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Spectral mapping theorem and the Taylor spectrum

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Abstract. In [6] Chō and Tanahashi showed new spectral mapping theorem of the Taylor spectrum for doubly commuting pairs of p-hyponormal operators and log-hyponormal operators. In this paper, we will show that same spectral mapping theorem holds for commuting n-tuples.

1. Introduction and preparation

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\lambda \in \mathbb{C}$ belong to the residual spectrum $\sigma_r(T)$ of T if there exists $c > 0$ such that $\|(T-\lambda)x\| \geq c\|x\|$ for all $x \in \mathcal{H}$ and $(T-\lambda)\mathcal{H} \neq \mathcal{H}$. It is easy to see that if $\lambda \in \sigma_r(T)$, then $0 \in \sigma_p((T-\lambda)^*)$. It is well known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$. For an Hermitian operator $A \in B(\mathcal{H})$, we denote $A \geq 0$ if $(Ax, x) \geq 0$ for every $x \in \mathcal{H}$ and $A \geq B$ if $A - B \geq 0$. When $(Ax, x) > 0$ for every non-zero $x \in \mathcal{H}$, then we denote $T > 0$. For a given $p > 0$, $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. When $p = 1/2$, T is said to be semi-hyponormal. It means that T is semi-hyponormal if and only if $|T| \geq |T^*|$. T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$. It is well known that if T is invertible p-hyponormal for some $p > 0$, then T is log-hyponormal. If $\mathcal{M}$ is a reducing subspace for a p-hyponormal or log-hyponormal operator T, then so is $T|_{\mathcal{M}}$, respectively.

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we explain the Taylor spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ shortly. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, ..., e_n$. Let $E^n_k(\mathcal{H}) = \mathcal{H} \otimes E^n_k$. Define $D^n_k : E^n_k(\mathcal{H}) \longrightarrow E^n_{k-1}(\mathcal{H})$ by

$$D^n_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) := \sum_{i=1}^{k} (-1)^{i-1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

where $\check{e}_{j_i}$ means deletion. We denote D^n_k by D_k simply. We think Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ as follows:

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$$E(\mathbf{T}) \;:\; 0 \longrightarrow E_n^n(\mathcal{H}) \xrightarrow{D_n} E_{n-1}^n(\mathcal{H}) \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_2} E_1^n(\mathcal{H}) \xrightarrow{D_1} E_0^n(\mathcal{H}) \longrightarrow 0.$$

Since $E_k^n(\mathcal{H}) \cong \overbrace{\mathcal{H}\oplus\cdots\oplus\mathcal{H}}^{\binom{n}{k}=\frac{n!}{(n-k)!k!}}$ $(k = 1, ..., n)$, we set $E_k^n(\mathcal{H}) = \overbrace{\mathcal{H}\oplus\cdots\oplus\mathcal{H}}^{\binom{n}{k}}$ $(k = 1, ..., n)$.

Definition 1.1. *A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is said to be singular if and only if the Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ is not exact.*

Definition 1.2. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$ belongs to the Taylor spectrum $\sigma_T(\mathbf{T})$ of $\mathbf{T}$ if $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$ is singular.*

About the definition of the Taylor spectrum, see details J. L. Taylor [9] and [10]. In [7], Curto proved the following proposition.

Proposition 1.3. (Proposition 3.4, Curto [7]) *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $0 = (0, ..., 0) \notin \sigma_T(\mathbf{T})$ if and only if $D_k^* D_k + D_{k+1} D_{k+1}^*$ is invertible for all k.*

For a commuting pair $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, it is well known that, for polynomials $f_1, ..., f_m$ of n-variables, if $f(z_1, ..., z_n) = (f_1(z_1, ..., z_n), ..., f_m(z_1, ..., z_n))$, then it holds

$$\sigma_T(f(T_1, ..., T_n)) = f(\sigma_T(T_1, ..., T_n)),$$

where $\sigma_T(T_1, ..., T_n)$ is the Taylor spectrum of $\mathbf{T} = (T_1, ..., T_n)$. See Theorem 4.7 in [10].

In the paper [6], Chō and Tanahashi showed another spectral mapping theorem under the following assumption.

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with unitary U and f be a continuous function on the non-negative real line which contains $\sigma(|T|)$. Let $\mathcal{K}$ be Berberian extension of $\mathcal{H}$ and $\circ : B(\mathcal{H}) \ni T \to T^\circ \in B(\mathcal{K})$ be a faithful $*$-representation. We set the following conditions (1) and (2):

For a sequence $\{x_n\}$ of unit vectors, if $(T - z)x_n \to 0$, then $(T - z)^* x_n \to 0$. (1)

If a closed subspace $\mathcal{M}$ of $\mathcal{K}$ reduces T° and $re^{i\theta} \in \sigma(T^\circ|_{\mathcal{M}})$, (2)

then $\mathcal{M}$ reduces $U^\circ, |T|^\circ$ and $e^{-i\theta} f(r) \in \sigma_p\left((U^\circ|_{\mathcal{M}} f(|T|^\circ)|_{\mathcal{M}})^*\right)$.

Theorem 1.4. *Let $\mathbf{T} = (T_1, T_2)$ be a doubly commuting pair of operators and $T_j = U_j|T_j|$ $(j = 1, 2)$ be the polar decomposition. Let $f(t)$ be a continuous function on a open interval in the non-negative real line which contains $\sigma(|T_1|) \cup \sigma(|T_2|)$. Let $S_j = U_j f(|T_j|)$ $(j = 1, 2)$ and $\mathbf{S} = (S_1, S_2)$. Let T_1, T_2 and f satisfy* (1) *and* (2)*. If $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \sigma_T(\mathbf{T})$, then $(e^{i\theta_1} f(r_1), e^{i\theta_2} f(r_2)) \in \sigma_T(\mathbf{S})$.*

See the details of Berberian extension [1]. That proof depends on the following Vasilescu's result.

Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ and let

$$\alpha(\mathbf{T} - \mathbf{z}) := \begin{pmatrix} T_1 - z_1 & T_2 - z_2 \\ -(T_2 - z_2)^* & (T_1 - z_1)^* \end{pmatrix} \text{ on } \mathcal{H}\oplus\mathcal{H}.$$

Then Vasilescu proved the following result.

Proposition 1.5. (Theorem 1.1, Vasilescu [11]) *Let $\mathbf{T} = (T_1, T_2) \in B(\mathcal{H})^2$ be a commuting pair. Then*

$$\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T}) \text{ if and only if } \alpha(\mathbf{T} - \mathbf{z}) \text{ is not invertible.}$$

Therefore, we have $\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T})$ if and only if $0 \in \sigma(\alpha(\mathbf{T} - \mathbf{z}))$.

For an n-tuple $\mathbf{T} = (T_1, ..., T_n)$, the joint point spectrum $\sigma_{jp}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a non-zero vector $x \in \mathcal{H}$ which satisfies $T_j x = z_j x$ $(\forall j = 1, ..., n)$ and the joint approximate point spectrum $\sigma_{ja}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a sequence $\{x_k\}$ of unit vectors of $\mathcal{H}$ which satisfies

$$(T_j - z_j)x_k \rightarrow 0 \text{ as } k \rightarrow \infty \ (\forall j = 1, ..., n).$$

Following proposition is due to Berberian [1] for a single operator case. It is easy to see a proof for n-tuples. See Berberian [1] and Chō [2].

Proposition 1.6. *Let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Then there exist an extension space $\mathcal{K}$ of $\mathcal{H}$ and a faithful $*$-representation of $B(\mathcal{H})$ into $B(\mathcal{K})$: $T \rightarrow T^\circ$ such that*

$$\sigma_{ja}(\mathbf{T}) = \sigma_{ja}(\mathbf{T}^\circ) = \sigma_{jp}(\mathbf{T}^\circ),$$

where $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and $\mathbf{T}^\circ = (T_1^\circ, ..., T_n^\circ)$.

Following results are well known.

Proposition 1.7. *Let $T = U|T|$ be the polar decomposition of T and f be a continuous function on the non-negative real line which contains $\sigma(|T|)$. For a sequence $\{x_n\}$ of unit vectors, if $(T - re^{i\theta})x_n \rightarrow 0$ and $(T - re^{i\theta})^* x_n \rightarrow 0$, then $(U - e^{i\theta})x_n \rightarrow 0$, $(|T| - r)x_n \rightarrow 0$ and $(f(|T|) - f(r))x_n \rightarrow 0$.*

See Lemma 1.2.4 in [13].

Proposition 1.8. *Let T be semi-hyponormal. Then $\sigma(T) = \{\overline{z} : z \in \sigma_a(T^*)\}$.*

See Theorem 1.2.6 in [13].

Remark. If T is p-hyponormal and $f(t) = t^{2p}$, then (2) holds by Theorem 4 of [3]. If T is log-hyponormal and $f(t) = \log t$, then (2) holds by Lemma 3 of [8]. About (3), since the mapping $\circ$ of Berberian method is a faithful $*$-representation, so is T° if T is p-hyponormal or log-hyponormal, respectively. Let $\mathcal{M}$ be a reducing subspace for T. It is clear that if T is p-hyponormal or log-hyponormal, then so is $T|_{\mathcal{M}}$, respectively.
(i) Let T be p-hyponormal and $T = U|T|$ be the polar decomposition of T and $f(t) = t^{2p}$. Then $S = U|T|^{2p}$ is semi-hyponormal and $\sigma(U|T|^{2p}) = \{r^{2p}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ by Theorem 3 of [4]. Hence (3) holds by Proposition 1.8.
(ii) Let $T = U|T|$ be log-hyponormal and $f(t) = \log t$. Then $S = U \log |T|$ is semi-hyponormal and $\sigma(U \log |T|) = \{e^{i\theta} \log r : re^{i\theta} \in \sigma(T)\}$ by Lemma 8 of [8]. Hence (3) holds by Proposition 1.8.
Therefore, if T is p-hyponormal or log-hyponormal and $f(t) = t^{2p}$ or $f(t) = \log t$, respectively, then T satisfies (2) and (3) for this f.

In this paper, we would like to prove the following theorem.

Theorem 1.9. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of operators and $T_j = U_j|T_j|$ $(j = 1, ..., n)$ be the polar decompositions. Let $f(t)$ be a continuous function on a open interval in the non-negative real line which contains $\sigma(|T_1|) \cup \cdots \cup \sigma(|T_n|)$. Let $S_j = U_j f(|T_j|)$ $(j = 1, ..., n)$ and $\mathbf{S} = (S_1, ..., S_n)$. Let $T_1, ..., T_n$ and f satisfy* (1) *and* (2). *If $(r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})$, then $(e^{i\theta_1} f(r_1), ..., e^{i\theta_n} f(r_n)) \in \sigma_T(\mathbf{S})$.*

2. Proof of the theorem

First we need the following lemma.

Lemma 2.1. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of operators and T_j has property (1) for $j = 1, ..., n$. Let $\{D_k\}$ be the chain complex of n-tuple $\mathbf{T} = (T_1, ..., T_n)$. If there exists some $k \in \{1, 2, \cdots, n-1\}$ and unit vectors $x_m = \oplus_{j=1}^{r} x_m^j \in E_k^n(\mathcal{H})$ where $r = \binom{n}{k}$, such that $(D_k^* D_k + D_{k+1} D_{k+1}^*) x_m \to 0$ as $m \to \infty$, then there exists $s \in \{1, 2, \cdots, r\}$ such that $\{x_m^s\}$ is a bounded below sequence of non-zero vectors of $\mathcal{H}$ satisfying $T_j^* x_m^s \to 0$ as $m \to \infty$ for $j = 1, \cdots, n$. Thus, by taking unit vector $y_m = \dfrac{x_m^s}{\|x_m^s\|} \in \mathcal{H}$, we have $T_j^* y_m \to 0$ as $m \to \infty$ for $j = 1, \cdots, n$.*

Proof. We show it by the mathematical induction.
(1) Let $n = 2$. Then the chain complex of doubly commuting pair $\mathbf{T} = (T_1, T_2)$ is

$$0 \longrightarrow \mathcal{H} \xrightarrow{D_2} \mathcal{H} \oplus \mathcal{H} \xrightarrow{D_1} \mathcal{H} \longrightarrow 0.$$

By the definition of the Koszul complex we have

$$D_2 = \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix} \text{ and } D_1 = \begin{pmatrix} T_1 & T_2 \end{pmatrix}.$$

Since T_1, T_2 are doubly commuting, we have

$$D_1^* D_1 + D_2 D_2^* = \begin{pmatrix} T_1^* T_1 + T_2 T_2^* & 0 \\ 0 & T_1 T_1^* + T_2^* T_2 \end{pmatrix}.$$

Let $x_m = x_m^1 \oplus x_m^2 \in E_1^2(\mathcal{H}) \cong \mathcal{H} \oplus \mathcal{H}$ be unit vectors and

$$(D_1^* D_1 + D_2 D_2^*) x_m = \begin{pmatrix} T_1^* T_1 + T_2 T_2^* & 0 \\ 0 & T_1 T_1^* + T_2^* T_2 \end{pmatrix} \begin{pmatrix} x_m^1 \\ x_m^2 \end{pmatrix}$$
$$= \begin{pmatrix} (T_1^* T_1 + T_2 T_2^*) x_m^1 \\ (T_1 T_1^* + T_2^* T_2) x_m^2 \end{pmatrix} \to 0 \text{ as } m \to \infty.$$

Since $\|x_m^1\|^2 + \|x_m^2\|^2 = 1$ for all m, we may assume (i) $x_m^1 \nrightarrow 0$ or (ii) $x_m^2 \nrightarrow 0$.

We assume (i). By taking subsequence, we may asume that there exists $0 < c$ that that $0 < c < \|x_m^1\| \leq 1$ for all m, i.e., bounded below. Then $(T_1^* T_1 + T_2 T_2^*) x_m^1 \to 0$ implies $T_1 x_m^1, T_2^* x_m^1 \to 0$ and $T_1^* x_m^1 \to 0$ by (1). Case (ii) is similar. Hence the statement holds for $n = 2$.

(2) We assume that the statement holds for $(n-1)$-tuples of doubly commuting operators. Asuume $(D_k^* D_k + D_{k+1} D_{k+1}^*) x_m \to 0$ as $m \to \infty$ for unit vectors $x_m \in E_k^n(\mathcal{H})$.

Let $\{F_k\}$ be the chain complex of $(n-1)$-tuple $\mathbf{T}' = (T_1, ..., T_{n-1})$ and $x_m = y_m \oplus z_m \in E_k^{n-1}(\mathcal{H}) \oplus E_{k-1}^{n-1}(\mathcal{H}) = E_k^n(\mathcal{H})$. By Curto's characterization (see p.132, Curto [7]) it holds $D_k = \begin{pmatrix} F_k & (-1)^{k+1} \mathrm{diag}(T_n) \\ 0 & F_{k-1} \end{pmatrix}$. Hence

$$(D_k^* D_k + D_{k+1} D_{k+1}^*) x_m = \begin{pmatrix} \left(F_k^* F_k + F_{k+1} F_{k+1}^* + \mathrm{diag}(T_n T_n^*) \right) y_m \\ \left(F_{k-1}^* F_{k-1} + F_k F_k^* + \mathrm{diag}(T_n^* T_n) \right) z_m \end{pmatrix} \to 0.$$

Since $\|y_m\|^2 + \|z_m\|^2 = 1$ for all m, we may assume (i) $y_m \nrightarrow 0$ or (ii) $z_m \nrightarrow 0$.

We assume (i).

Then $\left(F_k^* F_k + F_{k+1} F_{k+1}^* + \mathrm{diag}(T_n T_n^*) \right) y_m \to 0$ implies $\left(F_k^* F_k + F_{k+1} F_{k+1}^* \right) y_m \to 0$ and $(\mathrm{diag}(T_n T_n^*))\, y_m \to 0$. By taking subsequence, we may asume that there exists $0 < c$ that that $0 < c < \|y_m\| \leq 1$ for all m.

Let $v_m = \dfrac{y_m}{\|y_m\|}$. Then v_m are unit vectors and $\left(F_k^* F_k + F_{k+1}F_{k+1}^*\right) v_m \to 0$ and $(\mathrm{diag}(T_n T_n^*))\, v_m \to 0$. Let $v_m = \oplus_{s=1}^{\binom{n-1}{k}} v_m^s \in E_k^{n-1}(\mathcal{H})$. Then there exist $s \in \left\{1, 2, \cdots, \binom{n-1}{k}\right\}$ such that $v_m^s \in \mathcal{H}$ is a bounded below sequence of non-zero vectors and $T_j^* v_m^s \to 0$ for $j = 1, 2, \cdots, n-1$ and $T_n^* v_m^s \to 0$ as $m \to \infty$.

Case (ii) is similar. Hence the statement holds for n. It completes the proof. □

Theorem 2.2. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of operators which satisfy that every T_j ($j = 1, ..., n$) has property (1). If $z = (z_1, ..., z_n) \in \sigma_T(\mathbf{T})$, then there exists unit vectors $y_m \in \mathcal{H}$ such that $(T_j - z_j)^* y_m \to 0$ as $m \to \infty$, that is, $\overline{z} = (\overline{z_1}, ..., \overline{z_n}) \in \sigma_{ja}(\mathbf{T}^*)$, where $\mathbf{T}^* = (T_1^*, ..., T_n^*)$.*

Proof. Since $z = (z_1, ..., z_n) \in \sigma_T(\mathbf{T})$, by the spectral mapping theorem of the Taylor spectrum, it holds

$$0 = (0, ..., 0) \in \sigma_T(\mathbf{T} - z),$$

where $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$. Since $\mathbf{T} - z$ is a doubly commuting n-tuple of operators which satisfy that every $T_j - z_j$ ($j = 1, ..., n$) has property (1) and the Koszul complex $E(\mathbf{T} - z)$ of n-tuple $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$ is not exact. Hence there exists k such that $(D_k^* D_k + D_{k+1} D_{k+1}^*)$ is not invertible. Since the operator $D_k^* D_k + D_{k+1} D_{k+1}^*$ is positive on the space $E_k^n(\mathcal{H})$, there exists a sequence $\{x_m\}$ of unit vectors of $E_k^n(\mathcal{H})$ such that $\left(D_k^* D_k + D_{k+1} D_{k+1}^*\right) x_m \to 0$ as $m \to \infty$. Hence, by Lemma 2.1 there exists a sequence $\{y_m\}$ of unit vectors of $\mathcal{H}$ such that

$$(T_j - z_j)^* y_m \to 0 \ \text{ as } m \to \infty \ \text{ for all } \ j = 1, ..., n.$$

It's completes the proof. □

Proof of Theorem 1.9.
(1) If $n = 2$, theorem holds by Theorem 2.3 of [6].
(2) We assume that the statment holds for $(n-1)$-tuple. Since $(r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})$, by Theorem 2.2 there exists a sequence $\{x_m\}$ of unit vectors of $\mathcal{H}$ such that $(T_j - r_j e^{i\theta_j})^* x_m \to 0$ as $m \to \infty$ for all $j = 1, ..., n$. Consider the Berberian extension $\mathcal{K}$ of $\mathcal{H}$. Then there exists $0 \neq x^\circ \in \mathcal{K}$ such that

$$(T_j^\circ - r_j e^{i\theta_j})^* x^\circ = 0 \ \text{ for all } \ j = 1, ..., n.$$

Let $\mathcal{M} = \ker(T_n^\circ - r_n e^{i\theta_n})^*$. Then $\mathcal{M}(\neq \{0\})$ is a reducing subspace for $T_1^\circ, ..., T_{n-1}^\circ$ and $(r_1 e^{i\theta_1}, ..., r_{n-1} e^{i\theta_{n-1}}) \in \sigma_T(\mathbf{T}^{\circ\prime}_{|\mathcal{M}})$, where $\mathbf{T}^{\circ\prime}_{|\mathcal{M}} = (T^\circ_{1|\mathcal{M}}, ..., T^\circ_{n-1|\mathcal{M}})$. By the induction there exists a non-zero vector $y^\circ \in \mathcal{M}$ such that

$$(S_j^\circ - e^{i\theta_j} f(r_j))^* y^\circ = 0 \ \text{ for all } \ j = 1, ..., n-1.$$

Let $\mathcal{N} = \bigcap_{j=1}^{n-1} \ker(S_j^\circ - e^{i\theta_j} f(r_j))^*$. Then $\mathcal{N}$ is a reducing subspace for T_n°. Let $\mathcal{R} = \mathcal{M} \bigcap \mathcal{N} \neq \{0\}$. Hence $r_n e^{i\theta_n} \in \sigma(T^\circ_{n|\mathcal{R}})$. By property (2) there exists a non-zero vector $z^\circ \in \mathcal{R}$ such that $(S^\circ_{n|\mathcal{R}} - e^{i\theta_n} f(r_n))^* z^\circ = 0$. Since this z° satisfies $(S^\circ_{j|\mathcal{R}} - e^{i\theta_j} f(r_j))^* z^\circ = 0$ for all $j = 1, ..., n-1$, we have $(e^{i\theta_1} f(r_1), ..., e^{i\theta_n} f(r_n)) \in \sigma_T(\mathbf{S})$. This completes the proof.

□

Corollary 2.3. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of p-hyponormal operators $(0 < p < 1)$. Let U_j be unitary for the polar decomposition of $T_j = U_j |T_j|$ ($j = 1, ..., n$) and $\mathbf{S} = \left(U_1 |T_1|^{2p}, ..., U_n |T_n|^{2p}\right)$. Then*

$$\sigma_T(\mathbf{S}) = \{(r_1^{2p} e^{i\theta_1}, ..., r_n^{2p} e^{i\theta_n}) : (r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = t^{2p}$ on the non-negative real line. Since $\mathbf{T}$ is a doubly commuting n-tuple of p-hyponormal operators and $f(t) = t^{2p}$, $T_1, ..., T_n$ and f satisfy (2) and (3). Hence, by Theorem 1.9 we have

$$\sigma_T(\mathbf{S}) \supset \{(r_1^{2p} e^{i\theta_1}, ..., r_n^{2p} e^{i\theta_n}) : (r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, put $g(t) = t^{\frac{1}{2p}}$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting pair of semi-hyponormal operators, S_1, S_2 and g satisfy (2) and (3). Then we have the converse inclusion by Theorem 1.9 and similar argument. □

Corollary 2.4. *Let* $\mathbf{T} = (T_1, ..., T_n)$ *be a doubly commuting n-tuple of* log*-hyponormal operators with* $\log |T_j| > 0$. *Let* U_j *be unitary for the polar decomposition of* $T_j = U_j |T_j|$ $(j = 1, ..., n)$ *and* $\mathbf{S} = (U_1 \log |T_1|, ..., U_n \log |T_n|)$. *Then*

$$\sigma_T(\mathbf{S}) = \{e^{i\theta_1} \log r_1, ..., e^{i\theta_n} \log r_n) : (r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = \log t$ on $(0, \infty)$. Since $\mathbf{T}$ is a doubly commuting n-tuple of log-hyponormal operators and $f(t) = \log t$, $T_1, ..., T_n$ and f satisfy (2) and (3). So by Theorem 1.9 we have

$$\sigma_T(\mathbf{S}) \supset \{e^{i\theta_1} \log r_1, ..., e^{i\theta_n} \log r_n) : (r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, let $g(t) = e^t$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting n-tuple of semi-hyponormal operators, $S_1, ..., S_n$ and g satisfy (2) and (3). Hence, we have the converse inclusion by similar argument. □

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