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# Hyers–Ulam–Rassias stability for a class of nonlinear convolution integral equations

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**Abstract.** In this paper, we study the Hyers–Ulam stability, the Hyers–Ulam–Rassias stability, and a new kind of stability, the $\sigma$–semi–Hyers–Ulam stability, for a class of nonlinear convolution integral equations. Using the fixed–point method, sufficient conditions are derived to establish these stabilities for the given class of integral equations. The analysis considers both the finite interval and the infinite interval cases. For illustrative purposes, three examples are presented to validate the theoretical outcomes.

## 1. Introduction and preliminaries

Stability theory provides a rigorous framework for analyzing the robustness of functional equations, enabling researchers to quantify the impact of small perturbations on solutions. The study of stability has profound implications for mathematical modeling, ensuring that solutions remain accurate and reliable despite inherent uncertainties or numerical errors. This stability criterion has been successfully applied in diverse fields, including image processing, machine learning, and numerical analysis, demonstrating its versatility and significance. Researchers continue to explore new aspects of stability, particularly its connections to other stability notions such as Mittag–Leffler stability [16, 45–49].

Understanding stability theory is essential for developing robust mathematical models that accurately describe real–world phenomena, making it a dynamic area of research. The theory has attracted significant interest within the mathematical community, leading to the development of novel analytical tools and techniques. The intersection of stability theory with other mathematical disciplines, such as differential equations and functional analysis, has yielded remarkable results. As a cornerstone of mathematical analysis, stability theory provides valuable insights into the behavior of functional equations [1–3, 23, 32, 39, 62, 71, 78].

In 1940, S. M. Ulam [53] introduced the concept of stability, which later became a key idea in mathematical analysis. The idea was aimed at finding when an approximation of a functional equation's solution is as close to the exact solution as feasible, as well as whether such a solution exists. In 1941, D. H.

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Hyers partially answered this question for Banach spaces, specifically for the additive Cauchy equation $f(x + y) = f(x) + f(y)$, see [11, 13, 14], thus giving rise to what we now call the Hyers–Ulam stability. In the 1970s, T. M. Rassias [68] expanded on Hyers work, introducing additional ideas and developing the Hyers–Ulam–Rassias stability. Rassias [69] contributions greatly enlarged the scope of stability research, resulting in various generalizations and applications. Other mathematicians, such as Gajda [79] and Aoki [70], further developed and improved the notions of Hyers–Ulam and Hyers–Ulam–Rassias stability. Their research looked into numerous norms, equations, and techniques for approximate solutions, resulting in a variety of stability generalizations.

The study of stability in functional equations has gained significant attention, particularly in the framework of Hyers–Ulam stability and its various extensions. Several studies have explored the stability of functional equations in various mathematical settings, including normed spaces [64], Banach spaces [24], and metric groups [65]. Notable works include those by Brzdek et al. [26] on Ulam–type stability and Cho et al. [76] on stability in random normed spaces. The Hyers–Ulam stability of differential equations has also been extensively studied, as seen in the works of Abdollahpour et al. [35, 36] on hypergeometric and Laguerre differential equations and Murali et al. [50] on second–order linear differential equations. Functional equations of trigonometric type [56], as well as their connections to Fibonacci sequences [9, 57], have also been explored. Other significant contributions include studies on cubic–quadratic–additive equations [77], Jensen–type equations [73], and inequalities related to convex functions [72]. Functional equations in probabilistic normed spaces [5], C*-algebras [10], and inner product spaces further highlight the broad applicability of stability theories. Classical references such as Aczel et al. [27] and Bourgin [15] provide a foundational perspective on transformations and functional equations, while more recent advances are presented in works by Czerwik [63], and Kannappan [43]. Additionally, extensive research on generalized stability conditions has been conducted by Lee et al. [74], Sahoo et al. [44], and Wang [28]. These contributions collectively illustrate the depth and scope of research on stability in functional equations, offering valuable insights across multiple mathematical and applied domains.

For a comprehensive treatment of the subject concerning Ulam stability, Hyers–Ulam stability, and Hyers–Ulam–Rassias stability, we refer the readers to the following works: for the existence and Ulam stability of quadratic integral equations, see Abbas and Benchohra [54]; for the Hyers–Ulam stability of integral equations, see Akkouchi [33], Castro and Guerra [29], Castro and Simões [30], Ciplea et al. [55], Jung [58], Ögrekçi et al. [59], Tunç and Tunç [40], and Tunç et al. [41]; for the Hyers–Ulam–Rassias stability of integral equations, see Bacşi et al. [75], Jung [60], and Otrocol and Ilea [12]; for the Hyers–Ulam and Hyers–Ulam–Rassias stability of ordinary differential equations with and without delay, see Graef et al. [25], Janfada and Sadeghi [34], and Tunç et al. [42]; for the Hyers–Ulam and Hyers–Ulam–Rassias stability of integral equations and integrodifferential equations, see Benzarouala and Oubbi [7], Jung [61], Tunç and Tunç and Biçer [8]; and for the Hyers–Ulam and Hyers–Ulam–Rassias stability of fractional differential equations with and without delay, see Develi and Duman [21], El–hady and Ögrekçi [17], El–hady et al. [18], Khan et al. [22], Makhlouf et al. [4], and Ouagueni and Arioua [37]. Numerous other researchers have rigorously established the stability of various equations, as detailed in [51, 52].

This work examines the Hyers–Ulam stability, the Hyers–Ulam–Rassias stability, and a new kind of stability, the $\sigma$–semi–Hyers–Ulam stability, for the nonlinear convolution integral equation, which is given by:

$$\phi(u(t)) = L(t) + \int_a^t P(t-s)u^m(s)\,ds \quad \text{for } t \in [a,b], \tag{1}$$

where $a$ and $b$ are fixed real numbers, $\phi > 0$, $L : [a,b] \to \mathbb{C}$, and $P : [a,b] \to [a,b]$ are continuous functions, and $u \in C([a,b])$.

Nonlinear convolution integral equations find wide–ranging applications in various scientific and engineering disciplines [19, 38, 66, 80]. For instance, in biological systems, nonlinear convolution equations are used to model the interaction between different biochemical substances over time. Here, the current concentration depends not only on the present inputs but also on the cumulative effects of past concentrations, often governed by nonlinear dynamics. In control theory, these equations describe systems with memory or delayed feedback, where the output at any given time is influenced by past behavior in a nonlinear manner. This is particularly relevant for modeling processes with hysteresis or saturation effects, such as those found in electrical circuits with nonlinear components. Similarly, in signal processing, nonlinear filters are designed for noise reduction or signal enhancement, where the output depends on past signals through a nonlinear relationship. In finance, these equations model asset price dynamics in markets with memory effects, where future prices depend not only on the current state but also on the historical path of the asset's value. Moreover, nonlinear convolution integral equations are crucial for understanding system behavior under nonlinearity and memory effects, making them valuable in the development of predictive models, control system design, and process optimization in real–world applications. Furthermore, stability analysis of these equations such as, Hyers–Ulam and Hyers–Ulam–Rassias stability frameworks ensures that these systems remain predictable and resilient to small perturbations or errors, which is vital for ensuring robustness in practical applications.

We now formally define the above specified stabilities for the nonlinear convolution integral Equation (1).

**Definition 1.1.** *Consider a continuous function $u$ on $[a,b]$ that satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \le \sigma(t), \quad t \in [a,b],$$

*where $\sigma$ is a non–negative function. If there exists a solution $u_0$ of the convolution integral equation and a constant $C > 0$, independent of $u$ and $u_0$, such that*

$$|u(t) - u_0(t)| \le C\sigma(t),$$

*for all $t \in [a,b]$, then the convolution integral Equation (1) is said to possess Hyers–Ulam–Rassias stability.*

**Definition 1.2.** *Consider a continuous function $u$ on $[a,b]$ that satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \le \theta, \quad t \in [a,b],$$

*where $\theta \ge 0$. If there exists a solution $u_0$ of the convolution integral equation and a constant $C > 0$, independent of $u$ and $u_0$, such that*

$$|u(t) - u_0(t)| \le C\theta,$$

*for all $t \in [a,b]$, then the convolution integral Equation (1) is said to have Hyers–Ulam stability.*

**Definition 1.3.** *Let $\sigma$ be a non–decreasing function defined on $[a,b]$. If every continuous function $u$ satisfying*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \le \theta, \quad t \in [a,b], \tag{2}$$

*where $\theta \ge 0$, admits a solution $u_0$ of the convolution integral equation and a constant $C > 0$, independent of $u$ and $u_0$, such that*

$$|u(t) - u_0(t)| \le C\sigma(t), \quad t \in [a,b], \tag{3}$$

*then the convolution integral Equation (1) is said to have $\sigma$–semi–Hyers–Ulam stability.*

**Definition 1.4 ([67]).** *Let $X$ be a nonempty set, and let $d : X \times X \to [0,+\infty]$ be a mapping. The function $d$ is referred to as a generalized metric on $X$ if and only if it satisfies the following conditions:*

(C1) $d(x,y) = 0$ *if and only if* $x = y$;

(C2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(C3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

**Theorem 1.5 ([20]).** *Let* $(X, d)$ *be a generalized complete metric space, and let* $T : X \to X$ *be a strictly contractive mapping, meaning that*

$$d(Tx, Ty) \leq Ld(x, y), \quad \forall x, y \in X,$$

*for some Lipschitz constant* $0 \leq L < 1$. *If there exists a non–negative integer k such that* $d(T^{k+1}x, T^k x) < \infty$ *for some* $x \in X$, *then the following properties hold:*

1. *The sequence* $\{T^n x\}_{n\in\mathbb{N}}$ *converges to a fixed–point* $x^*$ *of* $T$.
2. $x^*$ *is the unique fixed–point of* $T$ *in the set*

$$X^* = \{y \in X : d(T^k x, y) < \infty\}.$$

3. *If* $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1 - L} d(Ty, y). \tag{4}$$

The manuscript is structured as follows: Section 2 discusses the Hyers–Ulam–Rassias stability of the convolution integral Equation (1) in a finite interval. Section 3 examines the $\sigma$–semi–Hyers–Ulam and Hyers–Ulam stability of the convolution integral Equation (1) in a finite interval. Section 4 extends the discussion to the stabilities of the convolution integral Equation (1) in an infinite interval. Three illustrative examples are provided in Section 5, followed by the conclusion in Section 6.

## 2. Hyers–Ulam–Rassias stability in the finite interval case

In this section, we present sufficient conditions for the Hyers–Ulam–Rassias stability of the convolution integral Equation (1), where $t \in [a, b]$, for some fixed real numbers $a$ and $b$. We consider the space of continuous functions $C([a, b])$ on $[a, b]$, equipped with a generalized form of the Bielecki metric,

$$d(Tu, Tv) = \sup_{t\in[a,b]} \frac{|(Tu)(t) - (Tv)(t)|}{\sigma(t)} \tag{5}$$

where $\sigma$ is a non–decreasing continuous function $\sigma : [a, b] \to (0, \infty)$. In Equation (5), if $\sigma(t) = e^{p(t-a)}$ with $p > 0$, the metric reduces to the well–known Bielecki metric. In this study, we adopt a more general form of the metric to enhance its applicability.

We recall that the space $C([a, b])$, endowed with the generalized metric $d$, forms a complete metric space (cf. previous studies [6, 31]).

**Theorem 2.1.** *Let* $L : [a, b] \to \mathbb{C}$ *be a continuous function. Additionally, assume that* $P : [a, b] \to [a, b]$ *is a continuous function satisfying the condition that there exists* $M > 0$ *such that*

$$M = \sup_{t,s\in[a,b]} |P(t - s)|. \tag{6}$$

*Furthermore, suppose there exists* $K > 0$ *for which*

$$\int_a^t \sigma(s)\, ds \leq K\sigma(t), \tag{7}$$

*holds for all* $t \in [a, b]$.

*If $u \in C([a,b])$ satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \le \sigma(t), \quad t \in [a,b], \tag{8}$$

*and the condition $KM\phi^{-1} < 1$ holds, then there exists a unique function $u_0 \in C([a,b])$ that satisfies Equation (1), given by*

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right), \tag{9}$$

*such that*

$$|u(t) - u_0(t)| \le \frac{1}{1 - KM\phi^{-1}}\,\sigma(t), \tag{10}$$

*for all $t \in [a,b]$, which ensures that the convolution integral Equation (1) is Hyers–Ulam–Rassias stable.*

*Proof.* We define the operator $T : C([a,b]) \to C([a,b])$ by the relation

$$(Tu)(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u^m(s)\,ds\right), \tag{11}$$

for all $t \in [a,b]$ and $u \in C([a,b])$. It is important to note that if $u$ is a continuous function, then $Tu$ is also continuous. In fact,

$$\begin{aligned}
|(Tu)(t) - (Tu)(t_0)| &= \left|\phi^{-1}\left(L(t) + \int_a^t P(t-s)u^m(s)\,ds\right)\right. \\
&\quad \left. -\phi^{-1}\left(L(t_0) + \int_a^{t_0} P(t_0-s)u^m(s)\,ds\right)\right| \\
&\le \phi^{-1}\left\{|L(t) - L(t_0)|\right. \\
&\quad \left. +\left|\int_a^t P(t-s)u^m(s)\,ds - \int_a^{t_0} P(t_0-s)u^m(s)\,ds\right|\right\} \\
&= \phi^{-1}\left\{|L(t) - L(t_0)|\right. \\
&\quad +\left|\int_a^t P(t-s)u^m(s)\,ds - \int_a^t P(t_0-s)u^m(s)\,ds\right. \\
&\quad \left.\left. + \int_a^t P(t_0-s)u^m(s)\,ds - \int_a^{t_0} P(t_0-s)u^m(s)\,ds\right|\right\} \\
&\le \phi^{-1}\left\{|L(t) - L(t_0)|\right. \\
&\quad +\left|\int_a^t (P(t-s) - P(t_0-s))u^m(s)\,ds\right| \\
&\quad \left. +\left|\int_{t_0}^t P(t_0-s)u^m(s)\,ds\right|\right\} \to 0
\end{aligned}$$

when $t \to t_0$.

Under the given conditions, we now proceed to show that the operator $T$ is strictly contractive (with respect to the metric under consideration). Indeed, for all $u, v \in C([a,b])$, we have

$$\begin{aligned}
d(Tu, Tv) &= \sup_{t\in[a,b]} \frac{|(Tu)(t) - (Tv)(t)|}{\sigma(t)} \\
&= \phi^{-1} \sup_{t\in[a,b]} \frac{\left|\int_a^t P(t-s)u^m(s)\,ds - \int_a^t P(t-s)v^m(s)\,ds\right|}{\sigma(t)} \\
&= \phi^{-1} \sup_{t\in[a,b]} \frac{\int_a^t |P(t-s)|\,|u^m(s) - v^m(s)|\ ds}{\sigma(t)} \\
&\le M\phi^{-1} \sup_{t\in[a,b]} \frac{\int_a^t |u^m(s) - v^m(s)|\ ds}{\sigma(t)} \\
&= M\phi^{-1} \sup_{t\in[a,b]} \frac{\int_a^t \frac{|u^m(s)-v^m(s)|}{\sigma(s)}\sigma(s)\,ds}{\sigma(t)} \\
&\le M\phi^{-1} \sup_{s\in[a,b]} \frac{|u^m(s) - v^m(s)|}{\sigma(s)} \sup_{t\in[a,b]} \frac{\int_a^t \sigma(s)\,ds}{\sigma(t)} \\
&\le KM\phi^{-1} \sup_{s\in[a,b]} \frac{\left|u(u^{m-1}(s)) - v(v^{m-1}(s))\right|}{\sigma(s)} \\
&\le KM\phi^{-1} d(u, v).
\end{aligned}$$

Due to the fact that $KM\phi^{-1} < 1$, it follows that $T$ is strictly contractive. Therefore, we can apply the aforementioned Banach fixed–point theorem, which guarantees that the convolution integral equation is Hyers–Ulam–Rassias stable. Additionally, (10) follows from (4) and (8). Indeed, from (8), we have

$$|u(t) - Tu(t)| \le \sigma(t), \quad t \in [a,b]. \tag{12}$$

Now, we can apply the Banach fixed–point theorem again, and from (4), we obtain

$$d(u, u_0) \le \frac{1}{1 - KM\phi^{-1}}\, d(Tu, u). \tag{13}$$

From the definition of the metric $d$ and by (12), it follows that

$$\sup_{t\in[a,b]} \frac{|u(t) - u_0(t)|}{\sigma(t)} \le \frac{1}{1 - KM\phi^{-1}}, \tag{14}$$

and consequently, (10) holds. □

## 3. $\sigma$–semi–Hyers–Ulam and Hyers–Ulam stabilities in the finite interval case

In this section, we will provide sufficient conditions for the $\sigma$–semi–Hyers–Ulam stability as well as for the Hyers–Ulam stability of the convolution integral Equation (1).

**Theorem 3.1.** *Let us consider a continuous function $L : [a,b] \to \mathbb{C}$. Furthermore, assume that $P : [a,b] \to [a,b]$ is also a continuous function such that there exists $M > 0$ satisfying*

$$M = \sup_{t,s\in[a,b]} |P(t-s)|\,. \tag{15}$$

*In addition, suppose there exists $K > 0$ such that*

$$\int_a^t \sigma(s)\,ds \leq K\sigma(t), \tag{16}$$

*for all $t \in [a, b]$.*

*If $u \in C([a, b])$ satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \leq \theta, \quad t \in [a, b], \tag{17}$$

*where $\theta \geq 0$ and $KM\phi^{-1} < 1$, then there exists a unique function $u_0 \in C([a, b])$, solution of Equation (1), given by*

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right), \tag{18}$$

*such that*

$$|u(t) - u_0(t)| \leq \frac{\theta}{(1 - KM\phi^{-1})\sigma(a)}\,\sigma(t), \tag{19}$$

*for all $t \in [a, b]$, which means that the convolution integral Equation (1) is $\sigma$–semi–Hyers–Ulam stable.*

*Proof.* Following the same procedure as before, we establish that $T$ is strictly contractive with respect to the metric (5), owing to the fact that $KM\phi^{-1} < 1$.

Thus, by applying the Banach fixed–point theorem, we conclude that the convolution integral Equation (1) satisfies $\sigma$–semi–Hyers–Ulam stability.

On the other hand, considering (17) and the definition of $T$, we obtain

$$|u(t) - (Tu)(t)| \leq \theta, \quad t \in [a, b]. \tag{20}$$

Using (4), the definition of the metric $d$, and applying (20), it follows that

$$\sup_{(t)\in[a,b]} \frac{|u(t) - u_0(t)|}{\sigma(t)} \leq \frac{1}{1 - KM\phi^{-1}} \sup_{(t)\in[a,b]} \frac{\theta}{\sigma(t)}. \tag{21}$$

Consequently, by the definition of $\sigma$, we deduce that (19) holds. □

**Corollary 3.2.** *Let us consider a continuous function $L : [a, b] \to \mathbb{C}$. Moreover, assume that $P : [a, b] \to [a, b]$ is also a continuous function such that there exists $M > 0$ satisfying*

$$M = \sup_{t,s\in[a,b]} |P(t-s)|\,. \tag{22}$$

*Additionally, suppose there exists $K > 0$ such that*

$$\int_a^t \sigma(s)\,ds \leq K\sigma(t), \tag{23}$$

*for all $t \in [a, b]$.*

*If $u \in C([a, b])$ satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \leq \theta, \quad t \in [a, b], \tag{24}$$

*where $\theta \geq 0$ and $KM\phi^{-1} < 1$, then there exists a unique function $u_0 \in C([a,b])$, solution of Equation* (1)*, given by*

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right), \tag{25}$$

*such that*

$$|u(t) - u_0(t)| \leq \frac{\sigma(b)}{(1 - KM\phi^{-1})\sigma(a)}\,\theta, \tag{26}$$

*for all $t \in [a,b]$, which means that the convolution integral Equation* (1) *is Hyers–Ulam stable.*

## 4. Stabilities in the infinite interval case

Instead of considering a finite interval $[a,b]$ with $a,b \in \mathbb{R}$, we will now analyze the Hyers–Ulam–Rassias and the $\sigma$–semi–Hyers–Ulam stabilities of the convolution integral Equation (1) on the infinite interval $[a,\infty)$, for some fixed $a \in \mathbb{R}$. With the necessary adaptations, similar results can also be presented for infinite intervals like $(-\infty,a]$, with $a \in \mathbb{R}$, and $(-\infty,\infty)$. Now, let us focus on the convolution integral equation,

$$\phi(u(t)) = L(t) + \int_a^t P(t-s)u^m(s)\,ds \quad \text{for } t \in [a,\infty), \tag{27}$$

where $a$ is a fixed real number, $\phi > 0$, $L : [a,\infty) \to \mathbb{C}$, and $P : [a,\infty) \to [a,\infty)$ are bounded continuous functions, and $u \in C([a,\infty))$. Our strategy will rely on a recurrence procedure based on the previously obtained results for the corresponding finite interval case. Let us consider a fixed non–decreasing function $\sigma : [a,\infty) \to (\epsilon,\omega)$, for some $\epsilon,\omega > 0$, and the space $C_b([a,\infty))$ of bounded functions endowed with the metric

$$d_b(u,v) = \sup_{t\in[a,\infty)} \frac{|u(t) - v(t)|}{\sigma(t)}. \tag{28}$$

**Theorem 4.1.** *Let us consider a bounded continuous function $L : [a,\infty) \to \mathbb{C}$. Moreover, assume that $P : [a,\infty) \to [a,\infty)$ is also a continuous function such that there exists $M > 0$ so that*

$$M = \sup_{t,s\in[a,\infty)} |P(t-s)|\,. \tag{29}$$

*Furthermore, suppose that*

$$\int_a^t P(t-s)u^m(s)\,ds$$

*is a bounded continuous function for any bounded continuous function $u$.*

*In addition, suppose that there exists $K > 0$ such that*

$$\int_a^t \sigma(s)\,ds \leq K\sigma(t), \tag{30}$$

*for all $t \in [a,\infty)$.*

*If $u \in C_b([a,\infty))$ is such that*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \leq \sigma(t), \quad t \in [a,\infty), \tag{31}$$

*and* $KM\phi^{-1} < 1$, *then there is a unique function* $u_0 \in C([a,\infty))$, *the solution of Equation* (27), *that is*

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right) \tag{32}$$

*such that*

$$|u(t) - u_0(t)| \le \frac{1}{1 - KM\phi^{-1}}\,\sigma(t) \tag{33}$$

*for all* $t \in [a,\infty)$, *which means that the convolution integral Equation* (27) *is Hyers–Ulam–Rassias stable.*

*Proof.* For each $n \in \mathbb{N}$, define $I_n = [a, a+n]$. According to Theorem 2.1, there exists a unique continuous function $u_{0,n} \in C(I_n)$ that satisfies

$$u_{0,n}(t) = \phi^{-1}\left(L(t) + \int_a^t \rho(t-s)u_{0,n}^m(s)\,ds\right) \tag{34}$$

and

$$|u(t) - u_{0,n}(t)| \le \frac{1}{1 - KM\phi^{-1}}\,\sigma(t) \tag{35}$$

for all $t \in I_n$. The uniqueness of $u_{0,n}$ implies that for $t \in I_n$, we have

$$u_{0,n}(t) = u_{0,n+1}(t) = u_{0,n+2}(t) = \cdots. \tag{36}$$

For any $t \in [a,\infty)$, let $n(t) \in \mathbb{N}$ be defined as $n(t) = \min\{n \in \mathbb{N} : t \in I_n\}$. We also define a function $u_0 : [a,\infty) \to \mathbb{R}$ by

$$u_0(t) = u_{0,n(t)}(t). \tag{37}$$

For any $t_1 \in [a,\infty)$, let $n_1 = n(t_1)$. Then $t_1 \in \mathrm{Int}I_{n_1+1}$, and there exists an $\epsilon > 0$ such that $u_0(t) = u_{0,n_1+1}(t)$ for all $t \in (t_1 - \epsilon, t_1 + \epsilon)$, (where $\mathrm{Int}I_{n_1+1}$ denotes the interior of $I_{n_1+1}$). By Theorem 2.1, $u_{0,n_1+1}$ is continuous at $t_1$, and so is $u_0$.
Now, we will prove that $u_0$ satisfies

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right) \tag{38}$$

and

$$|u(t) - u_0(t)| \le \frac{1}{1 - KM\phi^{-1}}\,\sigma(t). \tag{39}$$

For an arbitrary $t \in [a,\infty)$, we choose $n(t)$ such that $t \in I_{n(t)}$. From (34) and (37), we have

$$u_0(t) = u_{0,n(t)}(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_{0,n(t)}^m(s)\,ds\right) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right). \tag{40}$$

Note that $n(\tau) \le n(t)$ for any $\tau \in I_{n(t)}$, and from (36), we conclude that $u_0(\tau) = u_{0,n(\tau)}(\tau) = u_{0,n(t)}(\tau)$, so the last equality in (40) holds.
To prove (33), using (37) and (35), we obtain for all $t \in [a,\infty)$,

$$|u(t) - u_0(t)| = |u(t) - u_{0,n(t)}(t)| \le \frac{1}{1 - KM\phi^{-1}}\,\sigma(t). \tag{41}$$

Finally, we will prove the uniqueness of $u_0$. Let us consider another bounded continuous function $u_1$, which satisfies (32) and (33) for all $t \in [a,\infty)$. By the uniqueness of the solution on $I_{n(t)}$ for any $n(t) \in \mathbb{N}$, we have that $u_0|_{I_{n(t)}} = u_{0,n(t)}$ and $u_1|_{I_{n(t)}}$ satisfies (32) and (33) for all $t \in I_{n(t)}$. Therefore, $u_0(t) = u_0|_{I_{n(t)}}(t) = u_1|_{I_{n(t)}}(t) = u_1(t)$. □

We will now provide sufficient conditions for the $\sigma$–semi–Hyers–Ulam stability of the convolution integral Equation (27).

**Theorem 4.2.** *Consider a bounded continuous function $L : [a,\infty) \to \mathbb{C}$. Additionally, assume that $P : [a,\infty) \to [a,\infty)$ is a continuous function such that there exists $M > 0$ for which the following holds:*

$$M = \sup_{t,s\in[a,\infty)} |P(t-s)|. \tag{42}$$

*Furthermore, suppose that*

$$\int_a^t P(t-s)u^m(s)\,ds$$

*is a bounded continuous function for any bounded continuous function $u$.*

*Additionally, assume that there exists $K > 0$ such that*

$$\int_a^t \sigma(s)\,ds \le K\sigma(t), \tag{43}$$

*for all $t \in [a,\infty)$.*

*Suppose $u \in C_b([a,\infty))$ satisfies*

$$\left|\phi(u(t)) - L(t) - \int_a^t P(t-s)u^m(s)\,ds\right| \le \theta, \quad t \in [a,\infty], \tag{44}$$

*where $\theta \ge 0$ and $KM\phi^{-1} < 1$, then there exists a unique function $u_0 \in C([a,\infty))$, which is the solution to Equation (27), given by*

$$u_0(t) = \phi^{-1}\left(L(t) + \int_a^t P(t-s)u_0^m(s)\,ds\right), \tag{45}$$

*such that*

$$|u(t) - u_0(t)| \le \frac{\theta}{(1-KM\phi^{-1})\sigma(a)}\,\sigma(t), \tag{46}$$

*for all $t \in [a,b]$, which implies that the convolution integral Equation (27) is $\sigma$–semi–Hyers–Ulam stable.*

*Proof.* By the same procedure as above and Theorem 3.1, the proof is straightforward, so we omit it here. □

## 5. Illustrative examples

To illustrate that the conditions of the above results are possible to attain, we will present some examples.

**Example 5.1.** *Consider the convolution integral equation for a continuous function $u : [0,2] \to \mathbb{C}$, given by*

$$10(u(t)) = \frac{t^3}{t^2+t+100} + \int_0^t \frac{1}{10}(t-s)u^m(s)\,ds, \quad t \in [0,2]. \tag{47}$$

*It is evident that all the conditions of Theorem 2.1 are satisfied in this case. Specifically, we define $\phi = 10$ and $L : [0,2] \to \mathbb{C}$ by*

$$L(t) = \frac{t^3}{t^2+t+100},$$

*which is a continuous function. Additionally, the function $P : [0,2] \to [0,2]$, where $P(t-s) = \frac{1}{10}(t-s)$, is also continuous and satisfies*

$$\sup_{t,s\in[0,2]} |P(t-s)| = \sup_{t,s\in[0,2]} \left|\frac{1}{10}(t-s)\right| \leq \frac{1}{2} = M.$$

*Furthermore, there exists a constant $K > 0$ such that*

$$\int_0^t \sigma(s)\,ds = \int_0^t 2^s\,ds = \frac{2^t}{\ln 2} - \frac{1}{\ln 2} \leq \frac{2^t}{\ln 2} = K\sigma(t), \quad t \in [0,2],$$

*where $\sigma : [0,2] \to (0,\infty)$ is a non–decreasing continuous function defined as $\sigma(t) = 2^t$.*

*Now, let $u \in C([0,2])$ satisfy*

$$\left|\frac{1}{3}(u(t)) - \frac{t^3}{t^2+t+100} + \int_0^t \frac{1}{10}(t-s)u^m(s)\,ds\right| \leq 2^t = \sigma(t),\ t \in [0,2].$$

*This confirms the Hyers–Ulam–Rassias stability of the convolution integral Equation* (47). *Moreover, by considering*

$$KM\phi^{-1} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{10}\right) = 0.0721 < 1,$$

*we deduce that*

$$|u(t) - u_0(t)| \leq \frac{\sigma(t)}{1 - KM\phi^{-1}}, \quad t \in [0,2].$$

**Example 5.2.** *Consider the convolution integral equation for a continuous function $u : [0,1] \to \mathbb{C}$, given by*

$$4(u(t)) = \sin(t) + t^2 + t + 1 + \int_0^t \frac{\sin(t-s)}{100} u^m(s)\,ds, \quad t \in [0,1]. \tag{48}$$

*It is evident that all the conditions of Theorem 2.1 are satisfied in this case. Specifically, we define $\phi = 4$ and $L : [0,1] \to \mathbb{C}$ by*

$$L(t) = \sin(t) + t^2 + t + 1,$$

*which is a continuous function. Additionally, the function $P : [0,1] \to [0,1]$, where $P(t-s) = \frac{\sin(t-s)}{100}$, is also continuous and satisfies*

$$\sup_{t,s\in[0,1]} |P(t-s)| = \sup_{t,s\in[0,1]} \left|\frac{\sin(t-s)}{100}\right| \leq \frac{1}{100} = M.$$

*Furthermore, there exists a constant $K > 0$ such that*

$$\int_0^t \sigma(s)\,ds = \int_0^t e^{7s}\,ds = \frac{e^{7t}}{7} - \frac{1}{7} \leq \frac{e^{7t}}{7} = K\sigma(t), \quad t \in [0,1],$$

*where $\sigma : [0,1] \to (0,\infty)$ is a non–decreasing continuous function defined as $\sigma(t) = e^{7t}$.*

*Now, let $u \in C([0,1])$ satisfy*

$$\left|4(u(t)) - \sin(t) - t^2 - t - 1 - \int_0^t \frac{\sin(t-s)}{100} u^m(s)\,ds\right| \leq e^{7t} = \sigma(t),\ t \in [0,1].$$

*This confirms the Hyers–Ulam–Rassias stability of the convolution integral Equation* (48). *Moreover, by considering*

$$KM\phi^{-1} = \left(\frac{1}{7}\right)\left(\frac{1}{100}\right)\left(\frac{1}{4}\right) = 0.0004 < 1,$$

*we deduce that*

$$|u(t) - u_0(t)| \leq \frac{\sigma(t)}{1 - KM\phi^{-1}}, \quad t \in [0,1].$$

**Example 5.3.** *Consider the convolution integral equation for a continuous function $u : [0,3] \to \mathbb{C}$, given by*

$$9(u(t)) = \frac{\tan^{-1}(t)}{100} + \int_0^t \frac{1}{11(t-s)} u^m(s)\, ds, \quad t \in [0,3]. \tag{49}$$

*It is evident that all the conditions of Theorem 2.1 are satisfied. Specifically, we have $\phi = 9$, and the function $L : [0,3] \to \mathbb{C}$ defined as*

$$L(t) = \frac{\tan^{-1}(t)}{100}$$

*is continuous. Additionally, the function $P : [0,3] \to [0,3]$ given by $P(t-s) = \frac{1}{11(t-s)}$ is continuous and satisfies*

$$\sup_{t,s\in[0,3]} |P(t-s)| = \sup_{t,s\in[0,3]} \left| \frac{1}{11(t-s)} \right| \leq \frac{1}{11} = M.$$

*Furthermore, there exists a constant $K > 0$ such that*

$$\int_0^t \sigma(s)\, ds = \int_0^t e^{3s}\, ds = \frac{e^{3t}}{3} - \frac{1}{3} \leq \frac{e^{3t}}{3} = K\sigma(t), \quad t \in [0,3],$$

*where $\sigma : [0,3] \to (0,\infty)$ is a non–decreasing continuous function defined by $\sigma(t) = e^{3t}$.*

*If $u \in C([0,3])$ satisfies*

$$\left| 9(u(t)) - \frac{\tan^{-1}(t)}{100} - \int_0^t \frac{1}{11(t-s)} u^m(s)\, ds \right| \leq e^{3t} = \sigma(t),\ t \in [0,3],$$

*then this confirms the Hyers–Ulam–Rassias stability of the convolution integral Equation (49). Moreover, since*

$$KM\phi^{-1} = \left(\frac{1}{3}\right)\left(\frac{1}{11}\right)\left(\frac{1}{9}\right) = 0.0034 < 1,$$

*we conclude that*

$$|u(t) - u_0(t)| \leq \frac{\sigma(t)}{1 - KM\phi^{-1}}, \quad t \in [0,3].$$

## 6. Conclusion

In conclusion, this paper investigates the Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and a new kind of stability, $\sigma$–semi–Hyers–Ulam stability for a class of nonlinear convolution integral equations. By employing the fixed–point method, we establish sufficient conditions to ensure these stabilities in both finite and infinite interval settings. The theoretical findings are further supported by three illustrative examples, demonstrating the applicability and effectiveness of the proposed stability results. These contributions provide a deeper understanding of stability properties in integral equations, paving the way for future research in this area.

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