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Constructions of partial geometric difference sets from spread-like partitions

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Abstract. Partial geometric difference sets (PGDSs) were introduced by O. Olmez. In this paper, we construct partial geometric difference sets by using spread-like constructions method. We also give some PGDS constructions by using cosets of a finite group. We call it partial coset construction methods.

1. Introduction

The difference set technique is a remarkable asset for constructing combinatorial objects with large automorphism groups. In particular, designs are the most constructed ones of those constructed with this technique [7, 10, 14, 17]. This connection has provided elegant solutions to engineering problems. A (v, k, λ)- *difference set* (DS) in a finite group G of order v is a k-subset D with the property that the multiset $\Delta D := \{d_1(d_2)^{-1} | d_1, d_2 \in D, d_1 \neq d_2\}$ contains every non-identity element precisely λ times. The set $\{gD : g \in G\}$ called *the development set* of a difference set is a *symmetric design* with a regular automorphism group. Difference sets of various types have interesting links to other combinatorial objects, including graphs, association schemes, codes, and functions for cryptography [15, 16]. For instance, relative difference sets can be used to construct bent functions and divisible designs and partial difference sets can be used to construct strongly regular graphs. A (v, b, k, λ)-*relative difference set* (RDS) in a finite group G of order vb relative to a (forbidden) subgroup U of order b is a $k-$ subset R with the property that the multiset $\Delta R := \{r_1(r_2)^{-1} | r_1, r_2 \in R, r_1 \neq r_2\}$ contains every element of $G \backslash U$ precisely λ times and does not contain any nonzero elements of U . The RDS is called *semiregular* if $v = k = b\lambda$.

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Table 1: The Parameters of PGDSs.

v	k	α	$\beta-\alpha$	Comments	Citation
$(t-1)m^2$	$(t-1)m$	$mt(t-2)$	m^2	$t\geq 3$, t is a positive integer	[6]
tm^2	$m(t-1)$	$m(t-2)(t-1)$	m^2	$t\geq 3$, t is a positive integer	[6]
$p^{2(r+d)}$	$p^{r+2d}t$	$p^{r+4d}t^2-p^{r+2d}$	$p^{2(r+d)}$	$r,d\geq 1$,p is prime,$1\leq t\leq p^r$	[6]
$2^{2d}r^2t$	$2^{2d-1}r^2t-2^dr$	$2^{4d-3}r^4t^2-3\cdot 2^{3d-2}r^3t+2^{2d}r^2$	$2^{2d}r^2$	$r,d\geq 1$,p is prime,$1\leq t\leq 4$	[6]
$2^{2d+3}t$	$2^{2d+1}t-2^{2d}$	$2^{4d}t^2-3\cdot 2^{4d-1}t+2^{4d-1}$	2^{4d}	$d\geq 1$,$1\leq t\leq \frac{2^{2d+1}+1}{3}$	[6]
$3^{d+1}t$	3^dt+3^d	$3^{2d-1}t^2+3^{2d}t+2\cdot 3^{2d-1}$	3^{2d}	$d\geq 1$,$1\leq t\leq \frac{3^{2d+1}-1}{2}$	[6]
$2^{2d+4}t$	$2^{2d+2}t+2^{2d+1}$	$2^{4d+2}t^2+3\cdot 2^{4d+1}t+2^{4d+1}$	2^{4d+2}	$d\geq 1$,$1\leq t\leq \frac{2^{2d+2}-1}{3}$	[6]
$q^{2d+2}t$	$\frac{1}{2}(q^2-q)q^{2d}t$ $+q^{2d+1}$	$\frac{1}{8}(q^4-3q^3+3q^2-q)q^{4d}t^2$ $+\frac{3}{4}(q^3-2q^2+q)q^{4d}t+(q^2-q)q^{4d}$	q^{4d+2}	$q=3^r$ or $q=p^{2r}$, $1\leq t\leq 4\frac{q^{2d+2}-1}{q^2-1}$	[6]
$2q^{2d+2}t$	$(q^2-q)q^{2d}t$ $+q^{2d+1}$	$\frac{1}{2}(q^4-3q^3+3q^2-q)q^{4d}t^2$ $+\frac{3}{2}(q^3-2q^2+q)q^{4d}t+(q^2-q)q^{4d}$	q^{4d+2}	$q=2^r$, $1\leq t\leq 2\frac{q^{2d+2}-1}{q^2-1}$	[6]
p^k	p^{k-1}	$p^{2k-3}-p^{k-2}$	p^{k-1}	p is an odd prime, $k>1$	[14]
p^{n+1}	p^n	$p^{2n-1}-p^{n-1}$	p^n	p is prime, n is integer	[2]
p^{n+m}	p^n	$p^{2n-m}-p^{n+s-m}$	p^{n+s}	p is an odd prime,$0\leq s\leq n$, n and m are integers	[2]
3^n	3^{n-1}	$3^{2n-3}-3^{n-2}$	3^{n-1}	$n\geq 3$	[2]
p^{2n}	p^n	p^n-p^s	p^{n+s}	p is an odd prime, $0\leq s\leq n$	[2]
mp^2	mp	$\frac{3}{4}m^2p$	$\frac{m^2p^2}{4}$	$p>2$ is a prime, $m=0 \mod 2$	[9]
$6p^2$	$4p$	$8p$	$12p$	p is an odd prime	[9]
n^2	n	$n-1$	n	n is an integer	[9]
$8l$	$4l$	$6l^2$	$4l^2$	$n=4l$ is a positive integer	[9]
$q^{t+1}ms$	q^tms	$m^2q^{2t-1}(s^2-1)$	m^2q^{2t}	$t\geq 1$, q is a prime, $s\leq \frac{q^{t+1}-1}{q-1}$	[3]
$q^{t+1}ms$	$q^tm(s-1)$	$m^2q^{2t-1}(s-1)(s-2)$	m^2q^{2t}	$t\geq 1$, q is a prime, $s\leq \frac{q^{t+1}-1}{q-1}$	[3]
$q^{2l}ms$	q^lms	$m^2q^l(s^2-1)$	m^2q^{2l}	$l\geq 1$,$s\leq q^l+1$	[3]
$q^{2l}ms$	$q^lm(s-1)$	$m^2q^l(s-1)(s-2)$	m^2q^2l	$l\geq 1$,$s\leq q^l+1$	[3]
mq^2s	mqs	$m^2q(s^2-1)$	m^2q^2	$s\leq q+1$, q is an odd prime	[3]
mq^2s	$mq(s-1)$	$m^2q(s^2-3s+2)$	m^2q^2	$s\leq q+1$, q is an odd prime	[3]
$3^{t+1}ms$	$3^tm(s+1)$	$3^{2t-1}m^2(s^2+3s+2)$	$3^{2t}m^2$	$t\geq 1$, m is an integer, $1\leq s\leq \frac{1}{2}(3^{t+1}-1)$	[3]
lq^{s+1}	lq^s	$(l^2-1)q^{2s-1}$	q^{2s}	$2\leq l\leq \frac{q^{s+1}-1}{q-1}$,$s$ is a positive integer, q is a prime	[13]
$r1^{s+1}$	$(r-1)q^s$	$q^{2s-1}(r-1)(r-1)$	q^{2s}	$2\leq l\leq \frac{q^{s+1}-1}{q-1}$,$s$ is a positive integer, q is a prime	[13]
lq^{2m}	lq^m	$(l^2-1)q^m$	q^{2m}	$2\leq l\leq \frac{2}{q^m+1}$,$m$ is an integer	[13]
rq^{2m}	mq^m	$q^m(r-2)(r-1)$	q^{2m}	$r=q^m+1$, q is a prime, m is an integer	[13]
$4u$	$2u$	u^2	$3u^2$	$u>1$	[12]

In this paper, we are interested in a certain type of difference set known as a partial geometric difference set. The notion of PGDSs (or $1\frac{1}{2}$-difference sets) was introduced in [12], and some existence and nonexistence results were given. Furthermore, a series of $1\frac{1}{2}$-designs was constructed. From these designs, strongly regular graphs were derived [1]. To facilitate the management of the parameters, $1\frac{1}{2}$ were investigated as PGDS and are now referred to by this nomenclature in the existing literature. In [6], the framework of extended building sets was used to find infinite families of PGDSs in abelian groups. In [14], it was demonstrated that the existence of a family of partial geometric difference sets is equivalent to the existence of a certain family of three-weight linear codes, and a link was also provided between ternary weakly regular bent functions, three-weight linear codes and partial geometric difference sets. In [2], the authors established a relationship between vectorial s-plateaued functions and partial geometric difference sets, leveraging this connection to provide a partition of $\mathbb{F}_{3^n}$ into partial geometric difference sets.In [9], using Galois rings and Galois fields, they constructed several infinite classes of partial geometric difference sets and partial geometric difference families with new parameters. In [3], the authors introduced several new constructions of partial geometric difference sets and partial geometric difference families by using cosets of a group in the direct product of two groups. In [13], they constructed an infinite family of PGDSs (or $1\frac{1}{2}$-difference sets) in non-cyclic abelian p-groups. We have presented the parameters obtained from the above-mentioned studies in Table 1.

2. Preliminaries

Let G be a group of order v and $S \subset G$ be a k-subset. We define $\delta(g)$ as

$$\delta(g) := |\{(s,t) \in S \times S \colon g = st^{-1}\}|$$

for each $g \in G$. Let v and k be positive integers satisfying the condition $v > k > 2$. The following concept introduced in [12] for defining PGDSs will be used frequently in this article. A k-subset S of a group G of order v is called a PGDS in G with parameters $(v, k; \alpha, \beta)$ if there exist constants α and β such that, for each $x \in G$,

$$\sum_{y \in S} \delta(xy^{-1}) = \begin{cases} \alpha & \text{if } x \notin S, \\ \beta & \text{if } x \in S. \end{cases}$$

This definition generalizes the concept of (v, k, λ, n)-difference sets and semiregular relative difference sets. For instance the existence of a (v, k, λ, n)-DS, implies the existence of a $(v, k; k\lambda, n + k\lambda)$-PGDS and existence of an (m, u, k, λ) semiregular RDS, implies the existence of a $(mu, k; \lambda(k-1), k(\lambda+1) - \lambda)$-PGDS. The construction strategies we will consider in this paper requires standard and convenient tools from group rings.

Let G be a finite group and we call $\mathbb{Z}[G]$ be the group ring of G and define it as

$$\mathbb{Z}[G] := \left\{ \sum_{g \in G} a_g g | a_g \in \mathbb{Z} \right\}.$$

The addition and multiplication operations on $\mathbb{Z}[G]$ are presented as

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{r \in G} b_r r \right) = \sum_{g, r \in G} a_g b_r (g + r).$$

Let $\mathbb{Z}[G] := \{\sum_{g \in G} a_g g | a_g \in \mathbb{Z}\}$. If $S \subset G$, then the group ring element $\underline{S}$ will be defined using the normal abuse of notation as $\underline{S} = \sum_{s \in S} s$. Furthermore, the group ring elements $\underline{S}^{(-1)}$ and $\underline{G}$ will be defined as $\underline{S}^{(-1)} := \sum_{s \in S} s^{-1}$ and $\underline{G} := \sum_{g \in G} g$.

Let S be a subset of G. In [12] they showed that S is a $(v, k; \alpha, \beta)$-PGDS if it satisfies the following group ring equation.

$$\underline{S}\,\underline{S}^{(-1)}\underline{S} = (\beta - \alpha)\underline{S} + \alpha \underline{G}. \tag{1}$$

3. Spread-like Constructions of PGDS

In this section, we delve into the construction of PGDSs by leveraging partial k-spreads within specific vector spaces over finite fields. Meidl and Pirsic [8] introduced a family of bent functions by using spreads of vector spaces where B denotes an abelian group of order 2^k. In the following proposition, we will present a slightly modified version of the method for constructing a bent function, as outlined in [8]. It has been observed that the set produced by this method yields PGDS. This proposition constitutes a preliminary step for our main PGDS constructing method. We designate a k-dimensional vector subspace of V_n as a $k-$subspace. A partial $k-$ spread in V_n refers to a collection of $k-$ subspaces that intersect trivially pairwise.

Unless stated otherwise, the additive group of V_n is denoted by E throughout this paper. In particular, $\mathbb{E}_{2m}$ represent the additive group of the $2m$-dimensional vector space $\mathbb{V}_{2m}$ over $GF(2)$. Notably, there exist precisely $2^m + 1$ m-dimensional subspaces of $\mathbb{V}_{2m}$ that intersect trivially pairwise.

Let $H_1, H_2, H_3, \ldots, H_{2^m+1}$ denote the corresponding subgroups of E with the property that $H_i \cap H_j = \{e\}$ for all $i \neq j$. Consequently, we derive the following equations in the group ring $\mathbb{Z}E$.

$$H_1 + H_2 + H_3 + \cdots + H_r = 2^m e + E$$

$$H_i H_j = \begin{cases} 2^m H_i & \text{if } i = j \\ E & \text{if } i \neq j \end{cases}$$

Proposition 3.1. *Let B be an abelian group of order 2^l. Each nonzero element γ of B is associated with the union of exactly 2^{m-l} subgroups of $\mathbb{E}_{2m}$, excluding the identity element $e \in \mathbb{E}_{2m}$. i.e.,*

$$\bigcup_{i=1}^{2^{m-l}} (H_i', \gamma),$$

where $H' = H \setminus \{0\}$.

Then the set

$$S = \bigcup_{j=1}^{2^l} \bigcup_{i=1}^{2^{m-l}} (H'_{j+2^k(i-1)}, \gamma_j)$$

is a PGDS in $\mathbb{V}_{2m} \times B$ with parameters $\beta = 2^{2m-l}(2^{2m} - 3 \cdot 2^m + 2) + 2^{2m}$ and $\alpha = 2^{2m-l}(2^{2m} - 3 \cdot 2^m + 2)$.

We will give the proof of this proposition as an application of our main constructions.

Example 3.2. *Let's examine the situation where $m = 2$, $l = 2$, and $B = \mathbb{Z}_4$. In this case, each element of B corresponds to an element of each 2-dimensional subspace of $\mathbb{V}_4$. There are precisely 5 unique 2-dimensional subspaces in $\mathbb{V}_4$ that intersect only at $\{0\}$. Therefore, we label these subspaces as $\{H_1, H_2, H_3, H_4, H_5\}$, creating a spread of $\mathbb{V}_4$.*

Considering this setup, the collection

$$S = \{(H_1', 0), (H_2', 1), (H_3', 2), (H_4', 3)\}$$

forms a PGDS in $\mathbb{V}_4 \times B$ with parameters $(64, 12; 24, 40)$. We can demonstrate that

$$\underline{S}\,\underline{S}^{-1}\underline{S} = 16\underline{S} + 24\underline{G}$$

in the group ring where $G = \mathbb{V}_4 \times B$.

In this scenario,

$$H_1' = \{(1,0,0,0), (0,1,0,1), (1,1,0,1)\}$$

$$H_2' = \{(0,1,1,0), (1,0,0,1), (1,1,1,1)\}$$

$$H_3' = \{(0,1,0,0), (1,0,1,0), (1,1,1,0)\}$$

$$H_4' = \{(1,1,0,0), (1,0,1,1), (0,1,1,1)\}$$

Next, we will adopt a similar approach to that used in the bent function constructions 1 and 2, as given in [8], to construct PGDSs. Let V_n denote the n-dimensional vector space over $\mathbb{F}_2$, where the vectors are n-tuples. Our focus will be on analyzing collections of m-dimensional subspaces within V_n that intersect in a non-trivial manner.

Let $\mathbb{V}_n$ be an n-dimensional vector space, and consider m-dimensional subspaces of $\mathbb{V}_n$ that pairwise intersect in a fixed t-dimensional subspace. We immediately observe the following upper and lower bounds on t:

$$m + m - t \leq n,$$
$$2m - n \leq t < m.$$

We also note that an additional restriction on the integer t is required. Since we are working over GF(2), we need $\frac{2^n-2^t}{2^m-2^t}$ to be an odd integer to partition $\mathbb{V}_n$ into m-dimensional subspaces that intersect in a t-dimensional subspace and m-dimensional subspaces with the intersection subspace removed. Therefore, we require $(m-t) \mid (n-t)$. Now, let us give the main constructions of the paper.

Construction 3.3. *Consider N as the t-dimensional intersection subspace. Define $\mathcal{S} = \{H_i \mid 1 \leq i \leq (2^l-1)(2^{n-m-l})\}$ to be a partial m-spread of $\mathbb{V}_n$ with intersection N. Let B denote an abelian group of order 2^l where $1 \leq l \leq m-t$. Let the construction steps for the set S be as follows:*

- *Every nonzero element γ of B is associated with the union of exactly 2^{n-m-l} elements of $\mathcal{S}$ excluding N. That is,*

$$\bigcup_{i=1}^{2^{n-m-l}} (H_i', \gamma) \quad \textit{where} \quad H' = H \setminus N.$$

- *All other elements of G are associated with $0 \in B$.*

Then,

$$S = \left(\bigcup_{j=1}^{2^l-1} \bigcup_{i=1}^{2^{n-m-l}} \left(H'_{(2^{n-m-l})j-2^{n-m-l}+i}, \gamma_j \right) \right) \cup \left(\bigcup_{i=(2^l-1)(2^{n-m-l})+1}^{\frac{2^n-2^t}{2^m-2^t}} (H_i', 0) \right) \cup (N, 0)$$

yields a PGDS in $G = E \times B$.

Before proving the construction, let us present some equations in the group ring $\mathbb{Z}E$:

$$H_i'H_i' = (|H_i'| - |N|)H_i' + |H_i'|N \tag{2}$$

$$NN = |N|N \tag{3}$$

$$H_i'N = |N|H_i' \tag{4}$$

Proof. Let $G = E \times \mathbb{Z}_{2^l}$ be a group and let the subset $S \subseteq G$ be constructed as in the construction above. For the first condition, precisely 2^{n-m-l} subspaces $H \in \mathcal{S}$, with the intersection subspace removed, match with the elements of $B' = \{x \in B \mid x \neq 0\}$. For the second condition, exactly $\left(\frac{2^n-2^t}{2^m-2^t} - 2^{n-m-l}(2^l-1)\right)$ elements of $\mathcal{S}$ that do not match with the non-identity elements of B and the subspace N are paired with the identity element (chosen as 0) of B.

To demonstrate that the set S constitutes a PGDS, it is sufficient to show that it satisfies Equation 1. It is evident that the inverse of the subset H' is equal to itself. Indeed, the set H' is constructed by removing a certain subspace from the group H, and the inverse of each element remains within H'.

The proof is conducted by examining two distinct cases based on the intersection subspaces N of m-dimensional subspaces, specifically when considering the products of these subspaces with N removed.

These cases are distinguished by whether the ratio $\frac{2^{2m}+2^{2t}-2^{m+t+1}}{2^n+2^t-2^{m+1}}$ is an integer. This ratio pertains to the total number of elements obtained from the product of two distinct subspaces, excluding their intersection subspaces, relative to the remaining elements of E not contained in these subspaces.

Case I: $H_i^{'}H_j^{'} = |N|(E \setminus (H_i^{'} \cup H_j^{'} \cup N))$

In this case, for different i and j, the product of $H_i^{'}$ and $H_j^{'}$ consists of all $H_w^{'}$ where $i \neq w \neq j$. First, let us compute the product $\underline{S}\underline{S}^{-1}$. As can be inferred from Equations 2,3 and 4, $\underline{S}\underline{S}^{-1}$ contains only the elements $(N,0)$ and $G-(N,B)$. The coefficients of $(N,0)$ arise from the products $H_i^{'}\underline{H_i^{'}}$ and NN. From this, we obtain:

$$(2^m-2^t)\frac{2^n-2^t}{2^m-2^t}+2^t=2^n.$$

On the other hand, the coefficient of $G-(N,B)$ similarly arises from the products $H_i^{'}H_i^{'}$, $H_i^{'}H_j^{'}$, and $H_i^{'}N$. Thus, we have:

$$\frac{(2^m-2^t-2^t)+2\cdot 2^t+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}$$
$$=\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}.$$

Therefore,

$$\underline{S}\underline{S}^{-1}=2^n\underline{(N,0)}+\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}(\underline{G}-\underline{(N,B)}).$$

Now we can compute $\underline{S}\underline{S}^{-1}\underline{S}$.

$$\begin{aligned}
\underline{S}\underline{S}^{-1}\underline{S}&=\left(2^n\underline{(N,0)}+\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)(\underline{G}-\underline{(N,B)})\right)\underline{S}\\
&=2^n|N|\underline{S}+\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)(|S|\underline{G}-|N|\underline{G})\\
&=2^n2^t\underline{S}+\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)(2^n\underline{G}-2^t\underline{G})\\
&=2^{n+t}\underline{S}+\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)\left(2^n-2^t\right)\underline{G}.
\end{aligned}$$

Therefore, the set S is a PGDS in the group G with parameters $\alpha=\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\right)\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)(2^n-2^t)$ and $\beta=\left(\left(\frac{2^m+2^t\left(\left(\frac{2^n-2^t}{2^m-2^t}\right)-2\left(\frac{2^n-2^t}{2^m-2^t}\right)-1\right)}{2^l}\right)(2^n-2^t)\right)+2^{n+t}$.

Case 2: $H_i^{'}H_j^{'}=H_w^{'}$, $i \neq w \neq j$

In this case, for different i and j, the product of $H_i^{'}$ and $H_j^{'}$ yields a set $H_w^{'}$ such that $i \neq w \neq j$. Following a similar enumeration as in Case 1, from equations (2.5), (2.6), and (2.7), the product $\underline{S}\underline{S}^{-1}$ is obtained as:

$$\underline{S}\underline{S}^{-1}=2^n\underline{(N,0)}+2^n\underline{(H_w^{'},0)}+2^{n-l}(\underline{G}-\underline{(N,B)}-\underline{(H_w^{'},B)}).$$

Therefore,

$$\underline{S}\underline{S}^{-1}=2^n\underline{(H_w,0)}+2^{n-l}(\underline{G}-\underline{(H_w,B)}).$$

Now we can compute the product $\underline{SS^{-1}}\underline{S}$.

$$\begin{aligned}\underline{SS^{-1}}\,\underline{S} &= \left(2^n\underline{(H_w,0)} + 2^{n-l}(\underline{G} - \underline{(H_w,B)})\right)\underline{S} \\ &= 2^n|H_w|\underline{S} + 2^{n-l}(|S|\underline{G} - |H_w|\underline{G}) \\ &= 2^n2^m\underline{S} + 2^{n-l}(2^n\underline{G} - 2^m\underline{G}) \\ &= 2^{n+m}\underline{S} + (2^{2n-l} - 2^{n-l+m})\underline{G}.\end{aligned}$$

Therefore, the set S is a PGDS in the group G with parameters $(v = 2^{n+l}, k = 2^n; \alpha = 2^{2n-l} - 2^{n-l+m}, \beta = 2^{2n-l} - 2^{n-l+m} + 2^{n+m})$.

□

By a similar counting method, the construction method given below also specifies PGDSs with the same parameters as those specified in Construction 3.3.

Construction 3.4. *Consider N as the t-dimensional intersection subspace. Define* $\mathcal{S} = \{H_i \mid 1 \le i \le (2^l-1)(2^{n-m-l})+1\}$ *to be a partial m-spread of* $\mathbb{V}_n$ *with intersection N. Let B denote an abelian group of order* 2^l *where* $1 \le l \le m-t$. *Let the construction steps for the set S be as follows:*

- *an element* $\bar{\gamma}$ *of* B' *makes pairs with the union of exactly* $2^{n-m-l}+1$ *elements of* $\mathcal{S}$. *i.e.,* $\bigcup_{i=1}^{2^{n-m-l}+1}(H_i,\bar{\gamma})$. *Note that the intersection subspace makes pairs with* γ *too.*

- *All other nonzero elements except* $\bar{\gamma}$ *of* B' *makes pairs with the union of exactly* 2^{n-m-l} *different elements of* $\mathcal{S}$ *except from N. i.e.,* $\bigcup_{i=1}^{2^{n-m-l}}(H_i',\gamma)$ *where* $H' = H \setminus N$.

- $0 \in B$ *element makes pair with rest of the elements of G that is not used in two steps.*

Then,

$$S = \left(\bigcup_{i=1}^{2^{n-m-l}+1}(H_i,\bar{\gamma})\right) \cup \left(\bigcup_{i=(2^{n-m-l})+2,\gamma\ne\bar{\gamma}}^{(2^l-1)(2^{n-m-l})+1}(H_i',\gamma)\right) \cup \left(\bigcup_{i=(2^l-1)(2^{n-m-l})+2}^{\frac{2^n-2^t}{2^m-2^t}}(H_i',0)\right) \cup (N,0)$$

yields a PGDS in $G = E \times B$.

When $t = 0$, the construction will be a partial spread partition for subspaces that intersect trivially in pairs. According to Meidl and Pirsic, these constructions with $t = 0$ yield bent functions when $n = 2m$, see [8].

Now we can give the proof of Proposition 3.1 by using the results in Construction 3.3.

Proof. {Proof of Proposition 3.1} Let S_1 be the PGDS constructed as in Construction 3.3 where $n = 2m$ and $N = e$. Then it is enough to prove that the set $S = S_1 - (H_{2^m+1}, 0)$ is a PGDS with parameters as $\beta = 2^{2m-l}(2^{2m} - 3\cdot2^m + 2) + 2^{2m}$ and $\alpha = 2^{2m-l}(2^{2m} - 3\cdot2^m + 2)$ since the intersection element e and the elements of G that pair with $0 \in B$ are removed from the set S.

To do this, we compute $(S_1 - (H_{2^m+1},0))(S_1 - (H_{2^m+1},0))^{-1}(S_1 - (H_{2^m+1},0))$:

$$\underline{(S_1-(H_{2^m+1},0))}\,\underline{(S_1-(H_{2^m+1},0))}^{-1}\underline{(S_1-(H_{2^m+1},0))}=2^m\underline{(H_{2^m+1},0)})$$
$$+(\underline{S_1}\,\underline{S_1}^{-1}-2(2^{m-l}(\underline{G}-\underline{(H_{2^m+1},B)})+2^m\underline{(H_{2^m+1},0)})\underline{(S_1-(H_{2^m+1},0))}$$
$$=(\underline{S_1}\,\underline{S_1}^{-1}-2^{m-l+1}\underline{G}+2^{m-l+1}\underline{(H_{2^m+1},B)}-2^m\underline{(H_{2^m+1},0)})\underline{(S_1-(H_{2^m+1},0))}$$
$$=(\underline{S_1}\,\underline{S_1}^{-1}-2^{m-l+1}\underline{G}+2^{m-l+1}\underline{(H_{2^m+1},B)}-2^m\underline{(H_{2^m+1},0)})\underline{(S_1-(H_{2^m+1},0))}$$
$$=\underline{S_1}\,\underline{S_1}^{-1}\underline{S_1}-2^{3m-l+1}\underline{G}+2^{2m-l+1}\underline{G}-2^{2m-l}\underline{G}+2^{2m-l}\underline{(H_{2m+1},B)}-2^{2m}\underline{(H_{2m+1},0)}$$
$$-2^{2m}\underline{(H_{2m+1},0)}-2^{3m-l}\underline{G}+2^{2m-l}\underline{(H_{2m+1},B)}+2^{2m-l+1}\underline{G}-2^{2m-l+1}\underline{(H_{2m+1},B)}$$
$$+2^{2m}\underline{(H_{2m+1},0)}$$

Since this proposition fits in case 1 in Construction 3.3 we have $\underline{S_1}\,\underline{S_1}^{-1}\underline{S_1}=2^{2m}\underline{S_1}+(2^{4m-l}-2^{2m-l})\underline{G}$.

$$=2^{2m}\underline{S_1}+(2^{4m-l}-2^{2m-l}-2^{3m-l+1}+2^{2m-l+1}-2^{2m-l}-2^{3m-l}+2^{2m-l+1})\underline{G}$$
$$-2^{2m}\underline{(H_{2m+1},0)}$$
$$=2^{2m}(\underline{S_1-(H_{2m+1},0)})+(2^{2m-l}(2^{2m}-3\cdot 2^m+2))\underline{G}$$
$$=2^{2m}(\underline{S}+(2^{2m-l}(2^{2m}-3\cdot 2^m+2))\underline{G}.$$

□

With these construction methods, we can obtain PGDSs with different parameters by changing the dimensions of the vector space, its subspaces and the intersection space within the given rules. As an example, we will now examine the following 2 propositions as an application of these construction methods to be better understood by the reader.

For these corollaries, we will consider an n-dimensional vector space and its $n-2$-dimensional subspaces, which intersect pairwise in $n-4$ and $n-3$-dimensional subspaces of $\mathbb{V}_n$, respectively. For $n-4$-dimensional intersections, there are $\frac{2^n-2^{n-4}}{2^{n-2}-2^{n-4}}=5$ such $n-2$-dimensional subspaces. For $n-3$-dimensional intersections, there are $\frac{2^n-2^{n-3}}{2^{n-2}-2^{n-3}}=7$ such $n-2$-dimensional subspaces. Let $B=\mathbb{Z}_4$ be a cyclic group. Constructing S in the following manner will provide a PGDS in $G=E\times B$.

Corollary 3.5. *Let* $\{H_1,H_2,H_3,H_4,H_5\}$ *be a collection of* $(n-2)$*-dimensional subspaces of* $\mathbb{V}_n$ *that intersect in a fixed* $(n-4)$*-dimensional subspace* N*. For the cyclic group* B *of order* 4 *and* 2*, we have partial spreads* $\mathcal{S}_1=\{H_1,H_2,H_3,\}$ $\mathcal{S}_2=\{H_1,H_2\}$ *Define the sets* $S_1=\{(H_1^{'},1),(H_2^{'},2),(H_3^{'},3),(H_4^{'},0),(H_5^{'},0),(N,0)\}$ *and* $S_2=\{(H_1^{'},1),(H_2^{'},1),(H_3^{'},0),(H_4^{'},0),(H_5^{'},0),(N,0)\}$*. These sets are PGDSs with the following parameters:*

- *For* $G=E\times B$*, where* B *is a cyclic group of order* 4*,* S_1 *has parameters* $(2^{n+2},2^n;15\cdot 2^{2n-6},19\cdot 2^{2n-6})$.
- *For* $G=E\times B$*, where* B *is a cyclic group of order* 2*,* S_2 *has parameters* $(2^{n+1},2^n;15\cdot 2^{2n-5},17\cdot 2^{2n-5})$.

Note that this corollary falls into case I in Construction 3.3 since $\underline{H_i^{'}}\,\underline{H_j^{'}}=|N|(E\setminus(H_i^{'}\cup H_j^{'}\cup N))$.

Example 3.6. *Let $n = 5$. Consider the following subspaces of $\mathbb{V}_5$:*

$$\begin{aligned}
H_1 &= \{(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(1,1,0,0,0),(0,0,1,0,0),\\
&\quad (1,0,1,0,0),(0,1,1,0,0),(1,1,1,0,0)\},\\
H_2 &= \{(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,1),(1,1,0,0,1),(0,0,1,1,0),\\
&\quad (1,0,1,1,0),(0,1,1,1,1),(1,1,1,1,1)\},\\
H_3 &= \{(0,0,0,0,0),(1,0,0,0,0),(0,1,0,1,0),(1,1,0,1,0),(0,0,1,1,1),\\
&\quad (1,0,1,1,1),(0,1,1,0,1),(1,1,1,0,1)\},\\
H_4 &= \{(0,0,0,0,0),(1,0,0,0,0),(0,1,0,1,1),(1,1,0,1,1),(0,0,1,0,1),\\
&\quad (1,0,1,0,1),(0,1,1,1,0),(1,1,1,1,0)\},\\
H_5 &= \{(0,0,0,0,0),(1,0,0,0,0),(0,0,0,1,0),(1,0,0,1,0),(0,0,0,0,1),\\
&\quad (1,0,0,0,1),(0,0,0,1,1),(1,0,0,1,1)\},
\end{aligned}$$

where these subspaces intersect in the $n-4 = 1$-dimensional subspace $N = \{(0,0,0,0,0),(1,0,0,0,0)\}$. Construct the set

$$S = \{(H_1',1),(H_2',2),(H_3',3),(H_4',0),(H_5',0),(N,0)\}.$$

This set S forms a PGDS with parameters $\alpha = 240$ and $\beta = 304$ in the group $G = E \times B$, where B is a cyclic group of order 4.

Corollary 3.7. *Let $\{H_1,H_2,H_3,H_4,H_5,H_6,H_7\}$ be a collection of $(n-2)$-dimensional subspaces of $\mathbb{V}_n$, intersecting in a fixed $(n-3)$-dimensional subspace N. Define the set*

$$S = \{(H_1',1),(H_2',1),(H_3',0),(H_4',0),(H_5',0),(H_6',0),(H_7',0),(N,0)\}.$$

This set S forms a PGDS with parameters $(2^{n+1},2^n;3\cdot 2^{2n-3},5\cdot 2^{2n-3})$ in the group $G = E \times B$, where B is a cyclic group of order 2.

Note that this corollary falls into case II in Construction 3.3.

Example 3.8. *Let $n = 4$, $m = 2$, $t = 1$, and $l = 1$. Consider the following $(n-2)$-dimensional subspaces of $\mathbb{V}_n$:*

$$\begin{aligned}
H_1 &= \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(1,1,0,0)\},\\
H_2 &= \{(0,0,0,0),(1,0,0,0),(0,1,0,1),(1,1,0,1)\},\\
H_3 &= \{(0,0,0,0),(1,0,0,0),(0,1,1,0),(1,1,1,0)\},\\
H_4 &= \{(0,0,0,0),(1,0,0,0),(0,1,1,1),(1,1,1,1)\},\\
H_5 &= \{(0,0,0,0),(1,0,0,0),(0,0,1,1),(1,0,1,1)\},\\
H_6 &= \{(0,0,0,0),(1,0,0,0),(0,0,0,1),(1,0,0,1)\},\\
H_7 &= \{(0,0,0,0),(1,0,0,0),(0,0,1,0),(1,0,1,0)\}.
\end{aligned}$$

These subspaces have an intersection in the fixed $(n-3)$-dimensional subspace $N = \{(0,0,0,0),(1,0,0,0)\}$. Define the set

$$S = \{(H_1',1),(H_2',1),(H_3',0),(H_4',0),(H_5',0),(H_6',0),(H_7',0),(N,0)\}.$$

This set S forms a PGDS with parameters $\alpha = 96$ and $\beta = 160$ in the group $G = E \times B$, where B is a cyclic group of order 2.

As commonly observed in the literature, the general construction of PGDSs with m-dimensional subspaces of V_n intersect trivially is always used with the specific condition that $n = 2m$. This corollary demonstrates the efficacy of the construction method when $n = 3m$. The following corollary exemplifies the scenario where m-dimensional subspaces of $\mathbb{V}_n$ intersect pairwise trivially, with the specific condition that $n = 3m$.

Corollary 3.9. *Let $\mathbb{V}_n$ be an n-dimensional vector space with $n = 3m$, and consider its m-dimensional subspaces. Let B be a cyclic group of order 2^l. Assume there exists a fixed intersection subspace of dimension t, where $0 \leq t < m$. For $t = 0$, the set S constructed via Construction 3.3 forms a PGDS with parameters $(2^{3m+l}, 2^{3m}; 2^{6m-l} - 2^{3m-l+1}, 2^{6m-l} - 2^{3m-l+1} + 2^{6m+1})$.*

According to Constructions 3.3 and 3.4, we can derive various corollaries with different parameter variations. We have demonstrated some of these corollaries so far. In the following, we will explore other construction types for PGDSs. Specifically, we will utilize multiple subgroups of $\mathbb{E}$. The following constructions illustrate these approaches.

Proposition 3.10. *Let $\mathbb{V}_n$ be an n-dimensional vector space over $GF(2)$. Consider the $(n-1)$-dimensional vector subspaces having an intersection subset $|N| = 2^{n-2}$ of E. Let B be a cyclic abelian group of order 4. Define $\mathcal{S} = \{H_1', H_2', H_3', N\}$ as a distinct subset collection of $\mathbb{V}_n$, where H_i' denotes the subset $H_i \backslash N$. Then the set $S = \bigcup_{i=1}^{3}(H_i', i) \cup \bigcup_{i=1}^{3}(H_i', 0)$ forms a PGDS with parameters $\beta = 4 \cdot 2^{2n-2}$ and $\alpha = 3 \cdot 2^{2n-2}$ in $G = E \times B$.*

Proof. We denote the group G by $\underline{G}$ and the set S by $\underline{S} = \sum_{i=1}^{3}(\underline{(H_i', i)} + \underline{(H_i', 0)})$ in the group ring $\mathbb{Z}G$. Note that $\underline{S^{-1}} = \sum_{i=1}^{3}(\underline{(H_i', -i)} + \underline{(H_i', 0)})$ since elements of H_i' are in $\mathbb{V}_n$ over $GF(2)$. We have the following equations in the group ring $\mathbb{Z}E$:

$$H_i'H_j' = 2^{n-2}H_k' \quad \text{for } i \neq j \neq k,$$

$$H_i'H_i' = 2^{n-2}N.$$

Let us verify the PGDS equation in the group ring:

$$\begin{aligned}
\underline{S} \cdot \underline{S}^{-1} \cdot \underline{S} &= \left(\sum_{i=1}^{3}(\underline{(H_i', i)} + \underline{(H_i', 0)})\right)\left(\sum_{j=1}^{3}(\underline{(H_j', -j)} + \underline{(H_j', 0)})\right)\underline{S} \\
&= 2^{n-2}\left(4(N,0) + 2\left(\sum_{i=0}^{3}(H_1', i) + \sum_{i=0}^{3}(H_2', i) + \sum_{i=0}^{3}(H_3', i) + \sum_{i=0}^{3}(N, i)\right)\right)\underline{S} \\
&= \left(2^{n-1}\underline{G} + 2^{n}(N,0)\right)\underline{S} \\
&= 6(2^{n-1})(2^{n-2})\underline{G} + (2^{n})(2^{n-2})\underline{S} \\
&= 3(2^{2n-2})\underline{G} + (2^{2n-2})\underline{S}.
\end{aligned}$$

Since $N + H_1' + H_2' + H_3' = E$, the proof is complete. □

Proposition 3.11. *Let $\mathbb{V}_n$ be an n-dimensional vector space over $GF(2)$ and let $B = \mathbb{Z}_4$. Consider the collection $\{H_1, H_2, H_3, \ldots, H_{2^{n-m+1}}\}$ of m-dimensional subspaces of $\mathbb{V}_n$ that intersect in a fixed $(m-1)$-dimensional subspace N. Define $H_i' = H_i \backslash N$ for $i \in \{1, 2, \ldots, 2^{n-m+1}\}$.*

Then the set $S = \left(\bigcup_{i=1}^{2^{n-m+1}}(H_i', 1)\right) \cup \left(\bigcup_{i=1}^{2^{n-m+1}}(H_i', 2)\right)$ forms a PGDS with parameters $\beta = 3 \cdot 2^{2n}$ and $\alpha = 2^{2n}$ in the group $G = E \times B$.

Proposition 3.12. *Let $\mathbb{V}_n$ be an n-dimensional vector space over $GF(2)$, and let $B = \mathbb{Z}_2^l$ where $l \geq 2$. Consider the collection $\{H_1, H_2, H_3, \ldots, H_{2^{n-m+1}-1}\}$ of m-dimensional subspaces of $\mathbb{V}_n$ that intersect in a fixed $(m-1)$-dimensional subspace N. Denote $H_i' = H_i \backslash N$ for $i \in \{1, 2, \ldots, 2^{n-m+1} - 1\}$, and let $\mathcal{S} = \{H_1', H_2', H_3', \ldots, H_i', \ldots, H_j'\}$ be the corresponding subset collection of $\mathbb{V}_n$.*

Then the set $S = \bigcup_{i=1}^{2^{n-m+1}} (H_i', j)$, where $j \in \left\{\frac{1+(-1)^t}{2} + 2t \mid t \in [0, 2^{l-1} - 1]\right\}$, forms a PGDS with parameters $\beta = 3 \cdot 2^{2n+2(l-2)}$ and $\alpha = 2^{2n+2(l-2)}$ in the group $G = E \times B$.

Example 3.13. *Let $n = 3$, $m = 2$, and let B be a cyclic abelian group of order 8. Consider the 2-dimensional subspaces of $\mathbb{V}_3$:*

$$H_1 = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0)\},$$
$$H_2 = \{(0,0,0), (1,0,1), (0,1,0), (1,1,1)\},$$
$$H_3 = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\},$$

which have the intersection subset $\{(0,0,0), (0,1,0)\}$ in $\mathbb{V}_3$.

Define the set $\mathcal{S} = \{H_1', H_2', H_3', H_4'\}$, where:

$$H_1' = \{(1,0,0), (1,1,0)\},$$
$$H_2' = \{(1,0,1), (1,1,1)\},$$
$$H_3' = \{(0,0,1), (0,1,1)\},$$
$$H_4' = \{(0,0,0), (0,1,0)\}.$$

Then the set S is defined as:

$$S = \bigcup_{i=1}^{4} \{(H_i', 1), (H_i', 2), (H_i', 5), (H_i', 6)\}.$$

This set S forms a PGDS with parameters $\beta = 3(2^8)$ and $\alpha = 2^8$ in the group $G = E \times \mathbb{Z}_8$.

The proofs of Proposition 3.11 and Proposition 3.12 utilize the same counting techniques as those used in the proof of Proposition 3.10, and hence are not included here. Next, we present Proposition 3.15 as formulated in [13], followed by an alternative version for PGDS construction.

Lemma 3.14. *([13], Lemma 3.3) Let $U_1, U_2, \ldots, U_l$ be l distinct m-dimensional subspaces of $\mathbb{V}_{2m}$, where q is a prime and K is a group of order l. Assume $r \geq l \geq 2$, where r denotes the total number of such subspaces. Then the set $S = \bigcup_{i=1}^{l} (H_i, k_i)$ is a PGDS in the group $G = E \times K$.*

S forms a PGDS with parameters $\alpha = (l^2 - 1)q^m$ and $\beta = \alpha + q^{2m}$.

Proposition 3.15. *Let $H_1, H_2, \ldots, H_l$ be $n-2$-dimensional subspaces of $\mathbb{V}_n$ intersecting in 2^{n-4} elements, and let B be a group of order l where $2 \leq l \leq 5$. Consider the group $G = E \times B$.*

Define the set $S = \bigcup_{i=1}^{l} (H_i, b_i)$, where $b_i \in B$. Then, S forms a PGDS with parameters $(2^n l, 2^{n-2} l; 2^{2n-6}(l^2 - 1), (l^2 + 3)2^{2n-6})$ in the group G.

Before proving the proposition, we establish some equations in the group ring $\mathbb{Z}E$:

$$\underline{H_i H_i} = 2^{n-2} \underline{H_i}$$

$$\underline{H_i}\underline{H_j} = 2^{n-4} \underline{E} \quad \text{for } i \neq j$$

$$\underline{H_1} + \underline{H_2} + \cdots + \underline{H_l} = l\underline{N} + \underline{E}$$

where N denotes the $n-4$-dimensional intersection subspace. For $n = 4$, this scenario corresponds to construction 2 in [13].

Proof. We aim to demonstrate that the set S defined as $S = \sum_{i=1}^{l}(H_i, b_i)$ satisfies the group ring equation. Here $S^{-1} = \sum_{i=1}^{l}(H_i, b_i^{-1})$.

$$\begin{aligned}
\underline{S}\underline{S^{-1}}\underline{S} &= \left(\sum_{i=1}^{l} \underline{(H_i, b_i)}\right)\left(\sum_{j=1}^{l} \underline{(H_j, b_j^{-1})}\right)\left(\sum_{t=1}^{l} \underline{(H_t, b_t)}\right) \\
&= \left(\sum_{i=1}^{l} \underline{(H_i, b_i)} + \sum_{i\neq j} \underline{(H_iH_j, b_ib_j^{-1})}\right)\left(\sum_{t=1}^{l} \underline{(H_t, b_t)}\right) \\
&= \left(2^{n-2}\sum_{i=1}^{l} \underline{(H_i, e_B)} + l2^{n-4}(\underline{G} - \underline{(E, e_B)})\right)\left(\sum_{t=1}^{l} \underline{(H_t, b_t)}\right) \\
&= 2^{2n-4}\sum_{i=1}^{l} \underline{(H_i, b_i)} + 2^{n-2}\sum_{i\neq j} \underline{(H_iH_t, b_t)} + l^2 2^{2n-6}\underline{G} - l2^{2n-6}\underline{G} \\
&= 2^{2n-4}\sum_{i=1}^{l} \underline{(H_i, b_i)} + \left(2^{2n-6}(l-1) + l^2 2^{2n-6} - l2^{2n-6}\right)\underline{G}
\end{aligned}$$

Therefore, S is a PGDS with parameters $(2^n l, 2^{n-2}l; 2^{2n-6}(l^2-1), (l^2+3)2^{2n-6})$ in $G = E \times B$. □

Example 3.16. *Let $n = 6$ and H_1, H_2 be 4-dimensional subspaces of $\mathbb{V}_6$ intersecting in the set* $\{(0,0,0,0,0,0), (1,0,0,0,0,0), (0,1,0,0,0,0), (1,1,0,0,0,0)\}$. *Then the set $S = \{(H_1, b_1), (H_2, b_2)\}$ is a PGDS with parameters* $(128, 32; 192, 448)$ *in $G = E \times B$ where B is a cyclic group of order* 2.

Note that;
Let $\{H_1, H_2, \ldots, H_l\}$ denote a collection of $n-2$-dimensional subspaces of $\mathbb{V}_n$ intersecting in 2^{n-4} elements as in Proposition 3.15. If we choose a cyclic group B of order $l+1$ (not l), same set $S = \bigcup_{i=1}^{l}(H_i, b_i)$ in Proposition 3.15 also forms a PGDS with parameters $(2^n(l+1), 2^{n-2}l; 2^{2n-6}(l^2-l), (l^2-l+4)2^{2n-6})$.

4. Partial Coset Constructions of PGDS

Now, we will demonstrate that the partial spread constructions of vector spaces can be translated into the context of direct products of cyclic groups, a process we will refer to as partial coset construction. Before proceeding, it is essential to present the following significant theorem.

Theorem 4.1. *Let S be a PGDS with parameters $(v, k; \alpha, \beta)$ in a group G, and let H be a group of order n. Then the set $A = S \times H = \{(x, y) : x \in S, y \in H\}$ is a PGDS with parameters $(nv, nk; n^2\alpha, n^2\beta)$ in $G \times H$.*

Proof. Given that S is a PGDS with parameters $(v, k; \alpha, \beta)$ in the group G, we have the fundamental group ring equation:

$$\underline{S}\underline{S^{-1}}\underline{S} = (\beta - \alpha)\underline{S} + \alpha\underline{G}.$$

Our goal is to demonstrate that the set $A = S \times H$ is a PGDS in the group $G \times H$ with the corresponding parameters $(nv, nk; n^2\alpha, n^2\beta)$.

Consider the group ring $\mathbb{Z}(G \times H)$. We can express the set A in this group ring as:

$$\underline{A} = \sum_{x \in S, y \in H} (x, y).$$

Next, we need to verify that A satisfies the PGDS equation in $\mathbb{Z}(G \times H)$. First, observe the inverse set A^{-1}:

$$A^{-1} = \{(x^{-1}, y^{-1}) \mid (x, y) \in A\}.$$

Thus, we have:

$$\underline{A}^{-1} = \sum_{(x,y) \in A} (x^{-1}, y^{-1}).$$

Now, let's compute $\underline{AA}^{-1}\underline{A}$:

$$\underline{AA}^{-1}\underline{A} = \left(\sum_{(x,y) \in A} (x, y)\right)\left(\sum_{(x',y') \in A^{-1}} (x', y')\right)\left(\sum_{(x'',y'') \in A} (x'', y'')\right)$$
$$= \left(\sum_{x \in S, y \in H} (x, y)\right)\left(\sum_{x' \in S^{-1}, y' \in H} (x', y')\right)\left(\sum_{x'' \in S, y'' \in H} (x'', y'')\right).$$

Using the distributive property of the group ring, we can separate the contributions of S and H:

$$\underline{AA}^{-1}\underline{A} = \left(\underline{SS}^{-1}\underline{S}\right) \times \left(\sum_{y, y', y'' \in H} (yy'y'')\right).$$

Since H is a group of order n, we have $\underline{HH}^{-1} = \underline{HH} = n\underline{H}$ and thus,

$$\sum_{y, y', y'' \in H} (yy'y'') = |H|^2 \underline{H} = n^2 \underline{H}.$$

Therefore, the above expression simplifies to:

$$\underline{AA}^{-1}\underline{A} = \left((\beta - \alpha)\underline{S} + \alpha\underline{G}\right) \times n^2 \underline{H}.$$

Distributing the multiplication, we get:

$$\underline{AA}^{-1}\underline{A} = n^2(\beta - \alpha)\underline{A} + n^2\alpha\underline{G \times H}.$$

This confirms that A satisfies the group ring equation for a PGDS with parameters $(nv, nk; n^2\alpha, n^2\beta)$ in $G \times H$. Hence, $A = S \times H$ is indeed a PGDS with the desired parameters. □

This theorem is a valuable tool for constructing PGDSs from existing ones. The following lemma seems as an application of the theorem but gives a PGDS that has parameters $(9, 3; 2, 5)$ and it is not from the existing one.

Corollary 4.2. *Let A be a group of order $3n$ and N be a subgroup of A of order n. Consider the cosets N, H_1, H_2 of N as a coset partition of A. Further, let B be an additive group $\mathbb{Z}_3$. Then, the sets*

$$\{\{(H_1, i) \cup (H_1, j) \cup (H_2, k)\} : i, j, k \in \{0, 1, 2\}, i \neq j\}$$

and

$$\{\{(H_2, i) \cup (H_2, j) \cup (H_1, k)\} : i, j, k \in \{0, 1, 2\}, i \neq j\}$$

are PGDS in $A \times B$ with parameters $(3^2n, 3n; 2n^2, 5n^2)$.

Proof. Let H_1 and H_2 be the cosets of N. We will do the proof for an arbitrary set $S = \{(H_1, 1) \cup (H_1, 0) \cup (H_2, 2)\}$. We have following equations; $H_1^{-1} = H_2$, $H_2^{-1} = H_1$, $N^{-1} = N$, $H_1H_2 = nN$, $H_1H_1 = nH_2$ and $H_2H_2 = nH_1$. Then the rest of the proof is just to show that the set S satisfies the PGDS equation in the group ring $\mathbb{Z}(A \times B)$. Note that $S^{-1} = \{x^{-1} : x \in S\} = \{(H_2, 2), (H_1, 1), (H_2, 0)\}$.

$$
\begin{aligned}
\underline{S}\,\underline{S^{-1}}\,\underline{S} &= \left(\underline{(H_1,1)} + \underline{(H_2,2)} + \underline{(H_1,0)}\right)\left(\underline{(H_2,2)} + \underline{(H_1,1)} + \underline{(H_2,0)}\right)\underline{S} \\
&= \left((3n)\underline{(N,0)} + n\underline{(A,1)} + \underline{(A,2)}\right)S \\
&= 3n^2\underline{S} + 2n^2\underline{G}
\end{aligned}
$$

Therefore, S is a PGDS with parameters $(3^2n, 3n; 2n^2, 5n^2)$ in $G = E \times B$ in $A \times B$. □

Example 4.3. *Let $A = \mathbb{Z}_6$ and $B = \mathbb{Z}_3$. The set $\{(1, 1), (4, 1), (1, 2), (4, 2), (2, 2), (5, 2)\}$ is a PGDS in $A \times B$ with parameters $(18, 6; 8, 20)$.*

Corollary 4.4. *Let A be a group of order $4n$ and N be a subgroup of A of order n. Consider the cosets N, H_1, H_2, H_3 as coset partition of A. Further, let B be an additive group $\mathbb{Z}_3$. Then the set $S = \bigcup_{i=1}^{3}((H_i, i) \cup (H_i, 0))$ is a PGDS with parameters $(16n, 6n; 12n^2, 16n^2)$ in $A \times B$.*

In [11], they constructed partial geometric difference families in the group $\mathbb{Z}_n$ where $n = 4l$ for some integer l and in [9] generalized the idea for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ as in Theorem 4.5. We constructed PGDSs in Corollary 4.6 by modifying this theorem.

Theorem 4.5. *(Theorem 3.13 [9]) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_n$ where $n = 4l$ for some positive integer l. Let $H =< 4 >$ be the unique subroup of $\mathbb{Z}_n$ of order l. Define $H + i = \{z + i | z \in H\} = \{x \in \mathbb{Z}_n | x \equiv i \mod 4\}$ for $i = 0, 1, 2, 3$ (that is, the cosets of H in $\mathbb{Z}_n$). Then both $\{0\} \times (H \cup (H + 1)) \cup \{1\} \times (H \cup (H + 3))$ and $\{1\} \times (H \cup (H + 1)) \cup \{0\} \times (H \cup (H + 3))$ are partial sets of geometric differences in G with parameters $(8l, 4l; 6l^2, 10l^2)$.*

We also note that the set $\{0\} \times (H + 2) \cup \{1\} \times ((H + 2) \cup (H + 1) \cup (H + 3))$ is a PGDS in G with same parameters.

Corollary 4.6. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 6l$ for a positive integer l. Let $H =< 3 >$ be a unique subgroup of $\mathbb{Z}_n$ of order $2l$. Let us define the cosets of H as $H_i = \{x \in \mathbb{Z}_n : x \equiv i \mod 3\}$, $H_i^{odd} = \{x \in H_i : x \equiv 1 \mod 2\}$ and $H_i^{even} = \{x \in H_i : x \equiv 0 \mod 2\}$. Then the following sets are PGDS with parameters $(12l, 4l; 4l^2, 8l^2)$ in G.*

$$
\begin{aligned}
&H_1 \times \{0\} \cup H_2^{odd} \times \{0\} \cup H_2^{even} \times \{1\} \\
&H_1 \times \{0\} \cup H_2^{odd} \times \{1\} \cup H_2^{even} \times \{0\} \\
&H_2 \times \{0\} \cup H_1^{odd} \times \{0\} \cup H_1^{even} \times \{1\} \\
&H_2 \times \{0\} \cup H_1^{odd} \times \{1\} \cup H_1^{even} \times \{0\}
\end{aligned}
$$

The proof of the corollary is a straightforward counting, so it is omitted.

Example 4.7. *Let $G = \mathbb{Z}_6 \times \mathbb{Z}_2$ for $l = 1$. Then the set $\{(1, 1), (4, 0), (2, 1), (5, 1)\}$ is a PGDS in G with parameters $(12, 4; 4, 8)$.*

5. Conclusion

We listed partial geometric difference sets constructed in this paper in the following tables.

Table 2: The Parameters of PGDS Constructed in This Paper.

v	k	α	$\beta-\alpha$	Comment	Citiation
2^{2m+l}	$2^m(2^m-1)$	$2^{4m-l}-3\cdot 2^{3m-l}+2^{2m-l+1}$	2^{2m}	$m\geq 1, 1\leq l\leq m-t$	Proposition 3.1
2^{n+1}	2^n	$15(2^{2n-5})$	2^{2n-4}	$n\geq 2$	Corollary 3.5
2^{n+2}	2^n	$15(2^{2n-6})$	2^{2n-4}	$n\geq 2$	Corollary 3.5
2^{n+1}	2^n	$3(2^{2n-3})$	2^{2n-2}	$n\geq 2$	Corollary 3.7
2^{3m+l}	2^{3m}	$2^{6m-l}-2^{3m-l+1}$	2^{6m+1}	$m\geq 1, 1\leq l\leq m-t$	Corollary 3.9
2^{n+2}	$3\cdot 2^{n-1}$	$3\cdot 2^{2n-2}$	2^{2n-2}	$n\geq 2$	Proposition 3.10
2^{n+2}	2^{n+1}	2^{2n}	2^{2n+1}	$n\geq 2$	Proposition 3.11
2^{n+2}	2^n	$2^{2n+2l-4}$	$2^{2n+2l-3}$	$1\leq l\leq m-t, l\geq 2$	Proposition 3.12
$l\cdot 2^n$	$l\cdot 2^{n-2}$	$2^{2n-6}(l^2-1)$	2^{2n-4}	$2\leq l\leq 5, n\geq 2$	Proposition 3.15
$9n$	$3n$	$2n^2$	$3n^2$	$n\geq 1$	Corollary 4.2
$16n$	$6n$	$12n^2$	$4n^2$	$n\geq 1$	Corollary 4.4
$12l$	$4l$	$4l^2$	$4l^2$	$l\geq 1$	Corollary 4.6

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