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The influence of the Hardy potential and a convection term on a nonlinear degenerate elliptic equations

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Abstract. This paper is devoted to prove existence of renormalized solutions for a class of non–linear degenerate elliptic equations involving a non–linear convection term, which satisfies a growth properties, and a Hardy potential. Additionally, we assume that the right-hand side is an L^m function, with $m \geq 1$.

1. Introduction

Let $\mathcal{W}$ denote a bounded open subset of $\mathbb{R}^N (N \geq 3)$ such that $0 \in \mathcal{W}$. Consider the following model of nonlinear elliptic problem with principal part having degenerate coercivity

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla v|^{p-2}\nabla v}{(1+|v|)^{\theta(p-1)}} + c_0(x)|v|^{\lambda-1}v\right) = \gamma\dfrac{|v|^{s-1}v}{|x|^p} + f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{1.1}$$

here $p \in (1, +\infty)$ and θ, λ, γ and s are positive numbers, $c_0(x) \in L^{\frac{N}{p-1}}(\mathcal{W})$ and f is in $L^m(\mathcal{W})$ with $m \geq 1$. Let us assume that the operator has no convection term and no Hardy potential, i.e. $\theta = c_0 = \gamma = 0$, in this case the difficulties in studying problem (1.1) are due only to the right-hand side f. We recall that in the classical case $\theta = 0$, such kind of problems with convection term were studied well in the literature in a different frameworks for an exhaustive review of this topic, we refer to [10, 20, 22, 24, 26, 28–30]. Moreover, we recall also the works [1, 2, 16, 21, 27] where the classic boundary value problems involving the Hardy potential.

Given $k > 0$ and $\forall n \in \mathbb{N}^*$, denote by $\mathcal{T}_k$ the truncation function at level $\pm k$ define as

$$\mathcal{T}_k(t) = \min\{k, \max\{-k, t\}\}, \quad \forall t \in \mathbb{R}.$$

It is well known that the framework of renormalized or entropy solution makes a sense according to the following definition of the weak gradient

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Definition 1.1. *(See [11], Lemma 2.1). If v is a measurable function defined on $\mathcal{W}$ that is almost everywhere finite and satisfies $\mathcal{T}_k(v) \in W_0^{1,p}(\mathcal{W})$ for all $k > 0$, then there exists a unique measurable function $w : \mathcal{W} \to \mathbb{R}^N$ such that*

$$\nabla \mathcal{T}_k(v) = w\chi_{\{|u| \le k\}}.$$

Thus, we can define the generalized gradient ∇v of v as this function w, and denote $\nabla v = w$.

The same reasoning applies if we deal with the degenerate case, on condition that u is finite almost everywhere in $\mathcal{W}$ and such that

$$\nabla \mathcal{T}_k(v) \in (L^p(\mathcal{W}))^N \quad \text{for every } k > 0. \tag{1.2}$$

The degenerate case was firstly studied in [17]. In this paper, Boccardo and al have studied the existence and regularity for the following quasi-linear elliptic problem

$$\begin{cases} -\operatorname{div}(A(x,v)\nabla v) = f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{1.3}$$

here f is assumed to be in $L^m(\mathcal{W})$ with $m \ge 1$, and $A(x,t) : \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to x for every $t \in \mathbb{R}$, and continuous function with respect to t for almost every $x \in \mathcal{W}$, satisfying the following condition: there exist $\theta \in [0,1]$, α, $\beta \in (0,\infty)$ such that

$$\frac{\alpha}{(1+|t|)^\theta} \le A(x,t) \le \beta, \quad \text{for a.e. } x \in \mathcal{W}, \forall t \in \mathbb{R}.$$

Moreover, in the paper [14] the authors demonstrated the existence of a renormalized solutions for the problem (1.3) with datum $f \in L^1(\mathcal{W})$ and $A(x,t) : \mathcal{W} \times \mathbb{R} \to \mathbb{R}^{N\times N}$ is a Carathéodory function with values in the space of matrices on $\mathbb{R}$ and is not assumed to be symmetric. A result on the existence and regularity of weak and entropy solutions is obtained, by Alvino and al in [8], for a nonlinear degenerate elliptic problem of the form

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla v|^{p-2}\nabla v}{(1+|v|)^{\theta(p-1)}}\right) = f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases}$$

where f is a measurable function in $L^m(\mathcal{W})$ with $m \ge 1$.

One of the main points that we stress in this paper is to analyze the interaction between the convection term and the one singular at the origin, the so-called Hardy potential, to obtain the existence of a renormalized solution for the problem (1.1). The influence of Hardy potential in elliptic problems has been studied in several papers (see for example the book [27] for a more general framework).

Actually, if $c_0(x) = 0$ and f is a nonnegative function in $L^m(\mathcal{W})$ with $m \ge 1$, the authors established, in [25], an existence and non-existence result of non-negative renormalized solutions for a nonlinear degenerate elliptic problem of the form

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla v|^{p-2}\nabla v}{(d(x)+|v|)^{\theta(p-1)}}\right) = \gamma\dfrac{|v|^s}{|x|^p} + f & \text{in } \mathcal{W}, \\ u \ge 0 & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases}$$

where γ and s are positive numbers and $d : \mathcal{W} \to (0,+\infty)$ is a bounded measurable function. In addition, one of the most interesting phenomena that exhibit this problem if $s = (1-\theta)(p-1)$ and $f \in L^1(\mathcal{W})$ is the non-existence of solutions. This non-existence result can be illustrated by considering the following simplest problem, studied in [16],

$$\begin{cases} -\Delta v = \gamma\dfrac{v}{|x|^2} + f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}. \end{cases}$$

On the other hand, in the case where f is a Radon measure with bounded variation defined on $\mathcal{W}$, T. Del Vecchio and M.R. Posteraro demonstrated, using the symmetrization method, the existence of weak solutions for a class of nonlinear and noncoercive problem involving a lower order term, whose prototype is

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v + c_0(x)|v|^{\lambda}) + d(x)|\nabla v|^{\mu} = f & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{1.4}$$

in this context, the functions $d(x)$ and $c_0(x)$ belong to $L^N(\mathcal{W})$ and $L^{\frac{N}{p-1}}(\mathcal{W})$ respectively. Moreover, in the case when $\gamma = \mu = p-1$ they supposed that $\|d(x)\|_{L^N(\mathcal{W})}$ or $\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}$ is small enough. The most delicate case was to obtain a priori estimate for v and ∇v in the case when $\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}$ is not small. Recently, in the paper [22], O. Guibé and A. Mercaldo studied the problem (1.4) in the general framework of Lorentz spaces. The authors successfully demonstrated the existence of renormalized solutions under the conditions $0 \le \mu, \lambda \le p-1$ and $\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}$ is not small. This is done by proving the following uniform estimate

$$\forall \eta > 0, \exists \nu_\eta > 0 \quad meas\{|v| > \nu_\eta\} \le \frac{1}{\eta^p}, \tag{1.5}$$

which allowed them to derive the estimate (1.2) which implies an estimates on $|v|^{p-1}$ and $|\nabla v|^{p-1}$, thanks to the lemma 2.4, in some Lorentz-Marcinkiewicz space.

However, in the case $\theta \neq 0$ and $\gamma \neq 0$, one can not prove (1.5) but instead one only has

$$\forall \eta > 0, \exists \bar{\nu}_\eta > 0 \quad meas\{|\widetilde{\varrho}(v)| > \bar{\nu}_\eta\} \le \frac{1}{\eta^p}.$$

with $\widetilde{\varrho}(t)$ denotes the primitive of a decreasing continuous function given by

$$\varrho(t) = \frac{1}{(1+|t|)^{\theta}} \qquad \theta \in [0,1),$$

which satisfies the following behavior at ∞

$$\lim_{|t|\to+\infty} \widetilde{\varrho}(t) = \pm\infty. \tag{1.6}$$

Moreover, there exists a positive constant $\widetilde{C} > 0$ and a positive real number k_0 such that for every $|t| > k_0$, one has

$$\frac{|t|^{\lambda}}{(1+|\widetilde{\varrho}(t)|)^{p-1}} \le \widetilde{C}. \tag{1.7}$$

We stress that the method used in [22], is not apply directly in our case since the right hand side of our problem involving the Hardy potential. In order to overcome this difficulty, we prove an L^1−estimate on Hardy potential term by arguing as in [25]. This is enough, thanks to (1.6) and (1.7), to ensure that u satisfies (1.5).

Finally we explicitly remark that if we consider $0 < s, \lambda < (1-\theta)(p-1)$. Thanks to (1.7), the following inequalities hold

$$\int_{\mathcal{W}} \frac{|v|^s}{|x|^p}\, dx \le \widetilde{C} \int_{\mathcal{W}} \frac{(1+|\widetilde{\varrho}(v)|)^{p-1}}{|x|^p}\, dx,$$

and

$$\int_{\mathcal{W}} c_0(x)|v|^{\lambda}\, dx \le \widetilde{C} \int_{\mathcal{W}} c_0(x)(1+\widetilde{\varrho}(v))^{p-1}\, dx.$$

Therefore, the equation (1.1) may be equivalently written as

$$\begin{cases} -\operatorname{div}\left(|\nabla\widetilde{\varrho}(v)|^{p-2}\nabla\widetilde{\varrho}(v)+\widetilde{c}_0(x)|\widetilde{\varrho}(v)|^{p-1}\right)=\gamma\dfrac{|\widetilde{\varrho}(v)|^{p-1}}{|x|^p}+g & \text{in } \mathcal{W}, \\ \widetilde{\varrho}(v)=0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{1.8}$$

such that $\widetilde{c}_0(x) \in L^{\frac{N}{p-1}}(\mathcal{W})$ and the the right-hand side, $g \in L^m(\mathcal{W})$ with $m > 1$. In general, the problem (1.8) is not coercive and has no weak solution when $g \in L^1(\mathcal{W})$, $\gamma > 0$. Thus, in the present paper, we face the two difficulties arise from the presence of both the non–linear convection term and Hardy potential. This means that we will be dealing with all the difficulties previously described, at the same time. To our knowledge, the exploration of the combined impact of the non–linear convection term and the Hardy potential has not been undertaken before.

For ease of reading, in the Sec. 2, we recall some well-know preliminaries, properties and definitions of the Lorentz-Marcinkiewicz space, as well as we set our main assumptions. While in Sec. 3 we give the proof of the existence result.

2. Some preliminaries and definitions

Now, we give some basic tools for functional analysis that we will use in our study. The Lorentz space $L^{q,r}(\mathcal{W})$ is the space of Lebesgue measurable functions such that for any $(q,r) \in (1,\infty)^2$

$$\|f\|_{L^{q,r}(\mathcal{W})} = \left(\int_0^{meas(\mathcal{W})} \left[f^*(t)t^{\frac{1}{q}}\right]^r \frac{dt}{t}\right)^{1/r} < +\infty.$$

Where, f^* stands for the decreasing rearrangement of the function f which defined by

$$f^*(t) = \inf\{r \geq 0 : \text{ meas } \{x \in \mathcal{W} : |f(x)| > r\} < t\} \quad t \in [0, meas(\mathcal{W})].$$

We also mentioned that, for any $q \in [1,+\infty)$ the Lorentz-Marcinkiewicz space $L^{q,\infty}(\mathcal{W})$ is the set of measurable functions $f : \mathcal{W} \to \mathbb{R}$ such that

$$\|f\|_{L^{q,\infty}(\mathcal{W})} = \sup_t t\,(\text{ meas } \{x \in \mathcal{W} : |f| > t\})^{\frac{1}{q}} < +\infty. \tag{2.1}$$

Moreover for any z and q such that $1 \leq q < r < z \leq +\infty$, the following chain of continuous inclusions in Lebesgue spaces holds true

$$L^z(\mathcal{W}) \subset L^{r,\infty}(\mathcal{W}) \subset L^q(\mathcal{W}) \subset L^1(\mathcal{W}). \tag{2.2}$$

For references about rearrangements see, for example, [19].

The following lemmas introduce some well-known inequalities that will be very useful in a number of situations such as a priori estimates.

Lemma 2.1. *(Poincaré's inequality). Suppose $p \in [1,N)$ and $v \in W_0^{1,p}(\mathcal{W})$. Then there exist a constant $c(N,p)$ such that*

$$\|v\|_{L^p(\mathcal{W})} \leq c(N,p)\|\nabla v\|_{L^p(\mathcal{W})}, \tag{2.3}$$

Lemma 2.2. *(Sobolev's inequality). Suppose $p \in [1,N)$ and $v \in W_0^{1,p}(\mathcal{W})$. Then there exist a constant $\mathcal{S}$ such that*

$$\|v\|_{L^{p^*}(\mathcal{W})} \leq \mathcal{S}\|\nabla v\|_{L^p(\mathcal{W})},$$

with $p^* = \frac{Np}{N-p}$.

Lemma 2.3. *(Hardy's inequality). Suppose $p \in (1, N)$ and $u \in W_0^{1,p}(\mathcal{W})$. Then we have*

$$\int_{\mathcal{W}} \frac{|v|^p}{|x|^p}\, dx \leq \mathcal{H} \int_{\mathcal{W}} |\nabla v|^p\, dx,$$

with $\mathcal{H} = \left(\frac{p}{N-p}\right)^p$ optimal and not achieved constant.

Proof. See [21]. □

An important lemma which generalize a result of [11] is the following:

Lemma 2.4. *Assume that $\mathcal{W}$ is an open subset of $\mathbb{R}^N$ with finite measure and that $p \in (1, N)$. Let ψ be a measurable function satisfying $\mathcal{T}_k(\psi) \in W_0^{1,\upsilon}(\mathcal{W})$, for every positive k, and such that for some constants M and L we have the inequality*

$$\int_{\mathcal{W}} |\nabla \mathcal{T}_k(\psi)|^{\upsilon}\, dx \leq Mk^{\tau} + L, \quad \forall k > 0,$$

where $(\tau, \upsilon) \in (0, N)^2$ are given constants. Then $|\psi|^{\upsilon-\tau}$ belongs to $L^{\frac{N}{N-\upsilon},\infty}(\mathcal{W})$, $|\nabla\psi|^{\upsilon-\tau}$ belongs to $L^{\frac{N}{N-\tau},\infty}(\mathcal{W})$ and there exists a constant C which depending only on N and p such that

$$\left\| |\psi|^{\upsilon-\tau} \right\|_{L^{\frac{N}{N-\upsilon},\infty}(\mathcal{W})} \leq C(N, p)\left[M + meas(\mathcal{W})^{1-\frac{\upsilon}{\tau}} L^{\frac{\upsilon-\tau}{\upsilon}}\right], \tag{2.4}$$

$$\left\| |\nabla\psi|^{\upsilon-\tau} \right\|_{L^{\frac{N}{N-\tau},\infty}(\mathcal{W})} \leq C(N, p)\left[M + meas(\mathcal{W})^{\frac{\tau}{\upsilon}(1-\frac{\upsilon}{\tau})} L^{\frac{\upsilon-\tau}{\upsilon}}\right]. \tag{2.5}$$

Proof. See [22]. □

Consider a nonlinear elliptic problem which can be written as

$$\begin{cases} -\operatorname{div}(b(x, v, \nabla v) + \mathcal{B}(x, v)) = \gamma \dfrac{|v|^{s-1}v}{|x|^p} + f(x) & \text{in } \mathcal{W}, \\ u = 0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{2.6}$$

with γ and s are positive constants. Throughout the paper, we assume that the following assumptions hold true:

$b : \mathcal{W} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies assumptions:

$$b(x, \eta, \xi).\xi \geq \alpha\varrho^{p-1}(\eta)|\xi|^p, \tag{2.7}$$

$$|b(x, \eta, \xi)| \leq C(G(x) + |\eta|^{p-1} + |\xi|^{p-1}), \tag{2.8}$$

$$[b(x, \eta, \xi) - b(x, \eta, \xi')][\xi - \xi'] > 0, \tag{2.9}$$

for almost every $x \in \mathcal{W}$, for every $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, α and C are positive real number, and G is a nonnegative function in $L^{p'}(\mathcal{W})$.

$\mathcal{B} : \mathcal{W} \times \mathbb{R} \to \mathbb{R}^N$ is Carathéodory function satisfies the growth condition

$$|\mathcal{B}(x, \eta)| \leq c_0(x)|\eta|^{\lambda}, \text{ with } 0 \leq \lambda < (1-\theta)(p-1) \text{ and } c_0(x) \in L^{\frac{N}{p-1}}(\mathcal{W}). \tag{2.10}$$

$$0 \leq s < \frac{p(1-\theta)(p-1)}{p^*}, \ \gamma \geq 0 \text{ and } f \in L^m(\mathcal{W}), \text{ with } m \geq 1. \tag{2.11}$$

We first give the definition of a renormalized solutions to problem (2.6). Then, we will discuss the existence of a renormalized solutions to problem (2.6).

We will now provide the definition of a renormalized solutions to problem (2.6).

Definition 2.5. *A function $v : \mathcal{W} \to \mathbb{R}$ is considered a renormalized solutions to Problem* (2.6) *if it satisfies the following conditions:*

$$v \text{ is measurable and finite almost everywhere in } \mathcal{W}, \tag{2.12}$$

$$\mathcal{T}_k(\widetilde{\varrho}(v)) \in W_0^{1,p}(\mathcal{W}) \quad \forall k > 0, \tag{2.13}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{|\widetilde{\varrho}(v)| \le n\}} \varrho(v) b(x, v, \nabla v)\nabla v \, dx = 0, \tag{2.14}$$

and if for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, we have

$$\begin{aligned} &\int_{\mathcal{W}} b(x, v, \nabla v)\varrho(v)\nabla v h'(\widetilde{\varrho}(v))\varphi \, dx + \int_{\mathcal{W}} b(x, v, \nabla v)\nabla\varphi h(\widetilde{\varrho}(v)) \, dx \\ &\quad + \int_{\mathcal{W}} \mathcal{B}(x, v)\varrho(v)\nabla v h'(\widetilde{\varrho}(v))\varphi \, dx + \int_{\mathcal{W}} \mathcal{B}(x, v)\nabla\varphi h(\widetilde{\varrho}(v)) \, dx \\ &= \gamma \int_{\mathcal{W}} \frac{|v|^{s-1}v}{|x|^p} h(\widetilde{\varrho}(v))\varphi \, dx + \int_{\mathcal{W}} f h(\widetilde{\varrho}(v))\varphi \, dx \end{aligned} \tag{2.15}$$

for every $\varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W})$.

Remark 2.6. *We notice that, since $\widetilde{\varrho}(\pm\infty) = \pm\infty$ which means that the set $\{|\widetilde{\varrho}(v)| \le n\}$ may be equivalent to $\{|u| \le k_n\}$ with $k_n = \max\{\widetilde{\varrho}^{-1}(n), \widetilde{\varrho}^{-1}(-n)\}$, then, due to* (2.13) *we deduce that the condition* (2.14) *is well defined.*

Remark 2.7. *It is worth noting that growth assumption* (2.10) *on $\mathcal{B}$ together with* (2.12) – (2.14) *allow to prove that any renormalized solutions u verifies*

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\mathcal{W}} |\mathcal{B}(x, v)||\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx = 0. \tag{2.16}$$

Indeed, (1.6), (1.7) *and the growth assumption* (2.10) *imply that*

$$\begin{aligned} \int_{\mathcal{W}} |\mathcal{B}(x, v)||\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| \, dx &\le \int_{\mathcal{W}} c_0(x)|u|^\lambda |\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| \, dx, \\ &= \int_{\mathcal{W}} c_0(x) \frac{|u|^\lambda \left(1 + |\widetilde{\varrho}(v)|\right)^{p-1}}{\left(1 + |\widetilde{\varrho}(v)|\right)^{p-1}} |\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| \, dx, \\ &\le \widetilde{C} \int_{\mathcal{W}} c_0(x) \left(1 + |\widetilde{\varrho}(v)|\right)^{p-1} |\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| \, dx, \\ &\le \widetilde{C} c_p \int_{\mathcal{W}} c_0(x)|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx + \widetilde{C} c_p \int_{\mathcal{W}} c_0(x)|\mathcal{T}_n(\widetilde{\varrho}(v))|^{p-1}|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx, \end{aligned}$$

where $c_p = \max\left\{1, 2^{p-2}\right\}$.

By Hölder's and Sobolev's inequalities it follows that

$$\begin{aligned} \int_{\mathcal{W}} |\mathcal{B}(x, v)||\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx &\le \widetilde{C} \int_{\mathcal{W}} c_0(x)|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx + \widetilde{C} c_p \int_{\mathcal{W}} c_0(x)|\mathcal{T}_n(\widetilde{\varrho}(v))|^{p-1}|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))| dx, \\ &\le \widetilde{C} c_p \|c_0(x)\|_{L^{p'}(\mathcal{W})} \|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))\|_{L^p(\mathcal{W})} + \widetilde{C} c_p \|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})} \left\|\mathcal{T}_n(\widetilde{\varrho}(v))\right\|_{L^{p^*}(\mathcal{W})}^{p-1} \left\|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))\right\|_{L^p(\mathcal{W})}, \\ &\le \widetilde{C} c_p \|c_0(x)\|_{L^{p'}(\mathcal{W})} \|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))\|_{L^p(\mathcal{W})} + \widetilde{C} c_p \mathcal{S}^{p-1} \|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})} \left\|\nabla\mathcal{T}_n(\widetilde{\varrho}(v))\right\|_{L^p(\mathcal{W})}^p, \end{aligned}$$

which, using Young's inequality and (2.14)*, gives* (2.16).

Remark 2.8. *Note that the term* $\gamma \int_{\mathcal{W}} \frac{|v|^{s-1}v}{|x|^p} h(\widetilde{\varrho}(v))\varphi \, dx$ *is well-defined. Indeed, let* $k > 0$ *such that* $supp(h) \subset [-k, k]$*. It follows from* (1.7)*, Hölder's and Hardy Inequalities that*

$$\left|\gamma \int_{\mathcal{W}} \frac{|v|^{s-1}v}{|x|^p} h(\widetilde{\varrho}(v))v \, dx\right| \leq \gamma \int_{\mathcal{W}} \frac{|v|^s}{(1+|\widetilde{\varrho}(v)|)^{p-1}} \frac{(1+|\widetilde{\varrho}(v)|)^{p-1}}{|x|^p} |h(\widetilde{\varrho}(v))||\varphi| \, dx,$$
$$\leq C_\gamma \left[\left(\int_{\mathcal{W}} \frac{dx}{|x|^p} \, dx \right)^{\frac{1}{p'}} + \mathcal{H}^{\frac{1}{p'}} \left(\int_{\mathcal{W}} |\nabla \mathcal{T}_k(\widetilde{\varrho}(v))|^p \, dx \right)^{\frac{1}{p'}} \right]$$

where

$$C_\gamma = \gamma c_p \widetilde{C} \|h\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\mathcal{W})} \left(\int_{\mathcal{W}} \frac{dx}{|x|^p} \, dx \right)^{\frac{1}{p}}. \tag{2.17}$$

Then we have, from (2.13)*, that* $\frac{|v|^{s-1}v}{|x|^p} h(\widetilde{\varrho}(v))\varphi \in L^1(\mathcal{W})$.

Remark 2.9. *The renormalized equation* (2.15) *is formally obtained through a pointwise multiplication of* (2.6) *by* $h(v)\varphi$*. Let us observe that by the previous remark,* (2.13) *and the properties of* h*, every term in* (2.15) *makes sense.*

3. Existence of renormalized solutions

The main result of the present paper is the following existence result.

Theorem 3.1. *Let us assume that the assumptions* (2.7) − (2.11) *hold and suppose that* $f \in L^m(\mathcal{W})$ *with*

$$1 \leq m < m_\theta = \frac{pN}{pN - (N-p)(\theta(p-1)+1)}. \tag{3.1}$$

If $\lambda < (1-\theta)(p-1)$*,* $s < \frac{p(1-\theta)(p-1)}{p^*}$ *and* $\gamma \geq 0$*, then there exists a renormalized solutions of equation* (2.6)*.*

Proof. The proof of this result follows a classical approach that involves introducing a sequence of approximate problems. Subsequently, we establish a priori estimates for both the approximate solutions and their gradients in Lorentz-Marcinkiewicz spaces, thereby providing an estimate in $L^1(\mathcal{W})$ for the singular term. Next, we prove an energy estimate, which constitutes a crucial element for the subsequent stages of the proof. Moreover, in the fourth step of the proof, we establish the a.e. convergence of gradient in $\mathcal{W}$ by proving Lemma 3.6. Finally, we pass to the limit in the approximate problem.

• **First Step: Approximate problem**

Let's introduce a regularization of the data as follows: for a fixed $\varepsilon > 0$, let's define

$$b_\varepsilon(x, \eta, \xi) = b(x, \mathcal{T}_{\frac{1}{\varepsilon}}(\eta), \xi) \quad \forall \eta \in \mathbb{R}, \tag{3.2}$$

$$\mathcal{B}_\varepsilon(x, \eta) = \mathcal{B}(x, \mathcal{T}_{\frac{1}{\varepsilon}}(\eta)) \quad \forall \eta \in \mathbb{R}, \tag{3.3}$$

$$f^\varepsilon = \mathcal{T}_{\frac{1}{\varepsilon}}(f) \quad \text{and } f^\varepsilon \to f \text{ strongly in } L^m(\mathcal{W}). \tag{3.4}$$

Observe that

$$\begin{aligned} |b_\varepsilon(x, \eta, \xi)| &= |b(x, \mathcal{T}_{\frac{1}{\varepsilon}}(\eta), \xi)| \\ &\leq C(G(x) + |\mathcal{T}_{\frac{1}{\varepsilon}}(\eta)|^{p-1} + |\xi|^{p-1}), \\ &\leq C(G(x) + \frac{1}{\varepsilon^{p-1}} + |\xi|^{p-1}) \in L^{p'}(\mathcal{W}), \end{aligned} \tag{3.5}$$

and

$$b_\varepsilon(x,\eta,\xi)\xi = b(x,\mathcal{T}_{\frac{1}{\varepsilon}}(\eta),\xi)\xi \geq \frac{\alpha|\xi|^p}{(1+\frac{1}{\varepsilon})^{\theta(p-1)}} \geq \widetilde{\alpha}|\xi|^p, \tag{3.6}$$

moreover

$$|\mathcal{B}_\varepsilon(x,\eta)| = |\mathcal{B}(x,\mathcal{T}_{\frac{1}{\varepsilon}}(\eta))| \leq \frac{c_0(x)}{\varepsilon^\lambda} \in L^{p'}(\mathcal{W}). \tag{3.7}$$

Let $v_\varepsilon \in W_0^{1,p}(\mathcal{W})$ be a weak solution to the following approximate problem

$$\begin{cases} -div(b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)+\mathcal{B}_\varepsilon(x,v_\varepsilon)) = \gamma\dfrac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon} + f^\varepsilon & \text{in } \mathcal{W}, \\ v_\varepsilon = 0 & \text{on } \partial\mathcal{W}, \end{cases} \tag{3.8}$$

in the sens that

$$\int_{\mathcal{W}} b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla\varphi\,dx + \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla\varphi\,dx = \gamma\int_{\mathcal{W}} \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}\varphi\,dx + \int_{\mathcal{W}} f^\varepsilon\varphi\,dx \tag{3.9}$$

$\forall\varphi \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W})$.

According to (3.5), (3.6) and (3.7) the existence of a solution v_ε of (3.8) is a well-known result (see, e.g.,[23]).

• **Second step: A priori estimates**

In this step we deal with the approximate problem (3.8). We begin by proving some a priori estimates on $\widetilde{\varrho}(v_\varepsilon)$, $\nabla\widetilde{\varrho}(v_\varepsilon)$, v_ε, and ∇v_ε.

Proposition 3.2. *Suppose that $f \in L^m(\mathcal{W})$ with $1 \leq m < m_\theta$. Under the assumptions* (2.7) − (2.10) *and if $\lambda < (1-\theta)(p-1)$, $s < \frac{p(1-\theta)(p-1)}{p^*}$ and $\gamma \geq 0$. Then, every weak solution of the problem* (3.9) *satisfies*

$$\left\||\widetilde{\varrho}(v_\varepsilon)|^{p-1}\right\|_{L^{\frac{N}{N-p},\infty}(\mathcal{W})} \leq c_1\left(N,p,\alpha,meas(\mathcal{W}),c_0\right) \tag{3.10}$$

$$\left\||\nabla\widetilde{\varrho}(v_\varepsilon)|^{p-1}\right\|_{L^{\frac{N}{N-1},\infty}(\mathcal{W})} \leq c_2\left(N,p,\alpha,meas(\mathcal{W}),c_0\right) \tag{3.11}$$

$$\left\||v_\varepsilon|^{(1-\theta)(p-1)}\right\|_{L^{\frac{N}{N-p},\infty}(\mathcal{W})} \leq c_3\left(N,p,\alpha,meas(\mathcal{W}),c_0\right) \tag{3.12}$$

$$\left\||\nabla v_\varepsilon|^{(1-\theta)(p-1)}\right\|_{L^{\frac{N}{N-1-\theta(p-1)},\infty}(\mathcal{W})} \leq c_4\left(N,p,\alpha,meas(\mathcal{W}),c_0\right) \tag{3.13}$$

for some positive constants c_1, c_2, c_3 and c_4.

Proof. We will divide the proof of this proposition into two steps. Initially, we will establish that our sequence of weak solutions v_ε is almost everywhere finite in $\mathcal{W}$. Following this outcome, we will derive uniform bounds for $\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))$ and $\mathcal{T}_k(v_\varepsilon)$ in Lebesgue spaces. Let us mention that throughout the paper C_i, $i \in \mathbb{N}^*$, denotes positive constants independent of ε that are different from line to line. At last, for any measurable set $D \subset \mathbb{R}^N$, D^c denotes its complement.

• *Step 1: v_ε is finite a.e. in $\mathcal{W}$.*

This step is devoted to establish that assuming conditions (2.7), (2.10), and $f \in L^1(\mathcal{W})$, the sequence of weak solution satisfies

$$\forall\eta > 0, \exists k_\eta > 0 \quad meas\{|v_\varepsilon| > k_\eta\} \leq \frac{1}{\eta^p}, \quad \text{uniformaly w.r.t } \varepsilon. \tag{3.14}$$

To this aim, let us consider the real valued function $\psi_p : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_p(t) = \int_0^t \frac{dr}{(\beta_\varepsilon^{\frac{1}{p-1}} + |r|)^p},$$

where $\beta_\varepsilon > 1$, is a suitably chosen parameter.

We use $\psi_p(\widetilde{\varrho}(v_\varepsilon))$ as test function in (3.9), we get

$$\int_{\mathcal{W}} b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\nabla\psi_p(\widetilde{\varrho}(v_\varepsilon)) + \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)\nabla\psi_p(\widetilde{\varrho}(v_\varepsilon)) = \gamma \int_{\mathcal{W}} \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon}\psi_p(\widetilde{\varrho}(v_\varepsilon)) + \int_{\mathcal{W}} f^\varepsilon\psi_p(\widetilde{\varrho}(v_\varepsilon)). \tag{3.15}$$

Using the assumptions (2.7) and (2.10), we obtain

$$\alpha \int_{\mathcal{W}} \frac{|\nabla\widetilde{\varrho}(v_\varepsilon)|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)^p}\, dx \le \int_{\mathcal{W}} c_0(x)\frac{|v_\varepsilon|^\lambda}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)^{p-1}}\frac{|\nabla\widetilde{\varrho}(v_\varepsilon)|}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)}\, dx + \frac{1}{\beta_\varepsilon(p-1)}M_\varepsilon,$$

where

$$M_\varepsilon = \gamma \int_{\mathcal{W}} \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^s dx}{|x|^p + \varepsilon} + \left\|f^\varepsilon\right\|_{L^1(\mathcal{W})}.$$

Thanks to (1.7), for any $\lambda < (1-\theta)(p-1)$, we have

$$\frac{|v_\varepsilon|^\lambda}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)^{p-1}} \le \widetilde{C}.$$

Thus, applying Young's inequality, we obtain

$$\int_{\mathcal{W}} \frac{|\nabla\widetilde{\varrho}(v_\varepsilon)|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)^p}\, dx \le C_1 + \frac{p'}{\alpha\beta_\varepsilon(p-1)}M_\varepsilon.$$

Now, if we choose $\beta_\varepsilon = 1 + \dfrac{p'}{\alpha(p-1)}M_\varepsilon$, we deduce that

$$\int_{\mathcal{W}} \frac{|\nabla\widetilde{\varrho}(v_\varepsilon)|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |\widetilde{\varrho}(v_\varepsilon)|)^p}\, dx \le C_1 + 1,$$

which implies, applying Poincaré's inequality and for any $h > 0$, that

$$\begin{aligned}
meas\left\{|\widetilde{\varrho}(v_\varepsilon)| > h\beta_\varepsilon^{\frac{1}{p-1}}\right\} &= \frac{1}{[ln(1+h)]^p}\int_{\left\{|\widetilde{\varrho}(v_\varepsilon)|>h\beta_\varepsilon^{\frac{1}{p-1}}\right\}} [ln(1+h)]^p\, dx,\\
&\le \frac{1}{[ln(1+h)]^p}\int_{\left\{|\widetilde{\varrho}(v_\varepsilon)|>h\beta_\varepsilon^{\frac{1}{p-1}}\right\}} \left[ln\left(1 + \frac{|\widetilde{\varrho}(v_\varepsilon)|}{\beta_\varepsilon^{\frac{1}{p-1}}}\right)\right]^p dx,\\
&\le \frac{1}{[ln(1+h)]^p}\int_{\mathcal{W}} \left[ln\left(1 + \frac{|\widetilde{\varrho}(v_\varepsilon)|}{\beta_\varepsilon^{\frac{1}{p-1}}}\right)\right]^p dx,\\
&\le \frac{A}{[ln(1+h)]^p},
\end{aligned}$$

where

$$A = c(N,p)\,(1 + C_1)\,.$$

Then, for any $\eta > 0$, we have

$$meas\left\{|\widetilde{\varrho}(v_\varepsilon)| > \sigma_\eta(\varepsilon)\right\} \le \frac{1}{\eta^p}, \tag{3.16}$$

where

$$\sigma_\eta(\varepsilon) = \left(\exp\left(\eta A^{\frac{1}{p}}\right) - 1\right)\beta_\varepsilon^{\frac{1}{p-1}}. \tag{3.17}$$

Remark 3.3. *Notice that, from* (3.17) *and recalling the definition of β_ε, we can express σ_η as follows*

$$\sigma_\eta(\varepsilon) = \left(\exp\left(\eta A^{\frac{1}{p}}\right) - 1\right)\left(1 + \frac{p'}{\alpha(p-1)}M_\varepsilon\right)^{\frac{1}{p-1}}.$$

Therefore, it is crucial to emphasize the need to establish the boundedness of the term M_ε uniformly with respect to ε to prove that $\widetilde{\varrho}(v_\varepsilon)$ is finite almost everywhere in $\mathcal{W}$. Referring back to the definition of M_ε, we can deduce the following inequality

$$M_\varepsilon \le \gamma \left\| \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^s}{|x|^p} \right\|_{L^1(\mathcal{W})} + \left\|f^\varepsilon\right\|_{L^1(\mathcal{W})}.$$

To derive the desired estimate, our task is to prove the boundedness of the Hardy potential term in $L^1(\mathcal{W})$. To this aim, we adopt the approach outlined in [25].

For any given $k \ge 0$, let $\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))$ be chosen as a test function in (3.9). This yields

$$\int_{\mathcal{W}} b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\,dx + \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\,dx$$
$$= \gamma\int_{\mathcal{W}} \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon}\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\,dx + \int_{\mathcal{W}} f^\varepsilon\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\,dx.$$

Using (2.7) and (2.10), we get

$$\alpha\int_{\mathcal{W}} \left|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\right|^p dx \le \int_{\mathcal{W}} c_0(x)|v_\varepsilon|^\lambda|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|\,dx + kM_\varepsilon,$$
$$\le \int_{\mathcal{W}} c_0(x)\frac{|v_\varepsilon|^\lambda|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|}{(1 + |\widetilde{\varrho}(v_\varepsilon)|)^{p-1}}\,(1 + |\widetilde{\varrho}(v_\varepsilon)|)^{p-1}\,dx + kM_\varepsilon,$$

Therefore, through (1.7) and employing Hölder's, Young's and Sobolev's inequalities, we obtain

$$\alpha\int_{\mathcal{W}} \left|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\right|^p dx \le \widetilde{C}\int_{\mathcal{W}} c_0(x)|\widetilde{\varrho}(v_\varepsilon)|^{p-1}|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|\,dx + \widetilde{C}\int_{\mathcal{W}} c_0(x)|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|\,dx + kM_\varepsilon,$$
$$\le \widetilde{C}\int_{Z^c_{\eta,\varepsilon}} c_0(x)|\widetilde{\varrho}(v_\varepsilon)|^{p-1}|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|\,dx + \widetilde{C}\int_{Z_{\eta,\varepsilon}} c_0(x)|\widetilde{\varrho}(v_\varepsilon)|^{p-1}|\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|\,dx$$
$$+ C_2 + \frac{\alpha}{2p}\int_{\mathcal{W}} |\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|^p\,dx + kM_\varepsilon,$$
$$\le \widetilde{C}\mathcal{S}^{p-1}\left\|c_0(x)\right\|_{L^{\frac{N}{p-1}}(Z^c_{\eta,\varepsilon})}\int_{\mathcal{W}} |\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|^p\,dx + \frac{\alpha}{p}\int_{\mathcal{W}} |\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|^p\,dx + C_\varepsilon + kM_\varepsilon,$$

where

$$Z_{\eta,\varepsilon} = \{x \in \mathcal{W}, |\widetilde{\varrho}(v_\varepsilon(x))| \le \sigma_\eta(\varepsilon)\},$$

and

$$C_\varepsilon = C_2 + \left(\frac{2p}{\alpha}\right)^{\frac{p'}{p}} \frac{\left(\widetilde{C}\sigma_\eta^{p-1}(\varepsilon)\right)^{p'}}{p'} \int_{\mathcal{W}} |c_0(x)|^{p'}\, dx,$$

i.e.,

$$\frac{\alpha}{p'} \int_{\mathcal{W}} \left|\nabla \mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\right|^p dx \le \widetilde{C}\mathcal{S}^{p-1} \left\|c_0(x)\right\|_{L^{\frac{N}{p-1}}(Z^c_{\eta,\varepsilon})} \int_{\mathcal{W}} |\nabla \mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|^p\, dx + C_\varepsilon + kM_\varepsilon.$$

Note that, for every $\varepsilon > 0$, we can select $\eta = \bar{\eta}$, in (3.17), such that

$$\frac{p'}{\alpha}\widetilde{C}\mathcal{S}^{p-1}\, \|c_0(x)\|_{L^{\frac{N}{p-1}}\left(Z^c_{\bar{\eta},\varepsilon}\right)} \le \frac{1}{2}.$$

As a result, we derive that

$$\int_{\mathcal{W}} \left|\nabla \mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\right|^p dx \le L_\varepsilon + M'_\varepsilon k,$$

where

$$L_\varepsilon = \frac{2p'}{\alpha} C_\varepsilon \text{ and } M'_\varepsilon = \frac{2p'}{\alpha} M_\varepsilon.$$

By lemma 2.4, we get

$$\left\|\left|\widetilde{\varrho}(v_\varepsilon)\right|^{p-1}\right\|_{L^{\frac{N}{N-p},\infty}(\mathcal{W})} \le C(N,p)\left[M'_\varepsilon + meas(\mathcal{W})^{\frac{1}{p}} L_\varepsilon^{\frac{1}{p'}}\right].$$

Moreover, using the fact that $\widetilde{\varrho}(v_\varepsilon)$ exhibits behavior similar to $|v_\varepsilon|^{1-\theta}$ for any $\theta < 1$, we conclude that

$$\left\||v_\varepsilon|^{(1-\theta)(p-1)}\right\|_{L^{\frac{N}{N-p},\infty}(\mathcal{W})} \le C(N,p)\left[M'_\varepsilon + meas(\mathcal{W})^{\frac{1}{p}} L_\varepsilon^{\frac{1}{p'}}\right].$$

Given $s < \frac{p(1-\theta)(p-1)}{p^*}$, and utilizing (2.2) along with the application of Hölder's inequality (with exponents $\rho = \frac{(1-\theta)(p-1)}{s}$ and $\rho' = \frac{\rho}{\rho-1}$), we obtain

$$\begin{aligned}
\left\|\frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^s}{|x|^p}\right\|_{L^1(\mathcal{W})} &\le \left(\int_{\mathcal{W}} |v_\varepsilon|^{(1-\theta)(p-1)}\, dx\right)^{\frac{1}{\rho}} \left(\int_{\mathcal{W}} \frac{dx}{|x|^{p\rho'}}\right)^{\frac{1}{\rho'}}, \\
&\le C_3 \left\||v_\varepsilon|^{(1-\theta)(p-1)}\right\|^{\frac{1}{\rho}}_{L^1(\mathcal{W})}, \\
&\le C_3 \left\||v_\varepsilon|^{(1-\theta)(p-1)}\right\|^{\frac{1}{\rho}}_{L^{\frac{N}{N-p},\infty}(\mathcal{W})}, \\
&\le C_3 \left[C(N,p)\left[M'_\varepsilon + meas(\mathcal{W})^{\frac{1}{p}} L_\varepsilon^{\frac{1}{p'}}\right]\right]^{\frac{s}{p-1}}, \\
&\le C_4 M_\varepsilon^{\frac{s}{p-1}} + C_5 L_\varepsilon^{\frac{s}{p}}.
\end{aligned}$$

Remark 3.4. *Let us emphasize that we don't know if the value* $\frac{p(p-1)(1-\theta)}{p^*}$ *is optimal in order to obtain the above estimate. Thus, it would be very interesting to know what happen in the lacking set* $\frac{p(p-1)(1-\theta)}{p^*} \leq s < (p-1)(1-\theta)$. *This delicate case, will be dealt with in an upcoming paper, wherein the focus lies on exploring renormalized solutionss with a new approach.*

After easy calculations, we prove that

$$L_\varepsilon = \frac{2p'}{\alpha}\left(C_2 + \left(\frac{2p}{\alpha}\right)^{\frac{p'}{p}} \frac{\left(\widetilde{C}\sigma_\eta^{p-1}(\varepsilon)\right)^{p'}}{p'} \int_{\mathcal{W}} |c_0(x)|^{p'}\, dx\right) \leq C_6 + C_7 M_\varepsilon^{p'}.$$

Which implies, by applying Young's inequality, that

$$\begin{aligned} \left\| \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^s}{|x|^p} \right\|_{L^1(\mathcal{W})} &\leq C_9 + C_8 M_\varepsilon^{\frac{s}{p-1}}, \\ &\leq C_{10} + \frac{1}{2\gamma} M_\varepsilon, \end{aligned}$$

thus, we deduce that

$$\left\| \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^s}{|x|^p} \right\|_{L^1(\mathcal{W})} \leq C_{11}. \tag{3.18}$$

So that there exists a positive constant $\bar{\sigma}_\eta$, independent of ε, such that

$$\sigma_\eta(\varepsilon) \leq \left(\exp\left(\eta A^{\frac{1}{p}}\right) - 1\right)\left(1 + \frac{p'}{\alpha(p-1)}\left(\gamma C_{11} + \|f\|_{L^1(\mathcal{W})}\right)\right)^{\frac{1}{p-1}} = \bar{\sigma}_\eta.$$

Observe that, by the definition of $\bar{\sigma}_\eta$, it results

$$\lim_{\eta \to +\infty} \bar{\sigma}_\eta = +\infty.$$

Moreover, since $meas\left\{|\widetilde{\varrho}(v_\varepsilon)| > \bar{\sigma}_\eta\right\} \leq meas\left\{|\widetilde{\varrho}(v_\varepsilon)| > \sigma_\eta(\varepsilon)\right\}$, and recalling (3.16), we affirm that $\widetilde{\varrho}(v_\varepsilon)$ is finite a.e. in $\mathcal{W}$. Therefore, since $\widetilde{\varrho}$ is C^1 non decreasing function, there exists $k_\eta > 0$ such that

$$meas\left\{|v_\varepsilon| > k_\eta\right\} \leq \frac{1}{\eta^p}.$$

This implies, since $\widetilde{\varrho}(\pm\infty) = \pm\infty$, that v_ε is also finite a.e. in $\mathcal{W}$. Which is equivalent to (3.14).

• *Step 2: $\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))$ and $\mathcal{T}_k(v_\varepsilon)$ are bounded in $W_0^{1,p}(\mathcal{W})$.*

Choosing $\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))$ as test function in (3.9), and following the same reasoning as before, we obtain

$$\int_{\mathcal{W}} \left|\nabla \mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))\right|^p \, dx \leq L + Mk \qquad \forall k > 0, \tag{3.19}$$

where

$$L = C_6 + C_8\left(\gamma C_{11} + \|f\|_{L^1(\mathcal{W})}\right) \text{ and } M = \frac{2p'}{\alpha}\left(\gamma C_{11} + \|f\|_{L^1(\mathcal{W})}\right).$$

Remark 3.5. *We note that, in this step, we have employed the measurable set*

$$\bar{Z}_{\eta,\varepsilon} = \{x \in \mathcal{W}, |\widetilde{\varrho}(v_\varepsilon(x))| \leq \bar{\sigma}_\eta\}, \tag{3.20}$$

instead of the set $Z_{\eta,\varepsilon}$.

The estimate (3.19), together with the lemma 2.4, enables us to deduce (3.10) and (3.11).

Taking $\mathcal{T}_k(v_\varepsilon)$ as test function in (3.9), we obtain

$$\int_{\mathcal{W}} b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\nabla\mathcal{T}_k(v_\varepsilon)\, dx + \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)\nabla\mathcal{T}_k(v_\varepsilon)\, dx$$
$$= \lambda \int_{\mathcal{W}} \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon}\mathcal{T}_k(v_\varepsilon)\, dx + \int_{\mathcal{W}} f^\varepsilon \mathcal{T}_k(v_\varepsilon)\, dx.$$

By (2.7), (2.10), and utilizing Hölder's, Young's, and Sobolev's inequalities, we get

$$\alpha \int_{\mathcal{W}} |\nabla\mathcal{T}_k(v_\varepsilon)|^p\, dx \leq (1+k)^{\theta(p-1)}|\mathcal{W}|^{\frac{p-1-\lambda}{p^*}}\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}\left(\int_{\mathcal{W}} |\nabla\mathcal{T}_k(v_\varepsilon)|^p\, dx\right)^{\frac{\lambda+1}{p}}$$
$$+ k(k+1)^{\theta(p-1)}\left(\gamma C_{11} + \|f\|_{L^1(\mathcal{W})}\right).$$

$$\begin{aligned}
\alpha I_k &\leq \int_{\mathcal{E}_{\eta,\varepsilon}} c_0(x)|\mathcal{T}_k(v_\varepsilon))|^\lambda|\nabla\mathcal{T}_k(v_\varepsilon))|\, dx + \int_{\mathcal{E}^c_{\eta,\varepsilon}} c_0(x)|\mathcal{T}_k(v_\varepsilon)|^\lambda|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + k\mathcal{R},\\
&\leq \int_{\mathcal{E}_{\eta,\varepsilon}} c_0(x)\,(1+|\mathcal{T}_k(v_\varepsilon)|)^{p-1}\,\frac{|\mathcal{T}_k(v_\varepsilon)|^\lambda}{(1+|\mathcal{T}_k(v_\varepsilon))|)^{p-1}}|\nabla\mathcal{T}_k(v_\varepsilon))|\, dx + \int_{\mathcal{E}^c_{\eta,\varepsilon}} c_0(x)|\mathcal{T}_k(v_\varepsilon))|^\lambda|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + k\mathcal{R},\\
&\leq \int_{\mathcal{E}_{\eta,\varepsilon}} c_0(x)\,(1+|\mathcal{T}_k(v_\varepsilon)|)^{p-1}\,|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + \int_{\mathcal{E}^c_{\eta,\varepsilon}} c_0(x)|\mathcal{T}_k(v_\varepsilon)|^\lambda|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + k\mathcal{R},\\
&\leq (1+k_\eta^\lambda)\int_{\mathcal{W}} c_0(x)|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + \int_{\mathcal{E}_{\eta,\varepsilon}} c_0(x)|\mathcal{T}_k(v_\varepsilon)|^{p-1}|\nabla\mathcal{T}_k(v_\varepsilon)|\, dx + k\mathcal{R}\\
&\leq \frac{(1+k_\eta)^{p'}\|c_0(x)\|_{L^{p'}(\mathcal{W})}}{p'\alpha^{\frac{p'}{p}}}(1+k)^\theta + \frac{\alpha}{p}I_k + (1+k)^{\theta(p-1)}\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{E}_{\eta,\varepsilon})}I_k + k\mathcal{R},
\end{aligned}$$

where, for every $k_\eta > 0$,

$$I_k = \int_{\mathcal{W}} \frac{|\nabla\mathcal{T}_k(v_\varepsilon)|^p}{(1+|\mathcal{T}_k(v_\varepsilon)|)^{\theta(p-1)}}\, dx, \text{ and } \mathcal{E}_{\eta,\varepsilon} = \{x \in \mathcal{W}, |v_\varepsilon(x)| > k_\eta\}.$$

Now, we choose k_η (for fixed $\eta = \bar{\eta}$) such that

$$\frac{p'}{\alpha}C_{12}\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{E}_{\bar{\eta},\varepsilon})}(1+k)^{\theta(p-1)} \leq \frac{1}{2}.$$

Thus, for every $k > 0$, we obtain

$$I_k \leq C_\eta(1+k)^\theta + C_{12}k,$$

with

$$C_\eta = \frac{2(1+k_\eta)^\alpha\|c_0(x)\|_{L^{p'}(\mathcal{W})}}{\alpha^{p'}}.$$

which implies that

$$\int_{\mathcal{W}} |\nabla \mathcal{T}_k(v_\varepsilon)|^p \, dx \leq (1+k)^{\theta(p-1)} I_k, \\ \leq C_\eta (1+k)^{\theta p} + C_{12} k (1+k)^{\theta(p-1)}.$$

Noting that, for a suitable constant C depending on p and θ, we have

$$k(1+k)^{\theta(p-1)} \leq C\left(1 + k^{\theta(p-1)+1}\right) \qquad \forall\, k > 0.$$

This implies that

$$\int_{\mathcal{W}} |\nabla \mathcal{T}_k(v_\varepsilon)|^p \, dx \leq C_{13} + C_\eta (1+k)^{\theta p} + C_{14} k^{\theta(p-1)+1},$$

using Young inequality with the exponents

$$\kappa = \frac{\theta(p-1)+1}{\theta p} \quad \text{and} \quad \kappa' = \frac{\kappa}{\kappa - 1},$$

we obtain

$$\int_{\mathcal{W}} |\nabla \mathcal{T}_k(v_\varepsilon)|^p \, dx \leq L + k^{\theta(p-1)+1}, \quad \forall k > 0, \tag{3.21}$$

where

$$Ł = C_\eta + C_{12} C + \frac{C_\eta^{\kappa'}}{\kappa'} \text{ and } M = C_{12} C + \frac{1}{\kappa}.$$

Bearing in mind that $\theta < 1$ and the estimate (3.21), we can apply Lemma 2.4 to extrapolate that the estimates (3.12) and (3.13) are hold. So, the proof of the proposition 3.2 is now complete. □

Based on the arguments in [11, 18], it can be inferred from estimates (3.19) and (3.21) that, for a subsequence which is still indexed by ε, the following convergences hold:

$$v_\varepsilon \to v \text{ a.e. in } \mathcal{W}, \tag{3.22}$$

$$\widetilde{\varrho}(v_\varepsilon) \to \widetilde{\varrho}(v) \text{ a.e. in } \mathcal{W}, \tag{3.23}$$

$$\mathcal{T}_k(v_\varepsilon) \rightharpoonup \mathcal{T}_k(v) \text{ weakly in } W_0^{1,p}(\mathcal{W}), \tag{3.24}$$

$$\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon)) \rightharpoonup \mathcal{T}_k(\widetilde{\varrho}(v)) \text{ weakly in } W_0^{1,p}(\mathcal{W}), \tag{3.25}$$

$$b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\, \chi_{\{|u_\varepsilon| \leq k\}} \rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(\mathcal{W}))^N, \tag{3.26}$$

• **Third step: Energy formula**

Now we look for the following energy estimate of the approximating solutions v_ε

$$\lim_{n \to +\infty} \limsup_{\varepsilon \to 0} \frac{1}{n} \int_{\{|\widetilde{\varrho}(v_\varepsilon)| \leq n\}} \varrho(v_\varepsilon) b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) \nabla v_\varepsilon \, dx = 0. \tag{3.27}$$

Taking $v = \frac{1}{n}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))$ as test function in (3.9) yields that

$$\frac{1}{n} \int_{\mathcal{W}} (b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) + \mathcal{B}_\varepsilon(x, v_\varepsilon)) \nabla \mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon)) \, dx = \frac{\gamma}{n} \int_{\mathcal{W}} \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^{s-1} \mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon} \mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon)) \, dx + \frac{1}{n} \int_{\mathcal{W}} f^\varepsilon \mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon)) \, dx.$$

Utilizing (1.7), (2.7), (2.10), as well as Hölder's, Young's, and Sobolev's inequalities, we obtain

$$\begin{aligned}
\frac{\alpha}{n}\int_{\mathcal{W}}|\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))|^p\,dx &\le \frac{1}{n}\int_{\mathcal{W}}c_0(x)|v_\varepsilon|^\lambda|\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))|\,dx+\frac{\gamma}{n}\int_{\mathcal{W}}\frac{|v_\varepsilon|^s}{|x|^p}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx+\frac{1}{n}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx\\
&\le \frac{1}{n}\int_{\mathcal{W}}c_0(x)\frac{|v_\varepsilon|^\lambda}{(1+|\widetilde{\varrho}(v_\varepsilon)|)^{p-1}}(1+|\widetilde{\varrho}(v_\varepsilon)|)^{p-1}|\nabla\widetilde{\varrho}(v_\varepsilon)|\,dx+\frac{\gamma}{n}\int_{\mathcal{W}}\frac{|v_\varepsilon|^s}{|x|^p}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx+\frac{1}{n}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx\\
&\le \frac{C_{17}}{n}+\frac{\alpha}{np}\int_{\mathcal{W}}|\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))|^p\,dx+\frac{1}{n}2^{p-2}\widetilde{C}\mathcal{S}^{p-1}\left\|c_0(x)\right\|_{L^{\frac{N}{p-1}}(\bar{Z}^c_{\eta,\varepsilon})}\int_{\mathcal{W}}|\nabla\mathcal{T}_k(\widetilde{\varrho}(v_\varepsilon))|^p\,dx\\
&+\frac{\gamma}{n}\int_{\mathcal{W}}\frac{|v_\varepsilon|^s}{|x|^p}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx+\frac{1}{n}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx.
\end{aligned}$$

with $\bar{Z}_{\eta,\varepsilon}$ is the measurable set defined by (3.20).

By choosing $\eta=\overline{\eta}$ (since $\widetilde{\varrho}(v_\varepsilon)$ is finite a.e. in $\mathcal{W}$), such that

$$\frac{2^{p-2}p'}{\alpha}\widetilde{C}\mathcal{S}^{p-1}\,\|c_0(x)\|_{L^{\frac{N}{p-1}}(\bar{Z}^c_{\bar{\eta},\varepsilon})}\le\frac{1}{2},$$

we obtain

$$\frac{1}{n}\int_{\mathcal{W}}|\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))|^p\,dx\le\frac{p'C_{17}}{n\alpha}+\frac{\gamma p'}{n\alpha}\int_{\mathcal{W}}\frac{|v_\varepsilon|^s}{|x|^p}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx+\frac{p'}{n\alpha}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx.$$

Now, we claim that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to0}\frac{p'}{n\alpha}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx=0,\tag{3.28}$$

and

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to0}\frac{\gamma p'}{n\alpha}\int_{\mathcal{W}}\frac{|v_\varepsilon|^s}{|x|^p}\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx=0.\tag{3.29}$$

Once this claim is proved and by using (2.7) , it follows that (3.27) is hold.

For any $n\in\mathbb{N}^*$ and in view of (3.22) we have

$$\mathcal{T}_n(v_\varepsilon)\rightharpoonup\mathcal{T}_n(v),\quad\text{weak-* in } L^\infty(\mathcal{W}),\tag{3.30}$$

which implies, combining with (3.4), that

$$\limsup_{\varepsilon\to0}\frac{1}{n}\int_{\mathcal{W}}f^\varepsilon\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx=\frac{1}{n}\int_{\mathcal{W}}f\mathcal{T}_n(\widetilde{\varrho}(v))\,dx.$$

Moreover, due to (3.16), (3.23) and Fatou Lemma, we can easily get that

$$meas\left\{|\widetilde{\varrho}(v)|>\bar{\sigma}_\eta\right\}\le\liminf_{\varepsilon\to0}meas\left\{|\widetilde{\varrho}(v_\varepsilon)|>\bar{\sigma}_\eta\right\}\le\frac{1}{\eta^p},$$

thus, it follows that $\widetilde{\varrho}(v)$ is finite a.e. in $\mathcal{W}$. In addition, the sequence $\left\{\frac{\mathcal{T}_n(\widetilde{\varrho}(v))}{n}\right\}$ converges to 0 a.e. in $\mathcal{W}$ and bounded by 1. Hence, applying Lebesgue's dominated convergence theorem leads to (3.28).

Defining E as a measurable subset of $\mathcal{W}$ containing 0 and such that the $meas(E)$ is small enough, and according to (3.13) it yields that

$$\begin{aligned}\int_E \frac{|v_\varepsilon|^s}{|x|^p}dx &\le \left(\int_{\mathcal{W}} |v_\varepsilon|^{(1-\theta)(p-1)}\,dx\right)^{\frac{s}{(1-\theta)(p-1)}} \left(\int_E \frac{dx}{|x|^{\frac{p(1-\theta)(p-1)}{(1-\theta)(p-1)-s}}}\right)^{\frac{(1-\theta)(p-1)-s}{(1-\theta)(p-1)}},\\ &\le \left\||v_\varepsilon|^{(1-\theta)(p-1)}\right\|_{L^1(\mathcal{W})}^{\frac{s}{(1-\theta)(p-1)}} \left(\int_E \frac{dx}{|x|^{\frac{p(1-\theta)(p-1)}{(1-\theta)(p-1)-s}}}\right)^{\frac{(1-\theta)(p-1)-s}{(1-\theta)(p-1)}},\\ &\le C_{18}\left(\int_E \frac{dx}{|x|^{\frac{p(1-\theta)(p-1)}{(1-\theta)(p-1)-s}}}\right)^{\frac{(1-\theta)(p-1)-s}{(1-\theta)(p-1)}}.\end{aligned}$$

Then, for every $\delta > 0$ and since $s < \dfrac{p(1-\theta)(p-1)}{p^*}$, the absolute continuity of the Lebesgue integral allow us to conclude that

$$\left(\int_E \frac{dx}{|x|^{\frac{p(1-\theta)(p-1)}{(1-\theta)(p-1)-s}}}\right)^{\frac{(1-\theta)(p-1)-s}{(1-\theta)(p-1)}} \le \delta,$$

therefore, we get

$$\int_E \frac{|v_\varepsilon|^s}{|x|^p}dx \le C_{18}\delta,$$

then, the sequence $\left\{\dfrac{|v_\varepsilon|^s}{|x|^p}\right\}$ is equi-integrable.

This implies, by applying Vitali's theorem, that

$$\frac{|v_\varepsilon|^s}{|x|^p} \to \frac{|v|^s}{|x|^p} \quad \text{strongly in } L^1(\mathcal{W}), \tag{3.31}$$

and thus (3.29) holds, since the sequence $\left\{\dfrac{\mathcal{T}_n(\widetilde{\varrho}(v))}{n}\right\}$ converges to 0 weak-* in $L^\infty(\mathcal{W})$. As a conclusion the energy formula (3.27) is proved.

• **Fourth step: The a.e. convergence of the sequence** ∇v_ε

The main point of this step is proving the a.e. convergence of ∇v_ε in $\mathcal{W}$ and this is will be done by using an arguments similar to these used in [8]. Note that, here we use a slightly different techniques due to the existence of the convection term $-\operatorname{div}(\mathcal{B}(x,v))$ and the term of Hardy potential $\gamma\dfrac{|v|^{s-1}v}{|x|^p}$ in our operator.

Lemma 3.6. *Assuming v_ε is a sequence of solutions to the problems* (3.8) *with f^ε strongly converging to some f in $L^1(\mathcal{W})$. Suppose that:*

(i) $\mathcal{T}_k(v_\varepsilon)$ belongs to $W_0^{1,p}(\mathcal{W})$ for every $k > 0$,

(ii) v_ε converges almost everywhere in $\mathcal{W}$ to some measurable function v which is finite almost everywhere and $\mathcal{T}_k(v)$ belongs to $W_0^{1,p}(\mathcal{W})$ for every $k > 0$,

(iii) $|v_\varepsilon|^{(1-\theta)(p-1)}$ is bounded in $L^{\frac{N}{N-p},\infty}(\mathcal{W})$, and $|v|^{(1-\theta)(p-1)}$ belongs to $L^{\frac{N}{N-p},\infty}(\mathcal{W})$, and

(iv) $|\nabla v_\varepsilon|^{(1-\theta)(p-1)}$ is bounded in $L^{\frac{N}{N-1-\theta(p-1)},\infty}(\mathcal{W})$, and $|\nabla v|^{(1-\theta)(p-1)}$ belongs to $L^{\frac{N}{N-1-\theta(p-1)},\infty}(\mathcal{W})$.

Then, up to a subsequence, ∇v_ε converges almost everywhere in $\mathcal{W}$ to ∇v, the weak gradient of v.

Proof. Let $\sigma > 1$ and $\tau > 1$ such that

$$0 < \sigma\tau < \frac{N(1-\theta)(p-1)}{p(N-1-\theta(p-1))}. \tag{3.32}$$

Let us consider for any $0 < j < k$ the sets

$$C_k = \{x \in \mathcal{W} : |v(x)| \leq k\}, \quad D_{\varepsilon,k,j} = \{x \in \mathcal{W} : |v_\varepsilon(x) - \mathcal{T}_k(v(x))| \leq j\}.$$

And we define

$$\begin{aligned} I(\varepsilon) &= \int_{\mathcal{W}} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla v)] \nabla (v_\varepsilon - v)\}^\sigma, \\ &= \int_{C_k^c} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla v)] \nabla (v_\varepsilon - v)\}^\sigma + \int_{C_k} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla v)] \nabla (v_\varepsilon - v)\}^\sigma, \\ &= I_1(\varepsilon, k) + I_2(\varepsilon, k). \end{aligned}$$

By Hölder's inequality and the growth condition (2.8) we get

$$I_1(\varepsilon, k) \leq C_8 \left(\int_{C_k^c} 1 + |\nabla v_\varepsilon|^{\sigma p \tau} + |\nabla v|^{\sigma p \tau} \, dx \right)^{\frac{1}{\tau}} meas\{C_k^c\}^{1-\frac{1}{\tau}},$$

We now choose σ and r such that (3.32) is hold, therefore putting together (*iv*) and the inclusion (2.2) (with $r = \frac{N(1-\theta)(p-1)}{N-1-\theta(p-1)}$ and $q = \sigma p \tau$), we deduce that

$$I_1(\varepsilon, k) \leq C_8 meas\{C_k^c\}^{1-\frac{1}{\tau}}.$$

By (iii), and by the choice of σ , we thus have

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} I_1(\varepsilon, k) = 0. \tag{3.33}$$

For j fixed, we have

$$\begin{aligned} I_2(\varepsilon, k) &\leq \int_{\mathcal{W}} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla \mathcal{T}_k(v))] \nabla (v_\varepsilon - \mathcal{T}_k(v))\}^\sigma, \\ &\leq \int_{D^c_{\varepsilon,k,j}} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla \mathcal{T}_k(v))] \nabla (v_\varepsilon - \mathcal{T}_k(v))\}^\sigma \\ &\quad + \int_{D_{\varepsilon,k,j}} \{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla \mathcal{T}_k(v))] \nabla (v_\varepsilon - \mathcal{T}_k(v))\}^\sigma + I_3(\varepsilon, k) + I_4(\varepsilon, k), \end{aligned}$$

thanks to (iii), we can affirm that

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} meas\{D^c_{\varepsilon,k,j}\} \leq \lim_{k\to+\infty} meas\{|v - \mathcal{T}_k(v)| > j\} = 0,$$

thus, reasoning as for $I_1(\varepsilon, k)$, one has

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} I_3(\varepsilon, k) = 0. \tag{3.34}$$

For $I_4(\varepsilon, k)$, one can rewrite it as

$$I_4(\varepsilon, k) = \int_{\mathcal{W}} \left\{[b_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) - b_\varepsilon(x, v_\varepsilon, \nabla \mathcal{T}_k(v))] \nabla T_j (v_\varepsilon - \mathcal{T}_k(v))\right\}^\sigma.$$

By the Hölder's inequality (with exponents $\frac{1}{\sigma}$ and $\frac{1}{1-\sigma}$), we have

$$I_4(\varepsilon,k) \leq \left(\int_{\mathcal{W}} [b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon) - b_\varepsilon(x,v_\varepsilon,\nabla \mathcal{T}_k(v))]\, \nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\ dx\right)^\sigma meas(\mathcal{W})^{1-\sigma}.$$

To control the integral on the right-hand side of the previous inequality, we employ $T_j(v_\varepsilon - \mathcal{T}_k(v))$ as the test function in (3.9). This leads to

$$\begin{aligned}&\int_{\mathcal{W}} b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx + \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx\\ &= \gamma\int_{\mathcal{W}} \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v)}{|x|^p+\varepsilon} T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx + \int_{\mathcal{W}} f^\varepsilon T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx.\end{aligned}$$

After some simple calculations, we derive that

$$\begin{aligned}\int_{\mathcal{W}} (b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon) - b_\varepsilon(x,v_\varepsilon,\nabla\mathcal{T}_k(v)))\,\nabla T_j(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,dx &\leq j\gamma\int_{\mathcal{W}} \frac{|v_\varepsilon|^s}{|x|^p}\,dx + j\int_{\mathcal{W}} |f^\varepsilon|\,dx - \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx\\ &\quad - \int_{\mathcal{W}} a_\varepsilon(x,v_\varepsilon,\nabla\mathcal{T}_k(v))\nabla T_j(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,dx.\end{aligned}$$

Let us pass to the limit, as j and ε tend to 0, in all the term of the right hand side of the above inequality. First of all, thanks to the properties of f^ε we have

$$\lim_{j\to 0}\lim_{\varepsilon\to 0} j\int_{\mathcal{W}} |f^\varepsilon|\,dx = \lim_{j\to 0} j\int_{\mathcal{W}} |f|\,dx = 0.$$

From (3.31), we conclude that

$$\lim_{j\to 0}\lim_{\varepsilon\to 0} j\gamma\int_{\mathcal{W}} \frac{|v_\varepsilon|^s}{|x|^p}\,dx = \lim_{j\to 0} j\gamma\int_{\mathcal{W}} \frac{|v|^s}{|x|^p}\,dx = 0.$$

Noting that

$$\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx = \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,T_{j+k}(v_\varepsilon))\nabla T_j(T_{j+k}(v_\varepsilon) - \mathcal{T}_k(v))\,dx,$$

therefore, by the growth condition (2.10), we obtain that

$$|\mathcal{B}_\varepsilon(x,T_{j+k}(v_\varepsilon))| \leq (j+k)^\lambda c_0(x),$$

moreover, by Lebesgue's convergence theorem, we get

$$\mathcal{B}_\varepsilon(x,T_{j+k}(v_\varepsilon)) \to \mathcal{B}(x,T_{j+k}(v)) \quad \text{strongly in} \quad (L^{p'}(\mathcal{W}))^N,$$

thus, for $\dfrac{1}{\varepsilon} > j+k$, we have

$$\lim_{\varepsilon\to 0}\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx = \int_{\mathcal{W}} \mathcal{B}(x,T_{j+k}(v))\nabla T_j(T_{j+k}(v) - \mathcal{T}_k(v))\,dx,$$

since, for any $j \leq 1$, we have

$$\begin{cases}\nabla T_j(T_{k+j}(v) - \mathcal{T}_k(v)) \to 0 \quad \text{a.e. as} \;\; j\to 0\\ |\nabla T_j(T_{k+j}(v) - \mathcal{T}_k(v))| \leq |\nabla T_1(T_{k+1}(v) - \mathcal{T}_k(v))| \in L^p(\mathcal{W})\end{cases}$$

then, by Lebesgue's convergence theorem it follows that

$$\lim_{j\to 0}\lim_{\varepsilon\to 0}\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x,v_\varepsilon)\nabla T_j(v_\varepsilon - \mathcal{T}_k(v))\,dx = 0.$$

Using (2.8) and (3.22), we prove that

$$b_\varepsilon(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v)) \to b(x,\mathcal{T}_k(v),\nabla\mathcal{T}_k(v)) \quad \text{strongly in } (L^{p'}(\mathcal{W}))^N,$$

putting together the last convergence, (i) and (ii) we arrive that

$$\lim_{j\to 0}\lim_{\varepsilon\to 0}\int_{\mathcal{W}} b_\varepsilon(x,v_\varepsilon,\nabla\mathcal{T}_k(v))\nabla T_j(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v))\,dx = 0.$$

Hence, we obtain

$$\lim_{k\to+\infty}\limsup_{\varepsilon\to 0} I_4(\varepsilon,k) = 0. \tag{3.35}$$

Combining (3.33), (3.34) and (3.35), we deduce

$$\lim_{\varepsilon\to 0} I(\varepsilon) \le 0.$$

Therefore, by (2.9), we deduce that

$$\int_{\mathcal{W}} \{[b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon) - b_\varepsilon(x,v_\varepsilon,\nabla v)]\,\nabla(v_\varepsilon - v)\}^\sigma \to 0$$

that is

$$\nabla v_\varepsilon(x) \to \nabla v(x) \quad \text{a.e. in } \mathcal{W}. \tag{3.36}$$

□

• **Fifth Step : Passing to the limit**

In this step we prove that v is a renormalized solutions of (2.6). Observe that, due to (3.14), (3.23) and Fatou Lemma, we can easily get that

$$meas\left\{|v| > k_\eta\right\} \le \liminf_{\varepsilon\to 0} meas\left\{|v_\varepsilon| > k_\eta\right\} \le \frac{1}{\eta^p},$$

thus, v is finite a.e. in $\mathcal{W}$ and (2.12) is proved. Moreover, thanks to (3.25), we infer that (2.13) is hold.

Combining proposition 3.2 and lemma 3.6, we get

$$b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon) \to b(x,v,\nabla v) \quad \text{a.e. in } \mathcal{W},$$

and under the growth assumption (2.8), we deduce that

$$b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon) \rightharpoonup b(x,v,\nabla v) \quad \text{weakly in } (L^{p'}(\mathcal{W}))^N,$$

and for any $k > 0$, we have

$$b_\varepsilon(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v_\varepsilon)) \rightharpoonup b(x,\mathcal{T}_k(v),\nabla\mathcal{T}_k(v)) \quad \text{weakly in } (L^{p'}(\mathcal{W}))^N.$$

Moreover, for any $k_n \in (0,\frac{1}{\varepsilon})$(as in the remark 2.6), we can write

$$\begin{aligned}\frac{1}{n}\int_{\{|\widetilde{\varrho}(v_\varepsilon)|\le n\}} \varrho(v_\varepsilon) b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\,dx &= \frac{1}{n}\int_{\mathcal{W}} b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx,\\ &= \frac{1}{n}\int_{\mathcal{W}} b_\varepsilon(x,T_{k_n}(v_\varepsilon),\nabla T_{k_n}(v_\varepsilon))\nabla\mathcal{T}_n(\widetilde{\varrho}(v_\varepsilon))\,dx,\end{aligned}$$

thus, from (3.19) and (3.27), it follows that

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\mathcal{W}} b(x,T_{k_n}(v),\nabla T_{k_n}(v))\nabla\mathcal{T}_n(\widetilde{\varrho}(v)\ dx=0,$$

which gives (2.14).

Given $k>0$ and $\forall n\in\mathbb{N}^*$, denote by h_n, the truncation function defined as

$$h_n(t)=1-\frac{|\mathcal{T}_{2n}(t)-\mathcal{T}_n(t)|}{n},\quad \forall t\in\mathbb{R}.$$

Now we claim that (3.9) holds true. Let $h\in W^{1,\infty}(\mathbb{R})$ such that supp(h) is compact and let $\varphi\in D(\mathcal{W})$. For any $n\in\mathbb{N}^*$ the function $h_n(\widetilde{\varrho}(v_\varepsilon))\,h(\widetilde{\varrho}(v))\varphi$ belongs to $W_0^{1,p}(\mathcal{W})\cap L^\infty(\mathcal{W})$, and then it is an admissible test function in (3.9). It yields that

$$\begin{aligned}&\int_{\mathcal{W}} h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\varrho(v_\varepsilon)\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\ dx+\int_{\mathcal{W}} h_n(\widetilde{\varrho}(v_\varepsilon))\,a_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla[h(\widetilde{\varrho}(v))\varphi]\ dx\\&+\int_{\mathcal{W}}\mathcal{B}_\varepsilon(x,v_\varepsilon)\varrho(v_\varepsilon)\nabla v_\varepsilon h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\ dx+\int_{\mathcal{W}}\mathcal{B}_\varepsilon(x,v_\varepsilon)h_n(\widetilde{\varrho}(v_\varepsilon))\nabla[h(\widetilde{\varrho}(v))\varphi]\ dx\\&=\gamma\int_{\mathcal{W}}\frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}h_n(\widetilde{\varrho}(v_\varepsilon))\,h(\widetilde{\varrho}(v))\varphi\ dx+\int_{\mathcal{W}} f^\varepsilon h_n(\widetilde{\varrho}(v_\varepsilon))\,h(\widetilde{\varrho}(v))\varphi\ dx\end{aligned}\tag{3.37}$$

Let us pass to limit in (3.37) as ε goes to zero and as n goes to $+\infty$.

Recalling the definition of function h_n, we have

$$\begin{aligned}\int_{\mathcal{W}} h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\varrho(v_\varepsilon)\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\ dx&=\frac{1}{n}\int_{\{n<|\widetilde{\varrho}(v_\varepsilon)|<2n\}} sign(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\varrho(v_\varepsilon)\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\ dx,\\&\le\frac{1}{n}\int_{\{n<|\widetilde{\varrho}(v_\varepsilon)|<2n\}} h(\widetilde{\varrho}(v))\varphi\varrho(v_\varepsilon)\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\ dx,\end{aligned}$$

the bounded character of the term $h(\widetilde{\varrho}(v))\varphi$ and (3.27), allow us to conclude that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to0}\int_{\mathcal{W}} h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\varrho(v_\varepsilon)\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla v_\varepsilon\ dx=0.\tag{3.38}$$

Under the growth assumption (2.8) and according to the lemma 3.6, we have

$$h_n(\widetilde{\varrho}(v_\varepsilon))\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\rightharpoonup h_n(\widetilde{\varrho}(v))\,b(x,v,\nabla v)\quad\text{weakly in }(L^{p'}(\mathcal{W}))^N,$$

In addition of the fact that the function $h(\widetilde{\varrho}(v))\varphi$ is in $W_0^{1,p}(\mathcal{W})$, we get

$$\limsup_{\varepsilon\to0}\int_{\mathcal{W}} h_n(\widetilde{\varrho}(v_\varepsilon))\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla[h(\widetilde{\varrho}(v))\varphi]\ dx=\int_{\mathcal{W}} h_n(\widetilde{\varrho}(v))\,b(x,v,\nabla v)\nabla[h(\widetilde{\varrho}(v))\varphi]\ dx,$$

moreover, we have that

$$\begin{aligned}&h_n(\widetilde{\varrho}(v))\to1\quad\text{a.e. in }\mathcal{W},\\&b(x,v,\nabla v)\nabla[h(\widetilde{\varrho}(v))\varphi]\in L^1(\mathcal{W}),\\&\nabla[h(\widetilde{\varrho}(v))\varphi]=\varphi\varrho(v)h'(\widetilde{\varrho}(v))\nabla v+h(\widetilde{\varrho}(v))\nabla\varphi,\end{aligned}$$

so that, Lebesgue's convergence theorem allows us to derive that

$$\begin{aligned}\lim_{n\to+\infty}\limsup_{\varepsilon\to0}\int_{\mathcal{W}} h_n(\widetilde{\varrho}(v_\varepsilon))\,b_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\nabla[h(\widetilde{\varrho}(v))\varphi]\ dx&=\int_{\mathcal{W}} b(x,v,\nabla v)\varphi\varrho(v)h'(\widetilde{\varrho}(v))\nabla v\ dx\\&+\int_{\mathcal{W}} b(x,v,\nabla v)h(\widetilde{\varrho}(v))\nabla\varphi\ dx,\end{aligned}\tag{3.39}$$

by means of the definition of h_n, the growth assumption (2.10), Hölder's, Sobolev's inequalities and bounded character of $h(\widetilde{\varrho}(v))\varphi$, we deduce that

$$\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)\varrho(v_\varepsilon)\nabla v_\varepsilon h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\, dx \le \frac{1}{n}\|h\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\mathcal{W})}\int_{\{n<|\widetilde{\varrho}(v_\varepsilon)|<2n\}} |\mathcal{B}_\varepsilon(x, v_\varepsilon)||\varrho(v_\varepsilon)||\nabla v_\varepsilon|\, dx,$$
$$\le \frac{C_{\widetilde{\varrho}}}{n}\|h\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\mathcal{W})}\|c_0(x)\|_{L^{p'}(\mathcal{W})}\|\nabla\mathcal{T}_{2n}(\widetilde{\varrho}(v_\varepsilon))\|_{L^p(\mathcal{W})}$$
$$+ \frac{C_{\widetilde{\varrho}}\mathcal{S}^{p-1}}{n}\|h\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\mathcal{W})}\|c_0(x)\|_{L^{\frac{N}{p-1}}(\mathcal{W})}\left\|\nabla\mathcal{T}_{2n}(\widetilde{\varrho}(v_\varepsilon))\right\|^p_{L^p(\mathcal{W})},$$

where $C_{\widetilde{\varrho}}$ is a positive constant which does not depends on ε.

From the energy formula (3.27), we prove that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)\varrho(v_\varepsilon)\nabla v_\varepsilon h_n'(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\, dx = 0. \tag{3.40}$$

Since $supp(h_n) = [-2n, 2n]$, then there exists $k_{2n} \in (0, \frac{1}{\varepsilon})$ such that

$$\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, v_\varepsilon)h_n(\widetilde{\varrho}(v_\varepsilon))\nabla[h(\widetilde{\varrho}(v))\varphi]dx = \int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, \mathcal{T}_{k_{2n}}(v_\varepsilon))h_n(\widetilde{\varrho}(v_\varepsilon))\nabla[h(\widetilde{\varrho}(v))\varphi]dx.$$

As a consequence of (3.16), (3.22) and (3.23) we obtain that

$$\mathcal{B}_\varepsilon(x, \mathcal{T}_{k_{2n}}(v_\varepsilon))h_n(\widetilde{\varrho}(v_\varepsilon)) \to \mathcal{B}(x, v)h_n(\widetilde{\varrho}(v)) \quad \text{a.e. in } \mathcal{W},$$

and by the growth condition (2.10) we deduce that

$$\left|\mathcal{B}_\varepsilon(x, \mathcal{T}_{k_{2n}}(v_\varepsilon))h_n(\widetilde{\varrho}(v_\varepsilon))\right| \le (k_{2n})^\lambda c_0(x)\|h_n\|_{L^\infty(\mathbb{R})} \in L^{p'}(\mathcal{W}),$$

therefore, Lebesgue's convergence theorem allow us to conclude that

$$\mathcal{B}_\varepsilon(x, \mathcal{T}_{k_{2n}}(v_\varepsilon))h_n(\widetilde{\varrho}(v_\varepsilon)) \to \mathcal{B}(x, v)h_n(\widetilde{\varrho}(v)) \quad \text{strongly in } L^{p'}(\mathcal{W}),$$

hence for n large enough such that $k_{2n} \ge supp(h)$, we have

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_{\mathcal{W}} \mathcal{B}_\varepsilon(x, \mathcal{T}_{k_{2n}}(v_\varepsilon))h_n(\widetilde{\varrho}(v_\varepsilon))\nabla[h(\widetilde{\varrho}(v))\varphi]\, dx = \lim_{n\to+\infty}\int_{\mathcal{W}} \mathcal{B}(x, \mathcal{T}_{k_{2n}}(v))h_n(\widetilde{\varrho}(v))\nabla[h(\widetilde{\varrho}(v))\varphi]\, dx,$$
$$= \int_{\mathcal{W}} \mathcal{B}(x, v)h(\widetilde{\varrho}(v))\nabla\varphi\, dx + \int_{\mathcal{W}} \mathcal{B}(x, v)\varphi\varrho(v)h'(\widetilde{\varrho}(v))\nabla v\, dx. \tag{3.41}$$

By combining (3.18), (3.22), (3.23) and the fact that h_n is bounded by 1, we can apply Lebesgue convergence theorem to establish that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\gamma\int_{\mathcal{W}} \frac{\left|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)\right|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}h_n(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\, dx = \gamma\int_{\mathcal{W}} \frac{|v|^{s-1}v}{|x|^p}h(\widetilde{\varrho}(v))\varphi\, dx. \tag{3.42}$$

At last (3.4), (3.23) and the behavior of the sequence h_n together with Lebesgue convergence theorem lead to

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_{\mathcal{W}} f^\varepsilon h_n(\widetilde{\varrho}(v_\varepsilon))h(\widetilde{\varrho}(v))\varphi\, dx = \int_{\mathcal{W}} fh(\widetilde{\varrho}(v))\varphi\, dx \tag{3.43}$$

Finally, thanks to (3.38)-(3.43), we deduce that for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathcal{W}} b(x,v,\nabla v)\varrho(v)\nabla v h'(\widetilde{\varrho}(v))v\,dx + \int_{\mathcal{W}} b(x,v,\nabla v)\nabla v h(\widetilde{\varrho}(v))\,dx \\
&\quad + \int_{\mathcal{W}} \mathcal{B}(x,v)\varrho(v)\nabla v h'(\widetilde{\varrho}(v))v\,dx + \int_{\mathcal{W}} \mathcal{B}(x,v)\nabla v h(\widetilde{\varrho}(v))\,dx \\
&\quad = \gamma \int_{\mathcal{W}} \frac{|u|^{s-1}u}{|x|^p} h(\widetilde{\varrho}(v))v\,dx + \int_{\mathcal{W}} f h(\widetilde{\varrho}(v))v\,dx
\end{aligned}$$

for every $v \in W_0^{1,p}(\mathcal{W}) \cap L^\infty(\mathcal{W})$.

At least the limit v satisfies (2.12), (2.13), (2.14) and (2.15), which asserts that v is a renormalized solution of the problem 2.6, then the proof of Theorem 3.1 is now complete. □

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Declarations

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