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Structure of the rough Hausdorff limit set of sequences of balls in normed spaces

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Abstract. In this paper, we study the rough Hausdorff convergence of a sequence (B_n) of balls in normed spaces. We also discuss the following questions:

(Q_1) Does the rough Hausdorff limit set of the sequence (B_n) consist of balls?

(Q_2) Is the set of intersections or unions of sets taken from the rough Hausdorff limit set of the sequence (B_n) a ball?

1. Introduction

In this paper, we define the rough Hausdorff limit set of a sequence (B_n) of balls and examine its structure and some properties. While examining the structure of this set, we use the definitions of Chebyshev radii to show whether it is ball or not.

Papini and Wu [11] studied the Hausdorff convergence of sequences of both sets and balls in Banach spaces. They showed that if a sequence (B_n) of balls is Hausdorff convergent, then its limit set is a ball (see Proposition 6). They also proved that if (B_n) is a uniformly bounded increasing sequence, then $B = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} B_n\right)$ is a ball and $H(B_n, B) \to 0$ as $n \to \infty$ (see Proposition 7). Additionally, Albayrak [1] generalized some of the results given for sequences of sets in [11] by using the concept of ideal convergence.

On the other hand, the structure of the union of unbounded increasing sequences of balls in a normed space X has been examined to characterize some geometric properties of the dual space X^*. It has been shown that for the unbounded increasing sequence (B_n), $B_* = \bigcup_{n=1}^{\infty} B_n$ is a cone in finite dimensional normed spaces, but not in infinite dimensional normed spaces (see [5, Proposition 2.7 and Theorem 2.16]).

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In Examples 3.4 and 3.6, we show that the answers to questions (Q_1) and (Q_2) are "no" in finite and infinite dimensional normed spaces, respectively. We mention the properties of the rough Hausdorff limit set of the sequence (B_n) such as monotonicity, closedness and convexity. We also characterize the relationship between the Hausdorff convergence and the rough Hausdorff convergence of the sequence (B_n) (see Proposition 3.12).

2. Preliminaries

Let $(X, \|.\|)$ be a normed space. $Cl(X)$ and $\mathcal{B}(X)$ denote the class of all nonempty closed subsets and the class of all nonempty, closed and bounded subsets of X, respectively.

Let (x_n) be a sequence of real numbers and r be a nonnegative real number. The sequence (x_n) is said to be rough convergent to x, if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\|x_n - x\| < r + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $x_n \xrightarrow{r} x$ as $n \to \infty$ [12].

Define $LIM^r x_n = \{x \in X : x_n \xrightarrow{r} x\}$, which is called the *rough limit set* of (x_n) [12].

For a sequence (x_n), the *rough limit superior set* and the *rough limit inferior set* are defined as follows:

$$LIMSUP^r x_n = \underset{n\to\infty}{LIM^r} \sup_{k \geq n} x_k$$

and

$$LIMINF^r x_n = \underset{n\to\infty}{LIM^r} \inf_{k \geq n} x_k \text{ [4].}$$

The *open ball* $S(a, \varepsilon)$ with centre $a \in X$ and radius $\varepsilon > 0$ is defined as

$$S(a, \varepsilon) = \{x \in X : \|x - a\| < \varepsilon\}$$

whereas the *closed ball* $\mathrm{B}(a, \varepsilon)$ with centre $a \in X$ and radius $\varepsilon > 0$ is defined as

$$\mathrm{B}(a, \varepsilon) = \{x \in X : \|x - a\| \leq \varepsilon\}.$$

For $A, B \in Cl(X)$, the *Hausdorff distance* between A and B is defined by

$$H(A, B) = \max\{h(A, B), h(B, A)\},$$

where

$$h(A, B) = \sup_{a \in A} d(a, B) \text{ and } d(a, B) = \inf_{b \in B} \|a - b\|.$$

Equivalently,

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq S_\varepsilon(B) \text{ and } B \subseteq S_\varepsilon(A)\},$$

where

$$S_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}$$

is the *ε-enlargement* of A. The closure of ε-enlargement of A is denoted by $\overline{S}_\varepsilon(A)$. It is clear that $\overline{S}_\varepsilon(A) = \mathrm{B}(A, \varepsilon)$.

Let $A, A_n \in Cl(X)$ $(n \in \mathbb{N})$. A sequence (A_n) is said to be *Hausdorff convergent* to A if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H(A_n, A) = \max\{h(A_n, A), h(A, A_n)\} < \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $A_n \xrightarrow{H} A$ or $H(A_n, A) \to 0$ as $n \to \infty$.

The *diameter* $\delta(A)$ of a nonempty set A is defined to be

$$\delta(A) = \sup_{a_1, a_2 \in A} \|a_1 - a_2\|.$$

Now we give the definitions of Chebyshev radii, which are necessary to show whether a set is ball or not (see [2, 3, 7] for details).

Definition 2.1. *Let A be a nonempty bounded subset of X. We denote*

$$R(x, A) = \inf\{\varepsilon > 0 : \mathrm{B}(x, \varepsilon) \supset A\} = \sup_{a \in A} \|x - a\| \ (x \in X)$$

and

$$R'(x, A) = \sup\{\varepsilon > 0 : \mathrm{B}(x, \varepsilon) \subset A\} \ (x \in A).$$

•*The relative Chebyshev radius of A in Y is defined by $R_Y(A) = \inf_{y \in Y} R(y, A)$ $(Y \subseteq X)$.*
•*The Chebyshev radius of A is defined by $R(A) = R_X(A)$.*
•*The self Chebyshev radius of A is defined by $R_A(A) = \inf_{a \in A} R(a, A)$.*
•*The inner Chebyshev radius of A is defined by $R'(A) = \sup_{x \in A} R'(x, A)$.*

It is clear that

$$0 \leq R'(A) \leq R(A) \leq R_A(A) \leq \delta(A).$$

Example 2.2. *Consider the space $\mathbb{R}^2$ endowed with the taxicab norm. Let A be a trapezoid with vertices at the points $(-3, 1), (1, 1), (3, -3)$ and $(-5, -3)$, and let $Y = \left\{(x, y) \in \mathbb{R}^2 : y = -5\right\}$. Then, we have*

$$\delta(A) = 10, \ R(A) = 6, \ R_Y(A) = 8, \ R_A(A) = 6 \text{ and } R'(A) = 0.$$

Let us give a useful lemma expressing the criteria for a set to be ball.

Lemma 2.3 ([10]). *Let $A \in Cl(X)$. A is a ball if and only if*

$$R'(A) = \frac{\delta(A)}{2} \Leftrightarrow R(A) = R'(A) \Leftrightarrow R_A(A) = R'(A). \tag{1}$$

It is obvious that the set A in Example 2.2 is not a ball.

3. Main Results

In this section, we study on rough Hausdorff convergence of a sequence $(\mathrm{B_n})$ of balls. The concept of rough Hausdorff convergence of sequences of sets was first introduced by Ölmez et al. [9]. We express our results for sequences of balls by generalizing this concept given in metric spaces to normed spaces.

Throughout this paper, let (A_n) and $(\mathrm{B_n})$ denote the sequence of sets and the sequence of closed balls in X, respectively. For simplicity, we suppose that $A, A_n \in \mathcal{B}(X) (n \in \mathbb{N})$.

Definition 3.1. *A sequence $(\mathrm{B_n})$ is said to be $r-$Hausdorff convergent to A if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that*

$$H(\mathrm{B_n}, A) = \max\{h(\mathrm{B_n}, A), h(A, \mathrm{B_n})\} < r + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $\mathrm{B_n} \xrightarrow{r-H} A$ or $H(\mathrm{B_n}, A) \xrightarrow{r} 0$ as $n \to \infty$.

Although the limit of a convergent sequence is unique in classical theory, this is not the case in rough convergence theory. That is, the sequence $(\mathrm{B_n})$ can have many limit sets depending on the given degree of roughness. For this reason, the structure and properties of the set consisting of rough limit sets are very important. Now we will state the concept of rough Hausdorff limit set of the sequence $(\mathrm{B_n})$. We note that the following definitions are similar to definitions (2.1) and (2.3) in [9].

Definition 3.2. *Define*

$$1 - LIM^r\mathrm{B_n} = \{A \subset X : H(\mathrm{B_n}, A) \xrightarrow{r} 0\},$$

which is called the rough Hausdorff limit set of $(\mathrm{B_n})$.
Define

$$2 - LIM^r\mathrm{B_n} = \bigcap_{C \in \mathcal{L}_{(\mathrm{B_n})}} \mathrm{B}(C, r),$$

where $\mathcal{L}_{(\mathrm{B_n})}$ *is the family of all limit sets of* $(\mathrm{B_n})$.

The set $1 - LIM^r\mathrm{B_n}$ consists of sets to which the sequence $(\mathrm{B_n})$ is rough Hausdorff convergent. In other words, it corresponds to the set of sets. The set $2 - LIM^r\mathrm{B_n}$ corresponds to a single set. These sets cannot be compared to each other because their structures are different. Unlike in [12, Proposition 3.5(b)], this shows that the definitions do not coincide with each other.

First, we examine the structure of the rough Hausdorff limit set of the sequence $(\mathrm{B_n})$. Namely, when the sequence $(\mathrm{B_n})$ is rough Hausdorff convergent to A, we examine whether A is a ball. As in [11, Proposition 6], we expected that any set taken from $1 - LIM^r\mathrm{B_n}$ would correspond to a ball. However, as can be seen from the following example, the rough Hausdorff limit of the sequence $(\mathrm{B_n})$ is not always a ball.

Example 3.3. *In the space* $\mathbb{R}^2$ *with usual norm, let us the sequence* $(\mathrm{B_n})$ *be defined by*

$$\mathrm{B_n} = \mathrm{B}\left(\left(1 - \frac{1}{n}, 0\right), 2 + \frac{1}{n}\right).$$

Take $A = \mathrm{B}((1,0), 3)$. *Then,* $H(\mathrm{B_n}, A) \xrightarrow{r} 0$ *as* $n \to \infty$ *for* $r = 1$ *and so* A *is a ball. Moreover, let's consider a rectangle* D *with vertices* $(-1, \mp 1)$ *and* $(3, \mp 1)$ *in* A. *Then, we see that* $H(\mathrm{B_n}, D) \xrightarrow{r} 0$ *as* $n \to \infty$ *for* $r = 1$. *Since* $\delta(D) = 2\sqrt{5}$ *and* $R'(D) = 0$, *we have* $R'(D) \neq \frac{\delta(D)}{2}$. *From (1), we say that* D *is not a ball in* $\mathbb{R}^2$.

So, is the set of intersections or unions of sets to which a sequence $(\mathrm{B_n})$ be rough Hausdorff convergent a ball? That is, when the sequence $(\mathrm{B_n})$ is rough Hausdorff convergent, do the sets $\bigcap 1 - LIM^r\mathrm{B_n}$ and $\bigcup 1 - LIM^r\mathrm{B_n}$ correspond to a ball? Now we will give answers to these questions in the following example.

Example 3.4. *Consider the space* $\mathbb{R}^2$ *with the max norm. Define the sequence* $(\mathrm{B_n})$ *as follows:*

$$\mathrm{B_n} = \begin{cases} \mathrm{B}((-4,2), 2) & , \text{ if } n \text{ is odd} \\ \mathrm{B}((1,1), 1) & , \text{ if } n \text{ is even} \end{cases}.$$

Since the limit sets of the sequence $(\mathrm{B_n})$ *are two different balls, it is not Hausdorff convergent. For* $r = 6$, *we obtain*

$$2 - LIM^r\mathrm{B_n} = \overline{S}_r(\mathrm{B}((-4,2), 2)) \cap \overline{S}_r(\mathrm{B}((1,1), 1)) = D_*,$$

where D_* *is a rectangle with vertices at the points* $(-6, 8), (-6, -6), (4, -6)$ *and* $(4, 8)$. *Then we have* $H(\mathrm{B_n}, D_*) \xrightarrow{r} 0$ *as* $n \to \infty$ *for* $r = 6$. *In addition, we get* $\delta(D_*) = 14$ *and* $R'(D_*) = 5$. *By (1),* D_* *is not a ball. This implies that the set* $2 - LIM^r\mathrm{B_n}$ *is not always a ball.*
Let's consider the rectangles D_1 *and* D_2 *in* D_*, *whose vertices are*

$$(-6, -2), (-6, -6), (4, -6), (4, -2)$$

and

$$(-6,8),(-6,0),(4,0),(4,8),$$

respectively. Then we have $H(\mathrm{B_n}, D_1) \xrightarrow{r} 0$ *and* $H(\mathrm{B_n}, D_2) \xrightarrow{r} 0$ *as* $n \to \infty$ *for* $r = 6$*. We also obtain*

$$D_1 \cap D_2 = \varnothing$$

for the sets $D_1, D_2 \in 1 - LIM^r\mathrm{B_n}$*. It follows that*

$$\bigcap 1 - LIM^r\mathrm{B_n} = D_1 \cap D_2 \cap \ldots = \varnothing$$

for $r = 6$*. As a consequence, the set* $\bigcap 1 - LIM^r\mathrm{B_n}$ *is not always a ball.*
On the other hand, since the largest rough Hausdorff limit set of $(\mathrm{B_n})$ *for* $r = 6$ *is* D_**, we get*

$$\bigcup 1 - LIM^r\mathrm{B_n} = D_1 \cup D_2 \cup \ldots = D_*.$$

It is clear that the set $\bigcup 1 - LIM^r\mathrm{B_n}$ *is not always a ball.*

Remark 3.5. *As can be seen from Example 3.4, the sets* $\bigcap 1 - LIM^r\mathrm{B_n}$*,* $\bigcup 1 - LIM^r\mathrm{B_n}$ *and* $2 - LIM^r\mathrm{B_n}$ *are not always balls in finite dimensional normed spaces.*

The following example shows that sets $\bigcap 1 - LIM^r\mathrm{B_n}$ and $\bigcup 1 - LIM^r\mathrm{B_n}$ are not balls in infinite dimensional normed spaces.

Example 3.6. *Let* (e_n) *denote the standart basis in the space* c_0*. Define a sequence* $(\mathrm{B_n})$ *by*

$$\mathrm{B_n} = \mathrm{B}\left(\frac{x_n}{n}, 1 + \frac{1}{n}\right),$$

where

$$u_n = \sum_{k=1}^{n} e_k \text{ and } x_n = \sum_{k=1}^{n} u_k.$$

It's easy to see that

$$\mathrm{B_n} = \left\{(\alpha_1, \alpha_2, \ldots) \in c_0 : \alpha_i \in \left[-\frac{i}{n}, 2 - \left(\frac{i-2}{n}\right)\right],\ \forall i \le n;\ \alpha_i \in \left[-1 - \frac{1}{n}, 1 + \frac{1}{n}\right],\ \forall i > n\right\}.$$

Set

$$\begin{aligned} A_1 &= \left\{\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, 0, 0, \ldots\right)\right\}, \\ A_2 &= \left\{\left(-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, 0, 0, \ldots\right)\right\}, \\ A_3 &= \{(\alpha_1, \alpha_2, \ldots) \in c_0 : \alpha_i \in [-2, 3],\ \forall i \in \mathbb{N}\}. \end{aligned}$$

Then we have $H(\mathrm{B_n}, A_1) \xrightarrow{r} 0$*,* $H(\mathrm{B_n}, A_2) \xrightarrow{r} 0$ *and* $H(\mathrm{B_n}, A_3) \xrightarrow{r} 0$ *as* $n \to \infty$ *for* $r = 2$*. We also obtain*

$$A_1 \cap A_2 = \varnothing$$

for the sets $A_1, A_2 \in 1 - LIM^r\mathrm{B_n}$*. It follows that*

$$\bigcap 1 - LIM^r\mathrm{B_n} = A_1 \cap A_2 \cap A_3 \cap \ldots = \varnothing$$

for $r = 2$. Consequently, the set $\bigcap 1 - LIM^r \mathrm{B_n}$ is not always a ball in c_0.
On the other hand, we take

$$A_{n_0} = \left\{(\alpha_1, \alpha_2, \ldots) \in c_0 : \alpha_i \in \left[-2 - \frac{i}{n_0}, 4 - \left(\frac{i-2}{n_0}\right)\right],\ \forall i \leq n_0;\ \alpha_i \in [-2, 3]\,,\ \forall i > n_0\right\}.$$

Obviously, $H(\mathrm{B_n}, A_{n_0}) \xrightarrow{r} 0$ as $n \to \infty$ for $r = 2$. Then,

$$\bigcup_{n_0 \in \mathbb{N}} A_{n_0} = \{(\alpha_1, \alpha_2, \ldots) \in c_0 : \alpha_i \in [-2, 4]\,,\ \forall i \leq n;\ \alpha_i \in [-2, 3]\,,\ \forall i > n\}\,.$$

Therefore, we obtain

$$\bigcup 1 - LIM^r \mathrm{B_n} = \bigcup_{n_0 \in \mathbb{N}} A_{n_0}.$$

It is clear that the set $\bigcup 1 - LIM^r \mathrm{B_n}$ is not always a ball in c_0.

Now let's talk about some properties of the rough Hausdorff limit set of $(\mathrm{B_n})$.

Proposition 3.7. *The diameter of $1 - LIM^r \mathrm{B_n}$ is not greater than $2r$.*

Proof. We show that

$$\delta\left(1 - LIM^r \mathrm{B_n}\right) = \sup\left\{H(A_1, A_2) : A_1, A_2 \in 1 - LIM^r \mathrm{B_n}\right\} \leq 2r.$$

Conversely, assume that $\delta\left(1 - LIM^r \mathrm{B_n}\right) > 2r$. Then there exist $A_1, A_2 \in 1 - LIM^r \mathrm{B_n}$ such that $\alpha := H(A_1, A_2) > 2r$. Let $0 < \varepsilon < \frac{\alpha}{2} - r$. Since $A_1, A_2 \in 1 - LIM^r \mathrm{B_n}$, there exist $N_1(\varepsilon), N_2(\varepsilon) \in \mathbb{N}$ such that

$$H(\mathrm{B_n}, A_1) < r + \varepsilon \text{ for all } n \geq N_1(\varepsilon)$$

and

$$H(\mathrm{B_n}, A_2) < r + \varepsilon \text{ for all } n \geq N_2(\varepsilon)\,.$$

Define $\widetilde{N}(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. Then, we have

$$\begin{aligned} H(A_1, A_2) &\leq H(A_1, \mathrm{B_n}) + H(\mathrm{B_n}, A_2) \\ &< r + \varepsilon + r + \varepsilon = 2(r + \varepsilon) \\ &< 2\frac{\alpha}{2} = \alpha \end{aligned}$$

for all $n \geq \widetilde{N}(\varepsilon)$. It follows that $H(A_1, A_2) < H(A_1, A_2)$, which is a contradicts. □

Proposition 3.8. (i) *If $(\mathrm{B_{k_n}})$ is a subsequence of $(\mathrm{B_n})$, then*

$$1 - LIM^r \mathrm{B_n} \subseteq 1 - LIM^r \mathrm{B_{k_n}}.$$

(ii) *If $r_1 \leq r_2$ then*

$$1 - LIM^{r_1} \mathrm{B_n} \subseteq 1 - LIM^{r_2} \mathrm{B_n} \text{ and } 2 - LIM^{r_1} \mathrm{B_n} \subseteq 2 - LIM^{r_2} \mathrm{B_n}.$$

Proof. The proofs of (i) and (ii) are analogous to that of Propositions 2.3 and 2.4 in [9], respectively. □

Proposition 3.9. *The set $1 - LIM^r \mathrm{B_n}$ is closed.*

Proof. From $1 - LIM^r \mathrm{B_n}$, let's take an arbitrary sequence (A_m) satisfying $H(A_m, A_*) \to 0$ as $m \to \infty$. We show that $A_* \in 1 - LIM^r \mathrm{B_n}$. Given $\varepsilon > 0$. By assumption, there exists an $M(\varepsilon) \in \mathbb{N}$ such that

$$H(A_m, A_*) < \frac{\varepsilon}{2}$$

for all $m \geq M(\varepsilon)$.

Since $(A_m) \subset 1 - LIM^r \mathrm{B_n}$, we have $A_{M(\varepsilon)} \in 1 - LIM^r \mathrm{B_n}$. Then, there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H\left(\mathrm{B_n}, A_{M(\varepsilon)}\right) < r + \frac{\varepsilon}{2}$$

for all $n \geq N(\varepsilon)$. Define

$$\widetilde{N}(\varepsilon) := \max\left\{M(\varepsilon), N(\varepsilon)\right\}.$$

Then, we have

$$\begin{aligned} H(\mathrm{B_n}, A_*) &\leq H\left(\mathrm{B_n}, A_{M(\varepsilon)}\right) + H\left(A_{M(\varepsilon)}, A_*\right) \\ &< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon \end{aligned}$$

for all $n \geq \widetilde{N}(\varepsilon)$. Thus we obtain $A_* \in 1 - LIM^r \mathrm{B_n}$, which completes the proof. □

To prove the next proposition we need the following lemma.

Lemma 3.10 ([6]). *Let Y be a Banach space. If A, A_1, C, C_1 are nonempty bounded subsets of Y then*
(i) $H(\alpha A, \alpha C) = \alpha H(A, C)$ for all $\alpha \geq 0$,
(ii) $H(A + C, A_1 + C_1) \leq H(A, A_1) + H(C, C_1)$.

We know that every finite dimensional normed space is complete [8, Theorem 2.4.2]. Since Lemma 3.10 is satisfied in Banach spaces, the next proposition is given in finite dimensional normed spaces.

Proposition 3.11. *If X is a finite dimensional normed space, then $1 - LIM^r \mathrm{B_n}$ is convex.*

Proof. Let $A_1, A_2 \in 1 - LIM^r \mathrm{B_n}$ be arbitrary. Given $\varepsilon > 0$. Then, there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H(\mathrm{B_n}, A_1) < r + \varepsilon \text{ and } H(\mathrm{B_n}, A_2) < r + \varepsilon$$

for all $n \geq N(\varepsilon)$.

We show that $(1 - t)A_1 + tA_2 \in 1 - LIM^r \mathrm{B_n}$ for all $t \in [0, 1]$. Using Lemma 3.10 (i) and (ii), we have

$$\begin{aligned} H\left(\mathrm{B_n}, (1 - t)A_1 + tA_2\right) &= H\left((1 - t)\mathrm{B_n} + t\mathrm{B_n}, (1 - t)A_1 + tA_2\right) \\ &\leq H\left((1 - t)\mathrm{B_n}, (1 - t)A_1\right) + H\left(t\mathrm{B_n}, tA_2\right) \\ &= (1 - t)H(\mathrm{B_n}, A_1) + tH(\mathrm{B_n}, A_2) \\ &\leq (1 - t)(r + \varepsilon) + t(r + \varepsilon) = r + \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$. This implies that $(1 - t)A_1 + tA_2 \in 1 - LIM^r \mathrm{B_n}$. □

Proposition 3.12. *Assume that $r_1 \geq 0$, $r_2 > 0$ and $(\mathrm{B_n}), (\widetilde{\mathrm{B}}_\mathrm{n}) \subset X$. If there exists a sequence $(\widetilde{\mathrm{B}}_\mathrm{n})$ such that $A \in 1 - LIM^{r_1} \widetilde{\mathrm{B}}_\mathrm{n}$ and $H(\mathrm{B_n}, \widetilde{\mathrm{B}}_\mathrm{n}) \leq r_2$ for all $n = 1, 2, \ldots$, then $A \in 1 - LIM^{r_1 + r_2} \mathrm{B_n}$.*

Proof. Suppose $A \in 1 - LIM^{r_1}\widetilde{\mathrm{B}}_\mathrm{n}$. Given $\varepsilon > 0$. Then, there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$H(\widetilde{\mathrm{B}}_\mathrm{n}, A) < r_1 + \varepsilon$$

for all $n \geq N(\varepsilon)$. By assumption $H(\mathrm{B}_\mathrm{n}, \widetilde{\mathrm{B}}_\mathrm{n}) \leq r_2$ for all $n = 1, 2, \ldots$, we have

$$\begin{aligned} H(\mathrm{B}_\mathrm{n}, A) &\leq H(\mathrm{B}_\mathrm{n}, \widetilde{\mathrm{B}}_\mathrm{n}) + H(\widetilde{\mathrm{B}}_\mathrm{n}, A) \\ &< r_2 + r_1 + \varepsilon. \end{aligned}$$

It follows that $A \in 1 - LIM^{r_1+r_2}\mathrm{B}_\mathrm{n}$. □

Taking $r_1 = 0$ and $r_2 = r$ in Proposition 3.12, we can see that if $H(\widetilde{\mathrm{B}}_\mathrm{n}, A) \to 0$ as $n \to \infty$ and $H(\mathrm{B}_\mathrm{n}, \widetilde{\mathrm{B}}_\mathrm{n}) \leq r$ for all $n = 1, 2, \ldots$, then $H(\mathrm{B}_\mathrm{n}, A) \xrightarrow{r} 0$ as $n \to \infty$. This shows the relationship between Hausdorff convergence and rough Hausdorff convergence.

We recall that (B_n) is an *increasing sequence* of balls if $\mathrm{B}_\mathrm{n} \subseteq \mathrm{B}_{\mathrm{n}+1}$ for all $n \in \mathbb{N}$. Also, for an increasing sequence (B_n), if $\mathrm{B} = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathrm{B}_\mathrm{n}\right)$ is bounded, the sequence (B_n) is said to be *uniformly bounded*.

We know that if a sequence (A_n) is Hausdorff convergent then this sequence is $r-$Hausdorff convergent to the same set for each r [9]. This fact is also true for the sequence (B_n). Finally, we express the effect of some properties of the sequence (B_n) on the rough Hausdorff convergence in the following corollary. Its proof is clear from [11, Proposition 7].

Corollary 3.13. *Suppose that* (B_n) *be a uniformly bounded increasing sequence. Then* $H(\mathrm{B}_\mathrm{n}, \mathrm{B}) \xrightarrow{r} 0$ *as* $n \to \infty$, *where* $\mathrm{B} = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathrm{B}_\mathrm{n}\right)$ *is a ball.*

Conflict of Interest The authors declare that they have no conflicts of interest.

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