi(x) - m_2 \leq s(0) + cK\psi(K) - m_2 < 0. \tag{7}$$

Next, we construct a Lyapunov function to show the GAS of $P_1 = (K, 0)$ with respect to Λ_K under the condition $s(0) + cK\psi(K) < m_2$. In this sense, consider the function $V_1 : \Lambda_K \to \mathbb{R}_+$, defined as

$$V_1(x, y) = \frac{y^2}{2}. \tag{8}$$

In view of its definition, it is straightforward that $V_1(x, y)$ is C^1 and positive definite. In addition, its derivative along solutions of (1) is

$$\frac{dV_1}{dt} = y\dot{y} = y^2[s(y) + cx\psi(x) - m_2].$$

Taking into account (7), now we have that $dV_1/dt \leq 0$ for all $(x, y) \in \Lambda_K$ and

$$E = \Big\{(x, y) \in \Lambda_K | \dot{V}_1(x, y) = 0\Big\} \equiv \Big\{(x, y) \in \Lambda_K | y = 0\Big\}.$$

Observe that the largest invariant set contained in E is precisely E. Thus, in E, system (1) reduces to

$$\dot{x} = x\Big[r(x) - m_1\Big]. \tag{9}$$

Note that equation (9) has a unique positive equilibrium point $x = K$, since $r'(x) < 0$ and $r(0) > m_1$. At this point, we can construct another Lyapunov function

$$W(x) = x - K - K\ln\frac{x}{K}, \tag{10}$$

whose derivative is

$$\dot{W}(x) = \frac{x-K}{x}\dot{x} = (x-K)\Big[r(x) - m_1\Big].$$

It is clear that, for $x = K$, $\dot{W}(K) = 0$, while $\dot{W}(x) < 0$ for all $0 \le x \ne K$ since $r'(x) < 0$. Thus, using the Lyapunov stability theorem, it follows that $x = K$ is globally asymptotically stable, and, consequently, $\lim_{t\to\infty} x(t) = K$. At this point, as under the assumed conditions the predator extinction equilibrium $P_1 = (K, 0)$ is locally asymptotically stable, by the LaSalle Invariant Principle [24, 28], we get its GAS in $\Lambda_K - \text{Fix}^*(P_1)$.

Finally, we will demonstrate that the equilibrium point P_1 is also globally asymptotically stable with respect to the set $\Lambda^K - \text{Fix}^*(P_1)$. To do that, it is only necessary to prove that $P_1 = (K, 0)$ is globally attractive with respect to $\Lambda^K - \text{Fix}^*(P_1)$. To proceed with, we divide the proof in two cases.

Case 1. There exists $t_0 > 0$ such that $x(t_0) < K$.

Then, $\big(x(t_0), y(t_0)\big) \in \Lambda_K$. Thus, considering $\big(x(t_0), y(t_0)\big)$ as initial conditions and using the same reasoning as above, $\lim_{t\to\infty} x(t) = K$ and $\lim_{t\to\infty} y(t) = 0$.

Case 2. $x(t) \ge K$ for all $t > 0$.

Then, $\big(x(t), y(t)\big) \in \Lambda^K$ for all $t > 0$ and, consequently,

$$\begin{aligned}
\dot{x} &= x[r(x) - y\psi(x) - m_1] \\
&\le x[r(K) - m_1] - xy\psi(x) \\
&= -xy\psi(x) \\
&\le 0.
\end{aligned}$$

This means that $x(t)$ is decreasing and bounded from below by K.

In addition, considering $u(t) = x(t) + y(t)$, then

$$\begin{aligned}
\dot{u} &= \dot{x} + \dot{y} \\
&= x[r(x) - m_1] + y[s(y) - m_2] + (c-1)xy\psi(x) \\
&\le x[r(K) - m_1] + y[s(0) - m_2] + (c-1)xy\psi(x) \\
&= y[s(0) - m_2] + (c-1)xy\psi(x) \\
&< y[s(0) + cK\psi(K) - m_2] + (c-1)xy\psi(x) \\
&\le 0,
\end{aligned}$$

which implies that $u(t)$ is also decreasing and bounded from below.

Since $x(t), u(t)$ are bounded and decreasing functions and $y(t) = u(t) - x(t)$, the limits $\lim_{t\to\infty} x(t), \lim_{t\to\infty} y(t)$ exist. Assume that $\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*)$. Obviously, (x^*, y^*) must be an equilibrium point of the model (1).

Remember that when $r(0) > m_1$ and $s(0) + cK\psi(K) < m_2$, then the trivial equilibrium $(0, 0)$ is unstable and only coexists with the predator extinction equilibrium $(K, 0)$ which is locally asymptotically stable. That is, under these assumptions system (1) has no other non-trivial equilibria. On the other hand, $x(t)$ is bounded from below by K, what allows us to deduce that $(x^*, y^*) \ne (0, 0)$ and, consequently, $(x^*, y^*) = (K, 0)$.

As under the assumed conditions, $r(0) > m_1$ and $s(0) + cK\psi(K) < m_2$, $P_1 = (K, 0)$ is locally asymptotically stable, and has global attraction, we obtain its GAS in $\Lambda^K - \text{Fix}^*(P_1)$.

□

Theorem 3.2. *The prey extinction equilibrium $P_2 = (0, M)$ of system* (1) *is globally asymptotically stable with respect to the set $\Lambda - \text{Fix}^*(P_2)$, if $r(0) - M\psi(0) < m_1$ and $s(0) > m_2$.*

Proof. Here, we will use a similar technique as in Theorem 3.1. In order to prove this theorem, we will consider a partition of Λ into two different sets,

$$\Lambda^M = \Big\{(x, y) \in \Lambda \Big| y \geq M\Big\}$$

and

$$\Lambda_M = \Big\{(x, y) \in \Lambda \Big| y \leq M\Big\}.$$

Then, we will prove that the $P_2 = (0, M)$ is globally asymptotically stable with respect to $\Lambda^M - \text{Fix}^*(P_2)$, and also with respect to $\Lambda_M - \text{Fix}^*(P_2)$. Since, $\Lambda = \Lambda^M \bigcup \Lambda_M$, from both issues, we will obtain the complete GAS of P_2.

Firstly, we will demonstrate that the set

$$\Lambda^M = \Big\{(x, y) \in \Lambda \Big| y \geq M\Big\}$$

is a positively invariant set of the model (1). To do that, we will prove that $\Big(x(t), y(t)\Big) \in \Lambda^M$ for all $t > 0$ when $\Big(x(0), y(0)\Big) \in \Lambda^M$. Since Λ is a positively invariant set of system (1), $x(t) \geq 0$ for all $t > 0$.

Now, consider the change of variables $v = y - M$. Then, system (1) becomes

$$\begin{cases} \dot{x} &= x\Big[r(x) - (v + M)\psi(x) - m_1\Big], \\ \dot{v} &= (v + M)\Big[s(v + M) + cx\psi(x) - m_2\Big]. \end{cases} \tag{11}$$

Thus,

$$\dot{x}|_{x=0} = 0,$$
$$\dot{v}|_{v=0} = M\Big[s(M) + cx\psi(x) - m_2\Big] = cx\psi(x) \geq 0.$$

At this point, using the mentioned result in [43], we have that if $v(0) \geq 0$, then $v(t) \geq 0$ for all $t \geq 0$, what implies that $y(t) \geq M$ for all $t \geq 0$.

Secondly, we will demonstrate that, if $m_1 > r(0) - M\psi(0)$, then $P_2 = (0, M)$ is globally asymptotically stable with respect to the set $\Lambda^M - \text{Fix}^*(P_2)$. Since $r(0) - M\psi(0) < m_1$, $\dfrac{\partial f}{\partial x}(x, y)$, $\dfrac{\partial f}{\partial y}(x, y) < 0$, for all $(x, y) \in \Lambda^M$, we have

$$\begin{aligned} f(x, y) &= r(x) - y\psi(x) - m_1 \\ &\leq f(0, y) = r(0) - y\psi(0) - m_1 \\ &\leq r(0) - M\psi(0) - m_1 \\ &< 0. \end{aligned} \tag{12}$$

Next, we construct a Lyapunov function to show the GAS of P_2 with respect to Λ^M under the assumption $r(0) - M\psi(0) < m_1$. In this sense, consider the function $V : \Lambda_K \to \mathbb{R}$ defined as

$$V_2(x, y) = \frac{x^2}{2}. \tag{13}$$

Then, its derivative along the solutions of system (1) is

$$\frac{dV_2}{dt} = x\dot{x} = x^2(r(x) - y\psi(x) - m_1).$$

Taking into account (12), we have that $\dot{V}_2(x,y) \leq 0$ for all $(x,y) \in \Lambda^M$, and $\dot{V}_2 = 0$ if and only if $x = 0$.

Following a similar reasoning as before, for such a set, system (1) reduces to

$$\dot{y} = y\Big[s(y) - m_2\Big]. \tag{14}$$

As before, equation (14) has a unique positive equilibrium $y = M$, since $s(0) > m_2$ and $s'(y) < 0$, and it is globally asymptotically stable, i.e., $\lim_{t\to\infty} y(t) = M$. At this point, as under the assumed conditions the prey extinction equilibrium $P2 = (0,M)$ is locally asymptotically stable, by the LaSalle Invariant Principle we get its GAS with respect to $\Lambda^M - \text{Fix}^*(P_2)$.

Finally, we will demonstrate that the equilibrium point P_2 is globally asymptotically stable with respect to the set $\Lambda_M - \text{Fix}^*(P_2)$. To do that, it is only necessary to prove that $P_2 = (0,M)$ is globally attractive with respect to $\Lambda_M - \text{Fix}^*(P_2)$. To proceed with, we divide the proof in two cases.

Case 1. There exists $t_0 > 0$ such that $y(t_0) > M$.

Then, $\big(x(t_0), y(t_0)\big) \in \Lambda^M$. Thus, considering $\big(x(t_0), y(t_0)\big)$ as initial conditions and using similar arguments as before, $\lim_{t\to\infty} y(t) = M$ and $\lim_{t\to\infty} x(t) = 0$.

Case 2. $y(t) \leq M$ for all $t > 0$.

Then, $(x(t), y(t)) \in \Lambda_M$ for all $t > 0$ and we have

$$\begin{aligned}\dot{y} &= y[s(y) + cx\psi(x) - m_2]\\ &\geq y[s(M) - m_2] + cxy\psi(x)\\ &= cx(t)y\psi(x)\\ &\geq 0.\end{aligned}$$

This means that $y(t)$ is increasing and bounded from above by M. Thus, it allows to assure the existence of $\lim_{t\to\infty} y(t)$.

We shall also demonstrate the existence of $\lim_{t\to\infty} x(t)$, distinguishing two sub-cases according to the order relation between m_1 and $r(0)$.

(i) $m_1 \geq r(0)$. In this case,

$$\begin{aligned}\dot{x} &= x[r(x) - y\psi(x) - m_1]\\ &\leq x[r(0) - m_1 - y\psi(x)]\\ &\leq -xy\psi(x)\\ &\leq 0.\end{aligned}$$

This implies that $x(t)$ is decreasing and bounded from below. Thus, it allows to assure the existence of $\lim_{t\to\infty} x(t)$.

(ii) $m_1 < r(0)$. In this other case, the predator extinction equilibrium $P_1 = (K,0)$ also exists, and Λ_K is a positively invariant set of system (1). Therefore, if $x(0) \leq K$, then $x(t) \leq K$ for all $t > 0$. Consequently,

$$\begin{aligned}\dot{x} + \frac{1}{c}\dot{y} &= x\Big[r(x) - y\psi(x) - m_1\Big] + \frac{1}{c}y\Big[s(y) + cx\psi(x) - m_2\Big]\\ &= x\Big[r(x) - m_1\Big] + \frac{1}{c}\Big[s(y) - m_2\Big]\\ &> x[r(K) - m_1] + \frac{1}{c}[s(M) - m_2] = 0.\end{aligned}$$

Thus, $x(t) + \frac{1}{c}y(t)$ is increasing and bounded from below by $K + \frac{M}{c}$. This allows us to assure the existence of the limit of $x(t) + \frac{1}{c}y(t)$ as $t \to \infty$ and, as a consequence, the existence of $\lim_{t\to\infty}(x(t), y(t))$.

Otherwise, if $x(0) > K$, then, if we suppose that $x(t) \geq K$ for all $t > 0$,

$$\begin{aligned} \dot{x} &= x[r(x) - y\psi(x) - m_1] \\ &\leq x[r(K) - m_1 - y\psi(x)] \\ &= -xy\psi(y) \\ &\leq 0. \end{aligned}$$

This means that $x(t)$ is decreasing and bounded from below by K, what allows us to assure the existence of $\lim_{t\to\infty} x(t)$.

Therefore, we have demonstrated the existence of $\lim_{t\to\infty}(x(t), y(t))$, namely (x^*, y^*), which must be an equilibrium of system (1).
Remember that when $s(0) > m_2$ and $r(0) - M\psi(0) < m_1 < r(0)$, then the trivial equilibrium $P_0 = (0, 0)$ is unstable and it coexists with a unique predator extinction equilibrium $P_1 = (K, 0)$, and a unique prey extinction equilibrium $P_2 = (0, M)$, being P_1 unstable and P_2 locally asymptotically stable. Since $y(t)$ is increasing and bounded from above by M, $(x^*, y^*) \neq (0, 0)$ and $(x^*, y^*) \neq (K, 0)$.
On the other hand, when $r(0) < m_1$ and $s(0) > m_2$, then the trivial equilibrium $P_0 = (0, 0)$ is unstable and, coexists with a unique prey extinction equilibrium equilibrium $P_2 = (0, M)$, which is locally asymptotically stable. Thus, $(x^*, y^*) \neq (0, 0)$.

Thus, we deduce that the globally attractive equilibrium (x^*, y^*) is $(0, M)$, what allows us to infer its GAS with respect to $\Lambda_M - \mathrm{Fix}^*(P_2)$. □

Due to the unexplicit expression of P_3, we restrict our analysis to the case in which the recruitment follows a Beverton-Holt type, while the functional response a Holling type II type, which are among the most common in real-world applications [27]. Thus, the predator functional response $x\psi(x)$ and the per capita recruitment rates of prey $r(x)$ and predators $s(y)$ satisfy these similar equations:

$$\begin{aligned} xr(x) &= \frac{a_1 x}{b_1 + x}, \quad a_1,\ b_1 > 0, \\ x\psi(x) &= \frac{a_2 x}{b_2 + x}, \quad a_2,\ b_2 > 0, \\ ys(y) &= \frac{a_3 y}{b_3 + y}, \quad a_3,\ b_3 > 0. \end{aligned} \tag{15}$$

At this point, implementing these types in system (1), we have

$$\begin{cases} \dot{x} &= x\Big(\dfrac{a_1}{b_1 + x} - \dfrac{a_2 y}{b_2 + x} - m_1\Big), \\ \dot{y} &= y\Big(\dfrac{a_3}{b_3 + y} + c\dfrac{a_2 x}{b_2 + x} - m_2\Big). \end{cases} \tag{16}$$

This choice of types for the predator functional response and for the recruitment in (15) was also considered for numerical experiments in [27]. Actually, numerical simulation associated to (16) in [10, 27] suggested the GAS of P_3. These studies motivate the following analytical result.

Theorem 3.3. *The ecological stability equilibrium $P_3 = (x_3, y_3)$ of system* (16)*is globally asymptotically stable with respect to the set $\Lambda - \mathrm{Fix}^*(P_3)$, if it belongs to Λ.*

Proof. Suppose that $P_3 = (x_3, y_3) \in \Lambda$. Then, $0 \leq (x_3, y_3)$ and we have that

$$\begin{aligned} \frac{a_1}{b_1 + x_3} - \frac{a_2 y_3}{b_2 + x_3} &= m_1, \\ \frac{a_3}{b_3 + y_3} + c\frac{a_2 x_3}{b_2 + x_3} &= m_2. \end{aligned} \tag{17}$$

As a consequence, system (16) can be rewritten as

$$\begin{cases} \dot{x} &= x\Big(\dfrac{a_1}{b_1+x} - \dfrac{a_2 y}{b_2+x} - \dfrac{a_1}{b_1+x_3} + \dfrac{a_2 y_3}{b_2+x_3}\Big), \\ \dot{y} &= y\Big(\dfrac{a_3}{b_3+y} + c\dfrac{a_2 x}{b_2+x} - \dfrac{a_3}{b_3+y_3} - c\dfrac{a_2 x_3}{b_2+x_3}\Big), \end{cases} \tag{18}$$

which is equivalent to

$$\begin{aligned} \dot{x} &= x\Big[\Big(\frac{a_1}{(b_1+x)(b_1+x_3)} - \frac{a_2 y_3}{(b_2+x)(b_2+x_3)}\Big)(x_3 - x) - \frac{a_2 b_2 + a_2 x_3}{(b_2+x)(b_2+x_3)}(y - y_3)\Big], \\ \dot{y} &= y\Big[\frac{c a_2 b_2}{(b_2+x)(b_2+x_3)}(x - x_3) - \frac{a_3}{(b_2+x)(b_2+x_3)}(y - y_3)\Big]. \end{aligned} \tag{19}$$

At this point, we shall construct a Lyapunov function. Effectively, suppose that c_1 and c_2 represent positive (real) numbers and consider the following function

$$V(x,y) = c_1\Big(x - x_3 - x_3 \ln\frac{x}{x_3}\Big) + c_2\Big(y - y_3 - y_3 \ln\frac{y}{y_3}\Big). \tag{20}$$

Then, evaluating $\dot{V}$ along the solutions of system (16), from (19) and (20), we have

$$\begin{aligned} \dot{V} &= \frac{x - x_3}{x}\dot{x} + \frac{y - y_3}{y}\dot{y} \\ &= \Big[\Big(\frac{a_1}{(b_1+x)(b_1+x_3)} - \frac{a_2 y_3}{(b_2+x)(b_2+x_3)}\Big)(x_3 - x) - \frac{a_2 b_2 + a_2 x_3}{(b_2+x)(b_2+x_3)}(y - y_3)\Big](x - x_3) \\ &+ \Big[\frac{c a_2 b_2}{(b_2+x)(b_2+x_3)}(x - x_3) - \frac{a_3}{(b_2+x)(b_2+x_3)}(y - y_3)\Big](y - y_3) \\ &= -c_1\Big[\frac{a_1}{(b_1+x)(b_1+x_3)} - \frac{a_2 y_3}{(b_2+x)(b_2+x_3)}\Big](x_3 - x)^2 - c_2\frac{a_3}{(b_2+x)(b_2+x_3)}(y - y_3)^2 \\ &+ \Big[-c_1\frac{a_2 b_2 + a_2 x_3}{(b_2+x)(b_2+x_3)} + c_2\frac{c a_2 b_2}{(b_2+x)(b_2+x_3)}\Big](x - x_3)(y - y_3). \end{aligned} \tag{21}$$

At this point, if we suppose that c_1 and c_2 verify

$$c_1(a_2 b_2 + a_2 x_3) = c_2 c a_2 b_2,$$

then

$$\begin{aligned} \dot{V} &= -c_1\Big[\frac{a_1}{(b_1+x)(b_1+x_3)} - \frac{a_2 y_3}{(b_2+x)(b_2+x_3)}\Big](x - x_3)^2 \\ &- c_2\frac{a_3}{(b_2+x)(b_2+x_3)}(y - y_3)^2. \end{aligned} \tag{22}$$

The hypothesis $\dfrac{\partial f}{\partial x}(x,y) = r'(x) - y\psi'(x) < 0$ for all $x, y \geq 0$ implies that

$$\frac{a_1}{(b_1+x)(b_1+x_3)} - \frac{a_2 y_3}{(b_2+x)(b_2+x_3)} > 0.$$

From this, we deduce that the function V satisfies the Lyapunov stability theorem. Hence, the GAS of P_3 in $\Lambda - \mathrm{Fix}^*(P_3)$ is proved. □

4. Numerical simulations

To confirm the validity of the theoretical results, we will perform numerical simulations with two particular examples.

Example 4.1. *Consider the particular predator-prey system obtained from* (1) *by choosing an Ivlev type predator functional response and Beverton-Holt type recruitment for both species, given by*

$$xr(x) = \frac{15x}{x+10}, \quad x\psi(x) = \frac{1-e^{-x}}{30}, \quad ys(y) = \frac{5y}{y+10}.$$

We shall analyze numerically the six different cases associated with the parameters of m_1, m_2 listed in Corollary 1 and Figure 2 in [27], which determine all the possibilities of existence and coexistence of equilibria (see Table 1).

Since the predator-prey model possesses positive solutions, numerical approximations should satisfy this characteristic. For this reason, we utilize dynamically consistent nonstandard finite difference (NSFD) schemes proposed by Dang and Hoang [10] to simulate the dynamics of the model. More clearly, the following simple positivity-preserving NSFD scheme derived from [10] will be used

$$\begin{aligned} \frac{x_{n+1}-x_n}{\Delta t} &= x_n r(x_n) - x_{n+1} y_n \psi(x_n) - m_1 x_{n+1}, \\ \frac{y_{n+1}-y_n}{\Delta t} &= y_n s(y_n) + c y_n x_{n+1} \psi(x_{n+1}) - m_2 y_{n+1}, \end{aligned} \tag{23}$$

where Δt is the step size, $t_n = n\Delta t$ for $n = 0, 1, 2 \ldots$ and $\big(x_n, y_n\big)^T$ is the intended approximation for $\big(x(t_n), y(t_n)\big)^T$, respectively. The explicit form of the NSFD scheme (23) *is given by*

$$\begin{aligned} x_{n+1} &= \frac{x_n + \Delta t x_n r(x_n)}{1 + \Delta t y_n \psi(x_n) + m_1 \Delta t}, \\ y_{n+1} &= \frac{y_n + \Delta t y_n s(y_n) + \Delta t c y_n x_{n+1} \psi(x_{n+1})}{1 + m_2 \Delta t}. \end{aligned}$$

The NSFD scheme (23) *not only preserves the positivity but also has the ability to correctly preserve the dynamics of the model* (1) *for all finite step sizes. So, it can provide reliable approximations. In all the following numerical examples, we will consider the model* (1) *over the time interval* $[0, 500]$ *and use the step size* $\Delta t = 10^{-6}$.

Table 1: The parameters m_1, m_2 in Example 4.1.

Case	m_1	m_2	Source	Verified conditions	GAS equilibrium point
1	1.53	0.622	[27]	$m_1 > r(0)$ and $m_2 > s(0)$	$P_1 = (0, 0)$
2	1.53	0.4789	[27]	$m_1 > r(0)$ and $m_2 < s(0)$	$P_2 = (0, 0.4406)$
3	1.4925	0.4789	[27]	$m_2 < s(0)$ and $r(0) - M\psi(0) < m_1 < r(0)$	$P_2 = (0, 0.4406)$
4	1.38	0.4789	[27]	$m_2 < s(0)$ and $m_1 < r(0) - M\psi(0)$	$P_3 = (0.80, 0.44)$
5	0.3	0.501	[27]	$m_1 < r(0)$ and $s(0) < m_2 < s(0) + cK\psi(K)$	$P_3 = (39.6, 0.06)$
6	1.38	0.622	[27]	$m_1 < r(0)$ and $m_2 > s(0) + cK\psi(K)$	$P_1 = (0.86, 0)$

The figures 1-6 represent the phase portraits for the predator-prey model corresponding to the six mentioned cases in relation to the values of (m_1, m_2)*. In these figures, each blue curve represents a solution associated with a specific initial condition, the red arrows indicate the evolution (direction) of both species, and the green circle point the globally asymptotically stable equilibrium.*

As can be observed, all the solutions converge to the global asymptotically stable equilibrium, what confirms the GAS of the predator-prey system considered.

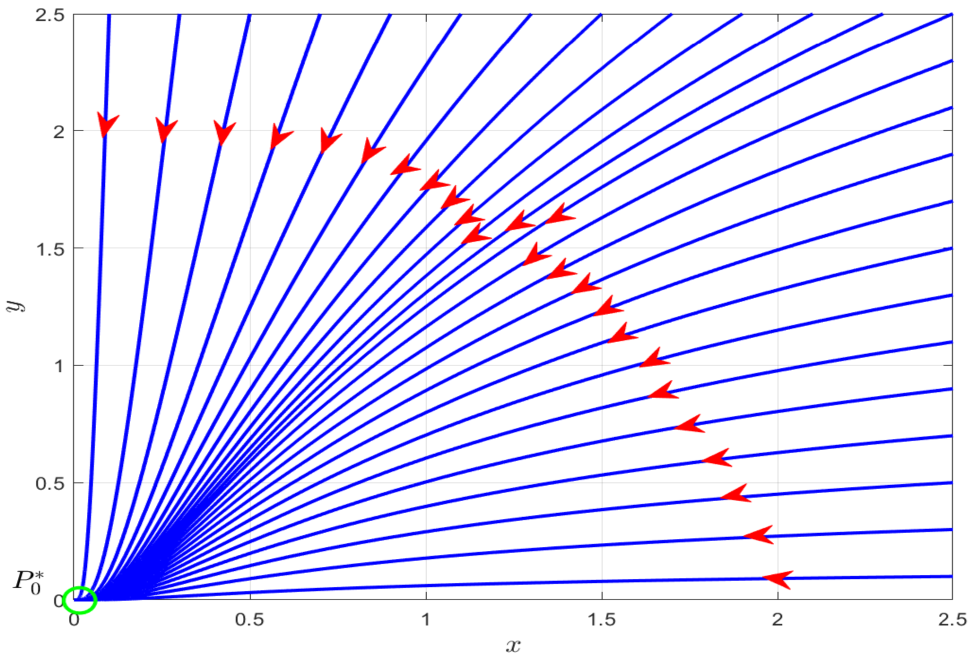

Figure 1: The phase planes of the predator-prey for Case 1 of Example 4.1.

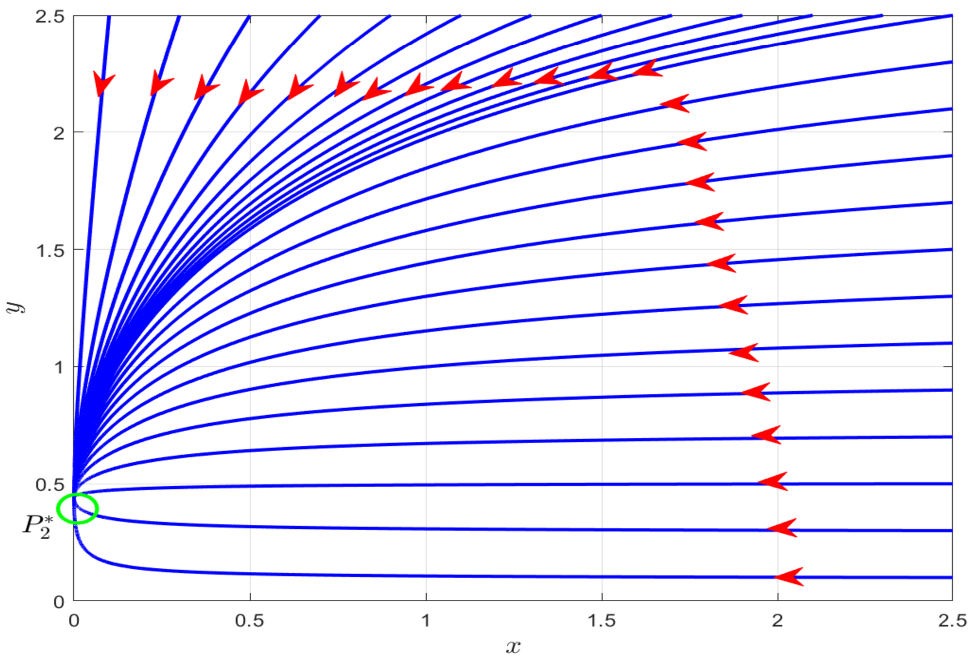

Figure 2: The phase planes of the predator-prey model for Case 2 of Example 4.1.

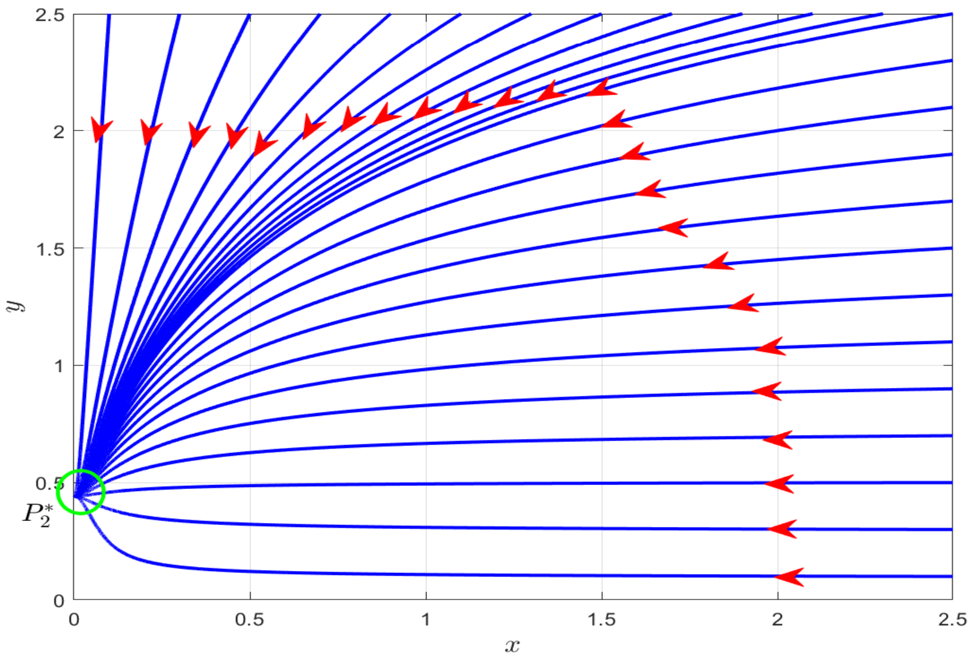

Figure 3: The phase planes of the predator-prey model for Case 3 of Example 4.1.

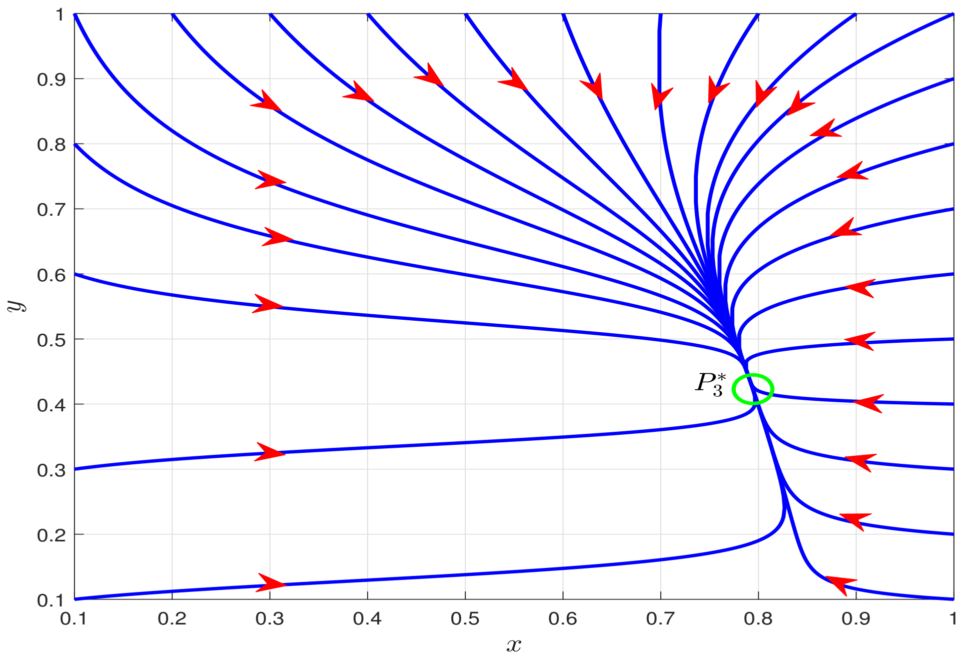

Figure 4: The phase planes of the predator-prey model for Case 4 of Example 4.1.

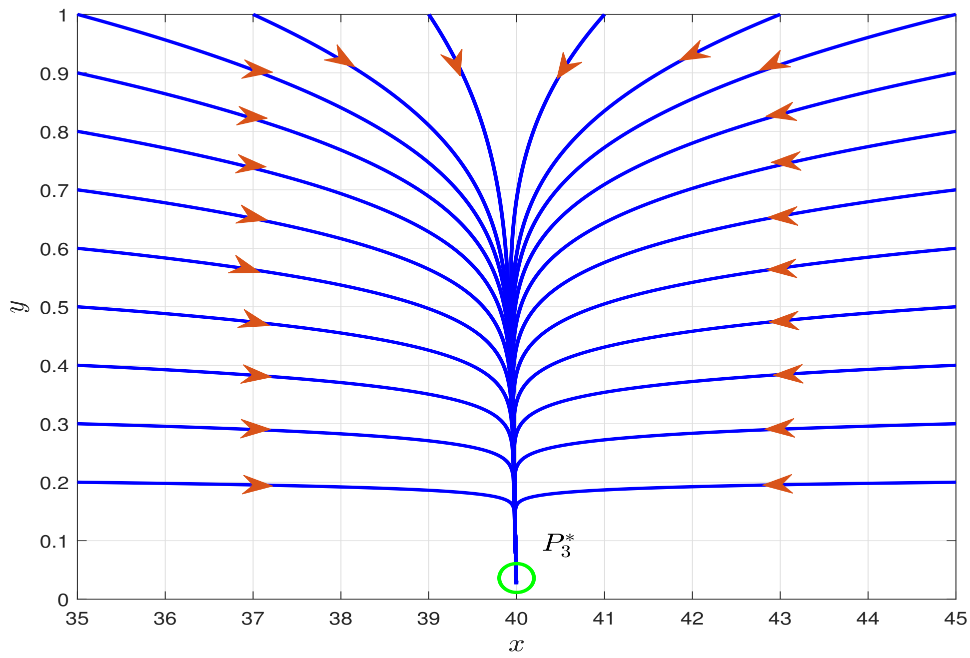

Figure 5: The phase planes of the predator-prey model for Case 5 of Example 4.1.

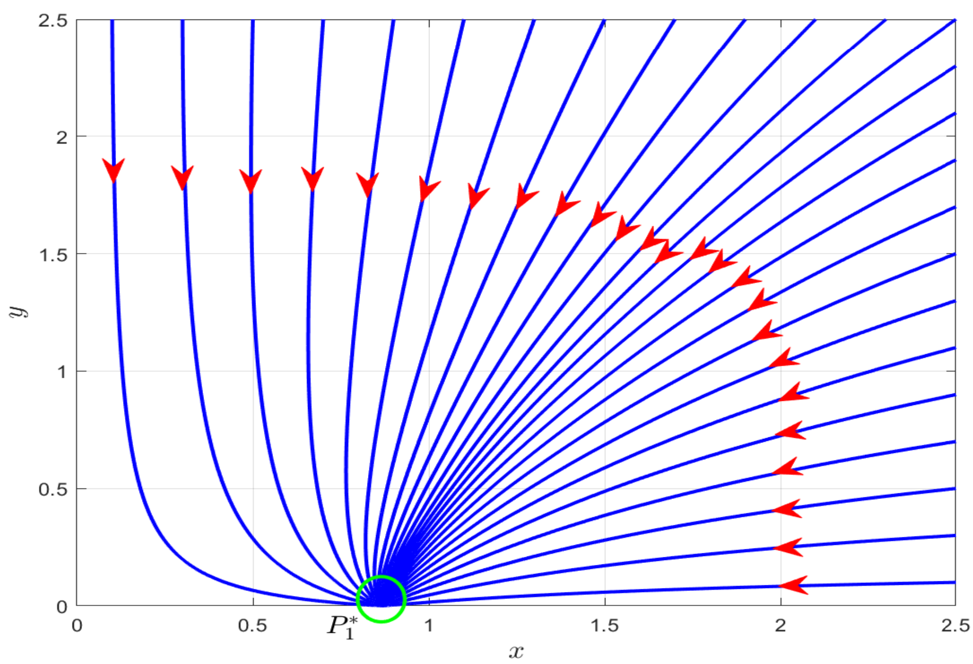

Figure 6: The phase planes of the predator-prey model for Case 6 of Example 4.1.

Example 4.2. *Consider the particular predator-prey system obtained from* (1) *by choosing a Holling type III type functional response and Beverton-Holt type recruitment for both species, given by*

$$xr(x) = \frac{15x}{x+10}, \quad x\psi(x) = \frac{x^2}{x^2+30}, \quad ys(y) = \frac{5y}{y+10},$$

We shall analyze numerically the cases associated with the parameters of (m_1, m_2) *listed in Table 2.*

Table 2: The parameters (m_1, m_2) in Example 4.2.

Case	m_1	m_2	Source	Verified conditions	GAS equilibrium point
1	1.53	0.622	[27]	$m_1 > r(0)$ and $m_2 > s(0)$	$P_1 = (0, 0)$
2	1.53	0.4789	[27]	$m_1 > r(0)$ and $m_2 < s(0)$	$P_2 = (0, 0.44)$
3	1.38	0.4789	[27]	$m_2 < s(0)$ and $m_1 < r(0) - M\psi(0)$	$P_3 = (0.78, 0.44)$
4	0.3	0.501	[27]	$m_1 < r(0)$ and $s(0) < m_2 < s(0) + cK\psi(K)$	$P_3 = (39.8, 0.04)$
5	1.38	0.622	[27]	$m_1 < r(0)$ and $m_2 > s(0) + cK\psi(K)$	$P_1 = (0.8696, 0)$

The phase portraits for the predator-prey model corresponding to the five cases of (m_1, m_2) *are shown in Figures 7-11, respectively. As in Example 4.1, it can be observed, that all the solutions converge to the global asymptotically stable equilibrium, what confirms the GAS of the predator-prey system considered.*

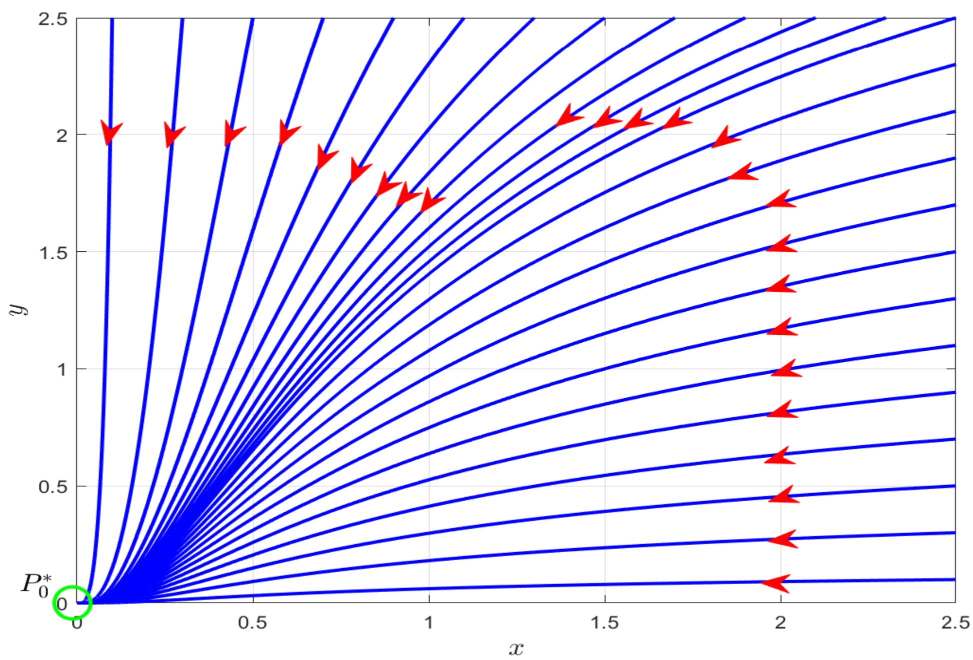

Figure 7: The phase planes of the predator-prey model for Case 1 of Example 4.2.

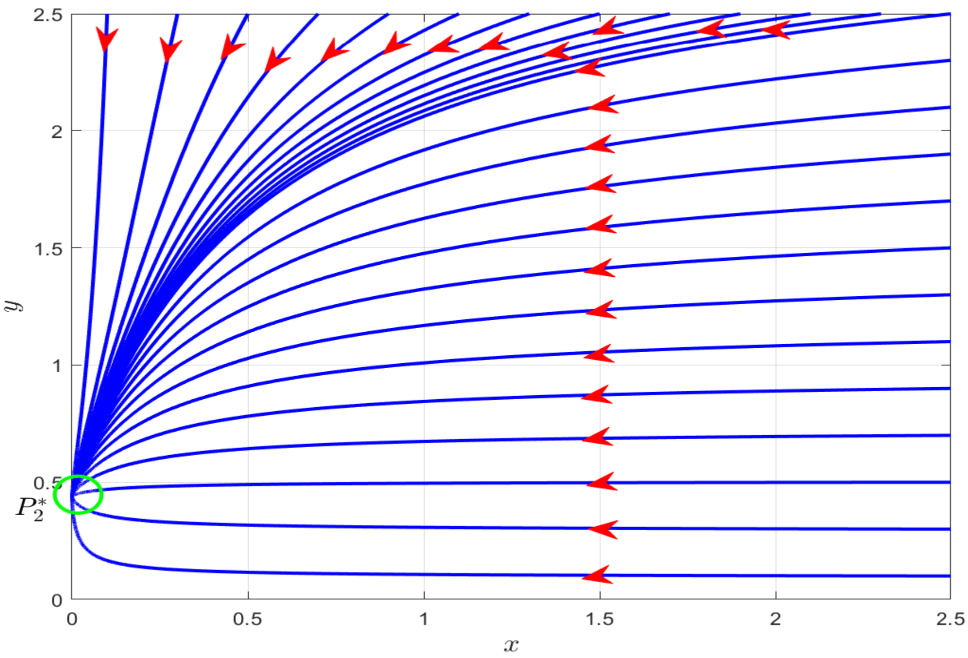

Figure 8: The phase planes of the predator-prey model for Case 2 of Example 4.2.

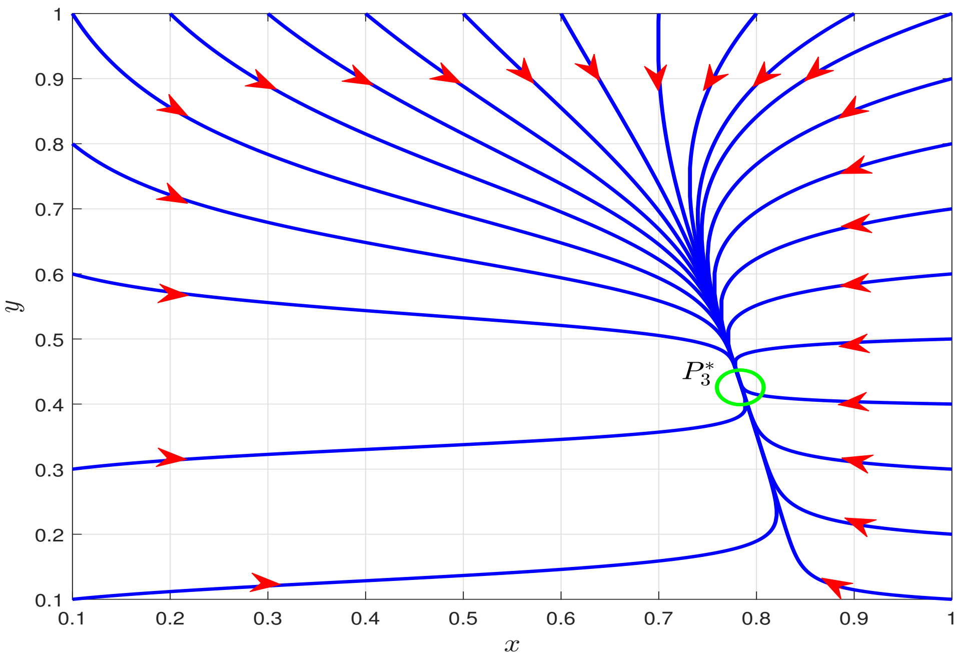

Figure 9: The phase planes of the predator-prey model for Case 3 of Example 4.2.

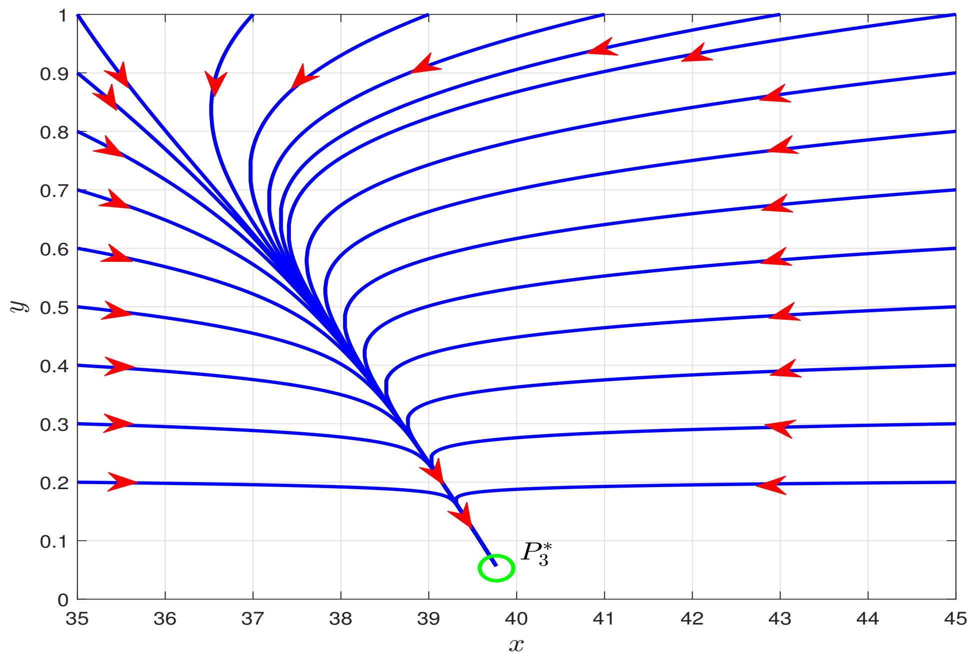

Figure 10: The phase planes of the predator-prey model for Case 4 of Example 4.2.

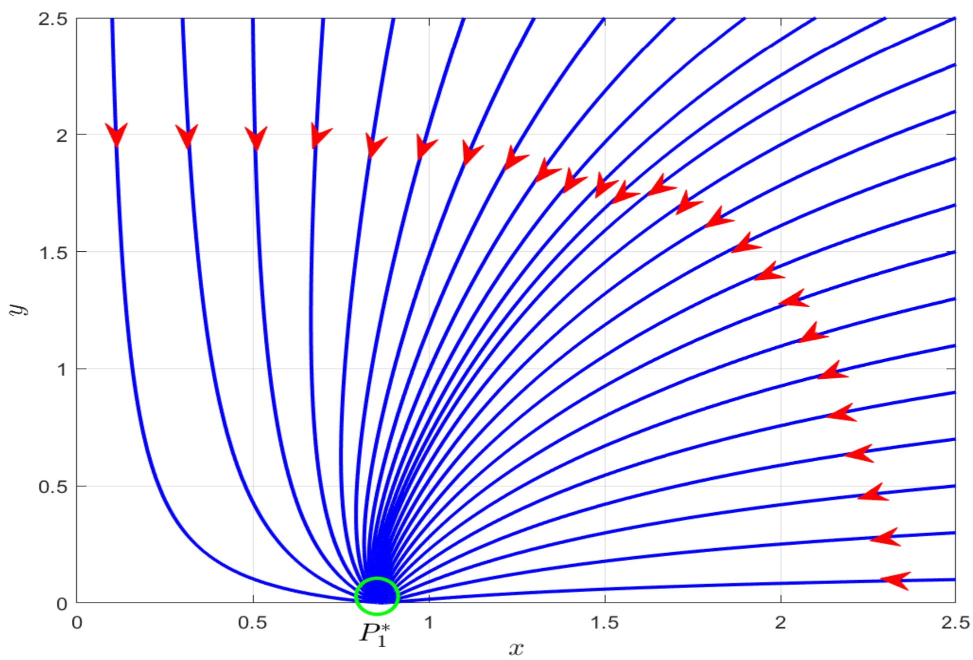

Figure 11: The phase planes of the predator-prey model for Case 5 of Example 4.2.

It is important to remark that Examples 4.1 and 4.2 suggest that the ecological stability equilibrium point may be globally asymptotically stable for general cases of functional response and recruitment. Therefore, it is reasonable to conjecture that the ecological stability equilibrium point of the general predator-prey model (1) is globally asymptotically stable.

5. Conclusions and open research directions

With this work, we show that the global dynamics of several kinds of predator-prey systems can be analyzed at the same time by only taking into account common (mathematical) characteristics of the different modalities of functions that appear in the definition of such systems. On the other hand, we give a useful idea to deal with the global asymptotic stability in a certain domain, consisting in dividing it in appropriate subsets with respect to the parametric values and construct respective Lyapunov function for these subsets. This allows us to complete the results on global asymptotic stability suggested in [27], on general predator-prey systems including recruitment and capture in both species. The numerical simulations in this work suggest that the ecological stability equilibrium is globally asymptotically stable when it belongs to the domain. We think that the methods and results in this work can be useful to guide the study of predator-prey models (1) given in other even more general or involved contexts. For instance, model (1) in the context of the Caputo fractional derivative with $\alpha \in (0, 1)$ becomes:

$$\begin{cases} \dfrac{d^{\alpha}x(t)}{dt} = x(t)f(x(t), y(t)) = x(t)\Big[r(x(t)) - y(t)\psi(x(t)) - m_1\Big], \\ \dfrac{d^{\alpha}y(t)}{dt} = y(t)g(x(t), y(t)) = y(t)\Big[s(y(t)) + cx(t)\psi(x(t)) - m_2\Big], \end{cases} \tag{24}$$

In view of the proofs in this paper, this extended model (24) should be analyzed by using comparison results [29, 38] and the Lyapunov stability theorem for fractional-order dynamical systems [1, 2, 15]. We consider this analyzed as future research direction of our work.

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References

[1] R. P. Agarwal, D. O'Regan, S. Hristova, *Stability of Caputo fractional differential equations by Lyapunov functions*, Applications of Mathematics **60** (2015), 653–676.

[2] N. Aguila-Camacho, A. M. Duarte-Mermoud, *J. A. Gallegos, Lyapunov functions for fractional order systems*, Communications in Nonlinear Science and Numerical Simulation **19** (2014), 2951–2957.

[3] L. J. S. Allen, *An Introduction to Mathematical Biology*, Prentice Hall, New Jersey, 2007.

[4] U. M. Ascher, L. R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, Society for Industrial and Applied Mathematics, Philadelphia, 1998.

[5] P. Auger, R. Mchich, T. Chowdhury, G. Sallet, M. Tchuente, J. Chattopadhyay, *Effects of a disease affecting a predator on the dynamics of a predator-prey system*, Journal of Theoretical Biology **258** (2009), 344–351.

[6] A. A. Berryman, *The origins and evolution of predator-prey theory*, Ecology **73** (1992), 1530-1535.

[7] F. Brauer, C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, New York, 2001.

[8] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent–II*, Geophysical Journal International **13** (1967), 529–539.
[9] J. Chattopadhyay, S. Pal, A. E. Abdllaoui, *Classical predator-prey system with infection of prey population-a mathematical model*, Mathematical Methods in the Applied Sciences **26** (2003), 1211–1222.
[10] Q. A. Dang, M. T. Hoang, *Nonstandard finite difference schemes for a general predator-prey system*, Journal of Computational Science **36** (2019) 101015.
[11] B. D. Deka, A. Patra, J. Tushar, B. Dubey, *Stability and Hopf-bifurcation in a general Gauss type two-prey and one-predator system*, Applied Mathematical Modelling **40** (2016), 5793–5818.
[12] D. T. Dimitrov, H. V. Kojouharov, *Nonstandard finite-difference methods for predator-prey models with general functional response*, Mathematics and Computers in Simulation **78** (2008), 1–11.
[13] D. T. Dimitrov, H. V. Kojouharov, *Positive and elementary stable nonstandard numerical methods with applications to predator-prey models*, Journal of Computational and Applied Mathematics **189** (2006), 98–108.
[14] D. T. Dimitrov, H. V. Kojouharov, *Complete mathematical analysis of predator-prey models with linear prey growth and Beddington-DeAngelis functional response*, Applied Mathematics and Computation **162** (2005), 523–538.
[15] M . A. Duarte-Mermoud, N. Aguila-Camacho, A. J. Gallegos, R. Castro-Linares, *Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems*, Communications in Nonlinear Science and Numerical Simulation **22** (2015), 650–659.
[16] P. Georgescu, Y. H. Hsieh, *Global Dynamics of a Predator-Prey Model with Stage Structure for the Predator*, SIAM Journal on Applied Mathematics **67** (2007), 1379–1395.
[17] X. He, S. Zheng, *Global boundedness of solutions in a reaction-diffusion system of predator-prey model with prey-taxis*, Applied Mathematics Letters **49** (2015), 73–77.
[18] C. M. Heggerud, K. Lan, *Local stability analysis of ratio-dependent predator-prey models with predator harvesting rates*, Applied Mathematics and Computation **270** (2015), 349–357.
[19] M. T. Hoang, *On the global asymptotic stability of a predator-prey model with Crowley-Martin function and stage structure for prey*, Journal of Applied Mathematics Computation **64** (2020), 765–780.
[20] S. Hsu, T. Huang, *Global Stability for a Class of Predator-Prey Systems*, SIAM Journal on Applied Mathematics **55** (1995), 763–783.
[21] D. Hu, H. Cao, *Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvesting*, Nonlinear Analysis: Real World Applications **33** (2017), 58–82.
[22] T.-W. Hwang, *Global analysis of the predator-prey system with Beddington-DeAngelis functional response*, Journal of Mathematical Analysis and Applications **281** (2003), 395–401,
[23] L. Edelstein-Keshet, *Mathematical Models in Biology, Society for Industrial and Applied Mathematics*, Philadelphia, 1998.
[24] Hassan K. Khalil, *Nonlinear Systems*, (3rd Edition), Pearson, 2002.
[25] A. Korobeinikov, *Stability of ecosystem: Global properties of a general predator-prey model*, Mathematical Medicine and Biology **26** (2009), 309–321.
[26] Z. Lajmiri, R. K. Ghaziani, I. Orak, *Bifurcation and stability analysis of a ratio-dependent predator-prey model with predator harvesting rate*, Chaos, Solitons & Fractals **106** (2018), 193–200.
[27] L. M. Ladino, E. I. Sabogal, J. C. Valverde, *General functional response and recruitment in a predator-prey system with capture on both species*, Mathematical Methods in the Applied Sciences **38** (2015), 2876–2887.
[28] J. La Salle, S. Lefschetz, *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.
[29] Y. Li, Y. Chen, I. Podlubny, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Computers and Mathematics with Applications **59** (2010), 1810–1821.
[30] Y. Li, H. Gao, *Existence, uniqueness and global asymptotic stability of positive solutions of a predator-prey system with Holling II functional response with random perturbation*, Nonlinear Analysis: Theory, Methods & Applications **68** (2008), 1694–1705.
[31] H. Li, Y. Takeuchi, *Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response*, Journal of Mathematical Analysis and Applications **374** (2011), 644–654.
[32] B. Lisena, *Global stability of a periodic Holling-Tanner predator-prey model*, Mathematical Methods in the Applied Sciences **41** (2018), 3270–3281.
[33] M. Liu, C. Bai, *Global asymptotic stability of a stochastic delayed predator-prey model with Beddington-DeAngelis functional response*, Applied Mathematics and Computation **226** (2014), 581–588.
[34] M. Liu, K. Wang, *Global stability of stage-structured predator-prey models with Beddington-DeAngelis functional response*, Communications in Nonlinear Science and Numerical Simulation, **16** (2011), 3792–3797.
[35] D. Luo, *The study of global stability of a periodic Beddington-DeAngelis and Tanner predator-prey model*, Results in Mathematics **74** (2019), 74–101.
[36] A. M. Lyapunov, *The general problem of the stability of motion*, Taylor & Francis, London, 1992.
[37] R. E. Mickens, *A nonstandard finite-difference scheme for the Lotka-Volterra system*, Applied Numerical Mathematics **45** (2003), 309–314.
[38] Z. M. Odibat, N. T. Shawagfeh, *Generalized Taylor's formula*, Applied Mathematics and Computation **186** (2007), 286–293.
[39] Y. Qi, Y. Zhu, *The study of global stability of a diffusive Holling-Tanner predator-prey model*, Applied Mathematics Letters **57** (2016), 132–138.
[40] G. Ren, Y. Shi, *Global boundedness and stability of solutions for prey-taxis model with handling and searching predators*, Nonlinear Analysis: Real World Applications **60** (2021), 103306.
[41] Y. Saito, J. Sugie, Y. H. Lee, *Global asymptotic stability for predator-prey models with environmental time-variations*, Applied Mathematics Letters, **24** (2011), 1973–1980,
[42] H. Shi, Y. Li, *Global asymptotic stability of a diffusive predator-prey model with ratio-dependent functional response*, Applied Mathematics and Computation **250** (2015), 71–77.

[43] H. L. Smith, P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, 1995.
[44] X. Song, H. Guo, *Global stability of a stage-structured predator-prey system*, International Journal of Biomathematics 01 (2008), 313–326.
[45] J. Sugie, Y. Saito, M. Fan, *Global asymptotic stability for predator-prey systems whose prey receives time-variation of the environment*, Proceedings of the American Mathematical Society **139** (2011), 3475–3483.
[46] D. Y. Trejos, J. C. Valverde, E. Venturino, *Dynamics of infectious diseases: A review of the main biological aspects and their mathematical translation*, Applied Mathematics and Nonlinear Sciences **7** (2022), 1–26.
[47] C. Wang, *Rich dynamics of a predator-prey model with spatial motion*, Applied Mathematics and Computation **260** (2015), 1–9.
[48] K. Wang, Y.L. Zhu, *Permanence and global asymptotic stability of a delayed predator-prey model with Hassell-Varley type functional response*, Bulletin of the Iranian Mathematical Society **37** (2011), 197–215.
[49] X. Yan, C. Zhang, *Global stability of a delayed diffusive predator-prey model with prey harvesting of Michaelis-Menten type*, Applied Mathematics Letters **114** (2021), 106904.
[50] S. Yu, *Global Asymptotic Stability of a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes*, Discrete Dynamics in Nature and Society **2012** (2012), Article ID 208167.
[51] J. Yuan, L. Zhao, C. Huang, M. Xiao, *Stability and bifurcation analysis of a fractional predator-prey model involving two nonidentical delays*, Mathematics and Computers in Simulation **181** (2021), 562–580.