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Approximation operators derived from groups

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Abstract. This study explores the integration of group-theoretic structures with approximation operators to develop a novel mathematical framework for handling uncertainty. Traditional rough set theory, based on equivalence relations, has been widely applied in data analysis, but its rigid structure limits adaptability in cases where data is inherently imprecise or structured by transformations. To overcome these limitations, we introduce new approximation operators derived from fundamental group-theoretic concepts, including cyclic subgroups, centralizers, normalizers, orbits, and stabilizers.

By defining lower and upper approximation sets using group actions, we establish a more flexible approach to capturing structural uncertainty. Furthermore, we demonstrate the applicability of these operators in digital image processing, particularly in tasks such as object recognition, noise reduction, and feature extraction. The integration of group theory with approximation methods not only enriches the theoretical foundation of rough sets but also enhances their practical utility in mathematical modeling, pattern analysis, and computer vision. This work provides a systematic study of these group-based operators, their fundamental properties, and their potential applications, offering a new perspective on uncertainty modeling in structured data environments.

1. Introduction and Motivation

Pioneered by Zdzisław Pawlak in the early 1980s [18–20], rough set theory offers a rigorous framework for knowledge discovery and information systems analysis, specifically tailored to address the challenges posed by imperfect data. Departing from the crisp set membership paradigm of classical set theory, rough set theory elegantly accommodates the inherent vagueness and uncertainty prevalent in real-world datasets. Central to this theory is the concept of approximating a set through two derived subsets; *the lower approximation*, comprising elements definitively belonging to the target set based on available information, and *the upper approximation*, encompassing elements potentially belonging to the target set, acknowledging data limitations and uncertainties.

Data is structured within information systems, effectively represented as decision tables. These tables organize data points (objects) as rows and their descriptive features (attributes) as columns, facilitating classification. The strength of rough set theory lies in its capacity to identify information deficiency and enable decision-making under uncertainty. By examining the discrepancy between the lower and upper approximations, rough sets illuminate regions of data ambiguity or insufficiency, highlighting knowledge

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gaps and directing further inquiry. Crucially, even with imperfect data, rough sets empower the extraction of actionable insights, supporting informed decision-making by leveraging existing information while explicitly recognizing its inherent limitations.

Due to its robust handling of imprecise data, rough set theory finds broad applicability across diverse domains, including data mining, decision support systems, and machine learning.

In Pawlak's seminal work [18, 19], rough sets are grounded in the concept of an approximation space, defined by a universal set U and an equivalence relation R on U. Formally, the pair (U, R) constitutes an *approximation space*. For any $x \in U$, the equivalence class of x under R, denoted $[x]_R$, is termed an *elementary set*. The collection of all such elementary sets forms the quotient set U/R. The equivalence relation R embodies the available information concerning U, enabling the construction of lower and upper approximations. Specifically, for any $X \subseteq U$, *the lower approximation* of X is defined as:

$$R^{\downarrow}(X) = \{x \in U | [x]_R \subseteq X\}$$

and *the upper approximation* of X is defined as:

$$R^{\uparrow}(X) = \{x \in U | [x]_R \cap X \neq \varnothing\}$$

The pair $(R^{\downarrow}(X), R^{\uparrow}(X))$ then defines *the rough set* of X. This rough set provides an approximation of X based on the knowledge encoded by R, facilitating the classification of the universe U. The classical theory of rough sets has been significantly advanced by numerous studies, as exemplified in [7, 20–23, 26, 27, 34]. As noted in [29, 31, 37, 38], the equivalence relation is often difficult to implement due to its restrictive nature.

Over time, many studies have been conducted on generalizations of rough set theory by replacing the equivalence relation given in the approximation space with an ordinary relation [16, 33–35, 39]. These generalizations have made the theory easily applicable to many fields. On the other hand, it has been shown [12–15] that approximation can be studied through a general framework based on neighborhood systems from general topology. Nevertheless, under an arbitrary binary relation, the successor elements of a given element may be considered its neighborhood [32]. The neighborhood concept, used to construct rough sets from such relations, broadens the scope of approximate equivalence relations, thus enabling a more flexible analysis of data patterns and relationships. More modern versions of rough sets have been derived in [1–3, 9] and [10] by employing new paradigms of the neighborhood concept, with applications shown in various fields.

Let us formally define the derivation of approximation operators from the neighborhood of a point with respect to a given relation. Consider an ordinary relation R on a non-empty universe U. The pair (U, R) is then termed an *approximation space*. Here, the definition of approximation operators is grounded in the concept of a point's neighborhood within U, as induced by R. Specifically, the neighborhood of a point $x \in U$ under the relation R is defined as:

$$n(x) = \{y \in U | (x, y) \in R\}$$

Clearly, n constitutes an operator of the form $n : U \to \mathcal{P}(U)$, termed the neighborhood operator. Leveraging this generalization, the lower and upper approximations of a subset $X \subseteq U$ with respect to R are defined as:

$$R^{\downarrow}(X) = \{x \in U \mid n(x) \subseteq X\} \text{ and } \tag{1}$$

$$R^{\uparrow}(X) = \{x \in U \mid n(x) \cap X \neq \varnothing\} \tag{2}$$

This generalization leads to the definition of a generalized rough set as the pair $(R^{\downarrow}(X), R^{\uparrow}(X))$. The difference between the upper and lower approximations, $R^{\uparrow}(X)$ and $R^{\downarrow}(X)$, is defined as the R-boundary of X, denoted $BN_R(X)$, such that $BN_R(X) = R^{\uparrow}(X) - R_{\downarrow}(X)$.

In [16], in a generalized approximation space (U, R), a point x is called a *solitary element* if $n(x) = \varnothing$, and the set of all solitary element in (U, R) is denoted by $S = \{x \in U \mid n(x) = \varnothing\}$. However, as we recall, if $n(x) \neq \varnothing$ for every $x \in U$, then the relation R on U is called a *serial relation*, and also, n is called *the serial neighborhood operator*. We can clearly see that; n is serial neighborhood if and only if $S = \varnothing$.

Considering an arbitrary universal set U, a relation R defined on it, and a set S of solitary elements, the following properties hold:

Proposition 1.1. *[16]*

(a) $R^{\downarrow}(\varnothing) = S$, $R^{\uparrow}(\varnothing) = \varnothing$, $R^{\downarrow}(U) = U$ *and* $R^{\uparrow}(U) = S^c$, *where* S^c *is the complement of* S.

(b) $S \subseteq R^{\downarrow}(X)$ *and* $R^{\uparrow}(X) \subseteq S^c$ *for all* $X \in \mathcal{P}(U)$.

(c) $R^{\downarrow}(X) - S \subseteq R^{\uparrow}(X)$ *for each* $X \in \mathcal{P}(U)$.

(d) $R^{\downarrow}(X) = U$ *iff* $\bigcup_{x \in U} n(x) \subseteq X$ *and* $R^{\uparrow}(X) = \varnothing$ *iff* $X \subseteq (\bigcup_{x \in U} n(x))^c$.

(e) *If* $S \neq \varnothing$, *then* $R^{\downarrow}(X) \neq R^{\uparrow}(X)$ *for each* $X \in \mathcal{P}(U)$.

(f) *Let* I *be an arbitrary index set. Then,* $R^{\downarrow}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} R^{\downarrow}(X_i)$ *and* $R^{\uparrow}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} R^{\uparrow}(X_i)$.

(g) *For each* $X, Y \subseteq U$, *if* $X \subseteq Y$ *then* $R^{\downarrow}(X) \subseteq R^{\downarrow}(Y)$ *and* $R^{\uparrow}(X) \subseteq R^{\uparrow}(Y)$.

(h) $R^{\downarrow}(X) \cup R^{\downarrow}(Y) \subseteq R^{\downarrow}(X \cup Y)$ *and* $R^{\uparrow}(X \cap Y) \subseteq R^{\uparrow}(X) \cap R^{\uparrow}(Y)$.

(i) $(R^{\downarrow}(X))^c = R^{\uparrow}(X^c)$ *and* $(R^{\uparrow}(X))^c = R^{\downarrow}(X^c)$.

(j) *There is some* $X \in \mathcal{P}(U)$ *such that* $R^{\downarrow}(X) = R^{\uparrow}(X)$ *iff* R *is serial.*

Additionally, for an arbitrary serial relation R on U, the following results are also obtained:

Proposition 1.2. *[16] The followings are equivalent.*

(a) R *is serial,*

(b) *For each* $X \in \mathcal{P}(U)$, $R^{\downarrow}(X) \subseteq R^{\uparrow}(X)$,

(c) $R^{\downarrow}(\varnothing) = \varnothing$,

(d) $R^{\downarrow}(U) = R^{\uparrow}(U)$,

(e) *The set* $\{X \mid R^{\downarrow}(X) = R^{\uparrow}(X)\}$ *is non-empty.*

In [36], the authors define approximation operators, which are obtained from a transformation called a neighborhood operator, independently of any structure on the universal set, more generally than the concepts given above. This has led to a more general definition of the generalized rough set concept presented above.

From [36], any map $n : U \to \mathcal{P}(U)$ is called a *neighborhood operator*. If $n(x) \neq \varnothing$ for all $x \in U$, n is called a serial neighborhood operator. If $x \in n(x)$ for all $x \in U$, n is called a *reflexive neighborhood operator*.

The second important mathematical tool we will use in this study is group theory. Group theory is a fundamental branch of abstract algebra with vast importance in mathematics and numerous applications across various scientific fields.

Group theory is the mathematical language of symmetry, providing a unifying framework for understanding transformations such as rotations and reflections. It simplifies complex problems by revealing hidden structures and relationships across diverse fields.

In mathematics, group theory is foundational to abstract algebra, ring theory, and Galois theory, shaping fundamental theorems and interconnections. In physics, it governs the symmetries of particles and physical laws, underpinning conservation principles through Noether's theorem. In chemistry, it predicts molecular structures and reactions. In cryptography, it ensures secure communication through finite group properties, while in computer science, it plays a crucial role in coding theory and data compression.

At its core, a group consists of a set G and a binary operation $*$ that satisfies four key axioms [8]:

Closure The operation always results in an element of G .

Associativity The grouping of elements does not affect the outcome.

Identity Element A special element e exists such that $x * e = e * x = x$ for all $x \in G$.

Inverse Element Each element x has an inverse x^{-1} such that $x * x^{-1} = x^{-1} * xe$.

If the operation is commutative ($x * y = y * x$), the group is Abelian. The order of a group is its number of elements, and any subset that forms a group under the same operation is a subgroup.

Group theory is a remarkably profound and extensive theory. A few group-theoretical definitions that form the basis of this study and will be used later are given below.

Let G be a group and S be a subgroup of G.

Cyclic Subgroup [8] The *cyclic subgroup* of G generated by an element $x \in G$ is defined to be

$$\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of integers.

Centralizer [17] The *centralizer* of an element $x \in G$ is defined to be

$$C(x) = \{y \in G \mid xy = yx\}.$$

Normalizer [17] The *normalizer* of an element $x \in G$ is defined to be

$$N(x) = \{y \in G \mid y^{-1}xy \in \langle x \rangle\}.$$

Left - Right Coset [8] The *left coset* of S in G is defined to be

$$xS = \{xy \mid y \in S\},$$

and the *right coset* of S in G is defined to be

$$Sx = \{yx \mid y \in S\},$$

where $x \in G$.

It should be noted that there is a relationship between the cyclic subgroup, the centralizer, and the normalizer, such that

$$\langle x \rangle \subseteq C(x) \subseteq N(x). \tag{3}$$

Let X be a non-empty set. It is called that $* : G \times X \to X$ is a *group action* of G on X that satisfies the following two axioms;

(a) $e * x = x$, for all $x \in X$ and $e \in G$ is an identity element of G.

(b) $(gh) * x = g * (h * x)$, for all $g, h \in G$ and for all $x \in X$.

It is called that X is *G-set* if G is acting on X.

Let G be a group acting on a set X. The *orbit* of an element $x \in X$ is defined as

$$\mathbf{Orb}(x) = \{y \in X \mid \exists g \in G, y = g * x\}.$$

That is $\mathbf{Orb}(x) = G * x$.

At the same time, for given $x \in X$, its *stabilizer* is defined as

$$\mathbf{Stab}(x) = \{g \in G \mid g * x = x\}$$

which is subset of G.

Note that, for each $x \in X$, $\mathbf{Stab}(x)$ is a subgroup of G. It is called a *stabilizer subgroup* of G or *isotropy subgroup* of x.

Let X and Y be two G-sets, and $f : X \to Y$ be a function. It is called that f is a *morphism of G-sets* or *G-function*, if $f(g * x) = g * f(x)$ for all $g \in G$ and $x \in X$. If f is a bijective G-function then we call that f is an *isomorphism*.

In any group G and for any $g \in G$, the element aga^{-1} is called *conjugate of g* with respect to a. The automorphism $f : G \to G$ such that $f(x) = axa^{-1}$ is called *conjugation*. If we define the operation $\cdot : G\times G \to G$ such that $g \cdot x = gxg^{-1}$, then we have an action $\cdot$, and G acts on itself. We call that $\cdot$ is *conjugation action*.

The motivation behind this study arises from the need to bridge group-theoretic concepts with approximation operators to develop a more robust mathematical framework for uncertainty modeling. Traditional rough set theory, pioneered by Pawlak, relies on equivalence relations to construct lower and upper approximations. However, these strict conditions often limit its applicability in real-world scenarios where data is inherently imprecise. To address this, we explore an alternative approach by leveraging group structures—such as cyclic subgroups, centralizers, normalizers, orbits, and stabilizers—as a foundation for defining approximation operators. This integration provides a novel perspective that enhances the flexibility and applicability of rough set theory, particularly in domains where symmetry and transformations play a crucial role. Furthermore, we illustrate the practical implications of our approach by applying these newly developed operators to digital image processing, demonstrating their potential for tasks like object recognition, noise reduction, and feature extraction. By unifying these mathematical tools, this work not only extends the theoretical landscape of approximation operators but also offers a concrete pathway for their implementation in real-world applications.

2. Approximation operators derived from group structures

Neighborhood operators can be obtained on the group using the group structural concepts given in the previous section, immediately.

Definition 2.1. *Assume that U is a group and S is a subgroup of U. Define set valued mappings as follows:*

(1) The mapping $cn : U \to \mathcal{P}(U)$ such that $cn(x) = \langle x\rangle$ for each $x \in U$ is called the cyclical neighborhood operation on U.

(2) The mapping $Cn : U \to \mathcal{P}(U)$ such that $Cn(x) = C(x)$ for each $x \in U$ is called the centralizing neighborhood operator on U.

(3) The mapping $Nn : U \to \mathcal{P}(U)$ such that $Nn(x) = N(x)$ for each $x \in U$ is called the normalizing neighborhood operator on U.

(4) The mapping $Ln : U \to \mathcal{P}(U)$ such that $Ln(x) = xS$ for each $x \in U$ is called the left cosetial neighborhood operator on U.

(5) The mapping $Rn : U \to \mathcal{P}(U)$ such that $Rn(x) = Sx$ for each $x \in U$ is called the right cosetial neighborhood operator on U.

In [36], using the neighborhood operator, the lower and upper approximations of a set based on its elements are defined as shown in Equations 1 and 2, respectively. Building on this and using the arguments in Definition 2.1, the lower and upper approximation operators are obtained as shown below.

Definition 2.2. *Let U be a group, S be a subgroup of U and X be non-empty subset of U.*

(1) *The lower approximation of X with respect to the cyclical neighborhood operator is defined as* $\underline{\mathbf{apr}}_{cn}(X) = \{x \in U \mid cn(x) \subseteq X\}$, *and the upper approximation of X with respect to the cyclical neighborhood operator is defined as* $\overline{\mathbf{apr}}_{cn}(X) = \{x \in U \mid cn(x) \cap X \neq \varnothing\}$.

(2) *The lower approximation of X with respect to the centralizing neighborhood operator is defined as* $\underline{\mathbf{apr}}_{Cn}(X) = \{x \in U \mid Cn(x) \subseteq X\}$, *and the upper approximation of X with respect to the centralizing neighborhood operator is defined as* $\overline{\mathbf{apr}}_{Cn}(X) = \{x \in U \mid Cn(x) \cap X \neq \varnothing\}$.

(3) *The lower approximation of X with respect to the normalizing neighborhood operator is defined as* $\underline{\mathbf{apr}}_{Nn}(X) = \{x \in U \mid Nn(x) \subseteq X\}$, *and the upper approximation of X with respect to the normalizing neighborhood operator is defined as* $\overline{\mathbf{apr}}_{Nn}(X) = \{x \in U \mid Nn(x) \cap X \neq \varnothing\}$.

(4) *The lower approximation of X with respect to the left cosetial neighborhood operator is defined as* $\underline{\mathbf{apr}}_{Ln}(X) = \{x \in U \mid Ln(x) \subseteq X\}$, *and the upper approximation of X with respect to the left cosetial neighborhood operator is defined as* $\overline{\mathbf{apr}}_{Ln}(X) = \{x \in U \mid Ln(x) \cap X \neq \varnothing\}$.

(5) *The lower approximation of X with respect to the right cosetial neighborhood operator is defined as* $\underline{\mathbf{apr}}_{Rn}(X) = \{x \in U \mid Rn(x) \subseteq X\}$, *and the upper approximation of X with respect to the right cosetial neighborhood operator is defined as* $\overline{\mathbf{apr}}_{Rn}(X) = \{x \in U \mid Rn(x) \cap X \neq \varnothing\}$.

Proposition 2.3. *Let U be a group and $X \subseteq U$. If $e \notin X$, then* $\underline{\mathbf{apr}}_{cn}(X) = \varnothing$.

Proof. Since $\langle x\rangle$ is a subgroup of U for any $x \in U$, then $e \in \langle x\rangle$. Since e is not an element of X by hypothesis, it follows that $\langle x\rangle$ cannot be a subset of X. Consequently, $\underline{\mathbf{apr}}_{cn}(X)$ is the empty set. □

It is noteworthy that the converse of Proposition 2.3 does not generally hold. To illustrate this, consider, for instance, the group $\mathbb{Z}_2 = \{0, 1\}$. Let us choose $X = \{0\}$. In this case, it is evident that $e = 0 \in X$. Let us compute the set $\underline{\mathbf{apr}}_{cn}(X)$. For $x = 0$, we have $\langle 0\rangle = \{0\} \subseteq X$, and for $x = 1$, we have $\langle 1\rangle = \{0, 1\} = \mathbb{Z}_2 \not\subseteq X$. Consequently, we obtain $\underline{\mathbf{apr}}_{cn}(X) = X$. As a result, if $e \in X$, then $\underline{\mathbf{apr}}_{cn}(X) \neq \varnothing$.

Proposition 2.4. *Let U be a group and $X \subseteq U$. If $e \in X$, then* $\overline{\mathbf{apr}}_{cn}(X) = U$.

Proof. Since $\langle x\rangle$ is a subgroup of U for any $x \in U$, then $e \in \langle x\rangle$. Since e is an element of X by hypothesis, it follows that $\langle x\rangle \cap X \neq \varnothing$. Therefore, we have $\overline{\mathbf{apr}}_{cn}(X) = U$. □

Furthermore, it is important to note that the converse of Proposition 2.4 is also not true. To demonstrate this, let us consider the group $\mathbb{Z}_2$, again. If we choose $X = \{1\}$, then upon calculating $\overline{\mathbf{apr}}_{cn}(X)$, we find that for $x = 0$,

$$\langle 0\rangle \cap X = \{0\} \cap \{1\} = \varnothing,$$

and for $x = 1$,

$$\langle 1\rangle \cap X = \mathbb{Z}_2 \cap \{1\} = \{1\} \neq \varnothing.$$

Consequently, when $e \notin X$, $\overline{\mathbf{apr}}_{cn}(X) \neq U$.

Proposition 2.5. *Let U be a group and $X \subseteq U$.* $\underline{\mathbf{apr}}_{cn}(X) \supseteq \underline{\mathbf{apr}}_{Cn}(X) \supseteq \underline{\mathbf{apr}}_{Nn}(X)$.

Proof. Assume that $x \in \underline{\mathbf{apr}}_{Nn}(X)$. So, we have $Nn(x) \subseteq X$. From Equation 3, since $Cn(x) \subseteq Nn(x)$, it follows that $Cn(x) \subseteq X$, and therefore $x \in \underline{\mathbf{apr}}_{Cn}(x)$. Hence, we obtain that $\underline{\mathbf{apr}}_{Cn}(X) \supseteq \underline{\mathbf{apr}}_{Nn}(X)$.

In a similar manner, it can be demonstrated that $\underline{\mathbf{apr}}_{cn}(X) \supseteq \underline{\mathbf{apr}}_{Cn}(X)$ holds. As a result, the desired outcome is achieved. □

Proposition 2.6. *Let U be a group and $X \subseteq U$.* $\overline{\mathbf{apr}}_{cn}(X) \subseteq \overline{\mathbf{apr}}_{Cn}(X) \subseteq \overline{\mathbf{apr}}_{Nn}(X)$.

Proof. Let x be an arbitrary element in $\overline{\mathbf{apr}}_{cn}(X)$. Then, $cn(x) \cap X \neq \emptyset$ and thus $\langle x \rangle \cap X \neq \emptyset$ follows. Since $\langle x \rangle \subseteq C(x)$ by Equation 3, then $C(x) \cap X \neq \emptyset$ and thus $Cn(x) \cap X \neq \emptyset$ is obtained. Therefore, $x \in \overline{\mathbf{apr}}_{Cn}(X)$. Hence, $\overline{\mathbf{apr}}_{cn}(X) \subseteq \overline{\mathbf{apr}}_{Cn}(X)$.

Similarly, it is also shown that $\overline{\mathbf{apr}}_{Cn}(X) \subseteq \overline{\mathbf{apr}}_{Nn}(X)$. □

It is straightforward to conclude from Proposition 2.3 and Proposition 2.5 that if the identity element e is not in X, then $\underline{\mathbf{apr}}_{Cn}(X)$ and $\underline{\mathbf{apr}}_{Nn}(X)$ are both empty sets.

Similarly, it is clear that as a consequence of Proposition 2.4 and Proposition 2.6, if any subset X of a group U contains the identity element e, then $\overline{\mathbf{apr}}_{Cn}(X) = \overline{\mathbf{apr}}_{Nn}(X) = U$.

Example 2.7. *Let us consider the symmetric group S_3 that its elements are denoted by $I = (a, b, c)$, $f_a = (a, c, b)$, $f_b = (c, b, a)$, $f_c = (b, a, c)$, $f_1 = (b, c, a)$ and $f_2 = (c, a, b)$. The operation table for S_3 is as in Table 2.7:*

Table 1: The operation table for S_3

$\circ$	I	f_a	f_b	f_c	f_1	f_2
I	I	f_a	f_b	f_c	f_1	f_2
f_a	f_a	I	f_2	f_1	f_c	f_b
f_b	f_b	f_1	I	f_2	f_a	f_1
f_c	f_c	f_2	f_1	I	f_b	f_a
f_1	f_1	f_b	f_c	f_a	f_2	I
f_2	f_2	f_c	f_a	f_b	I	f_1

The following results are obtained from the definitions of cyclic subgroups, centralizers, and normalizers:

$\langle I \rangle = \{I\}$, $\langle f_a \rangle = \{I, f_a\}$, $\langle f_b \rangle = \{I, f_b\}$, $\langle f_c \rangle = \{I, f_c\}$, $\langle f_1 \rangle = \{I, f_1, f_2\}$ and $\langle f_2 \rangle = \{I, f_1, f_2\}$ are all cyclic subgroups.

$C(I) = S_3$, $C(f_a) = \{I, f_a\}$, $C(f_b) = \{I, f_b\}$, $C(f_c) = \{I, f_c\}$, $C(f_1) = \{I, f_1, f_2\}$ and $C(f_2) = \{I, f_1, f_2\}$ are all centralizers.

$N(I) = S_3$, $N(f_a) = \{I, f_a\}$, $N(f_b) = \{I, f_b\}$, $N(f_c) = \{I, f_c\}$, $N(f_1) = S_3$ and $N(f_2) = S_3$ are all normalizers.

Consider the subset $X = \{I, f_c, f_1, f_2\}$ of S_3. Thus, we have the lower approximations of X with respect to cyclical neighborhood, centralizing neighborhood and normalizing neighborhood operators are as $\underline{\mathbf{apr}}_{cn}(X) = \{I, f_c, f_1, f_2\}$, $\underline{\mathbf{apr}}_{Cn}(X) = \{f_c, f_1, f_2\}$ and $\underline{\mathbf{apr}}_{Nn}(X) = \{f_c\}$. The upper approximations of X with respect to cyclical neighborhood, centralizing neighborhood and normalizing neighborhood operators are obtained as $\overline{\mathbf{apr}}_{cn}(X) = \overline{\mathbf{apr}}_{Cn}(X) = \overline{\mathbf{apr}}_{Nn}(X) = S_3$.

Let us consider another subset of S_3, namely $Y = \{f_a, f_2\}$. Then we have that $\underline{\mathbf{apr}}_{cn}(Y) = \underline{\mathbf{apr}}_{Cn}(Y) = \underline{\mathbf{apr}}_{Nn}(Y) = \emptyset$ from Proposition 2.3. All upper approximations of Y are $\overline{\mathbf{apr}}_{cn}(Y) = \{f_a, f_1, f_2\}$, $\overline{\mathbf{apr}}_{Cn}(Y) = \{I, f_a, f_1, f_2\}$ and $\overline{\mathbf{apr}}_{Nn}(Y) = \{I, f_a, f_1, f_2\}$.

In general, it is clear from Example 2.7 that $\underline{\mathbf{apr}}_{cn}(X)$, $\underline{\mathbf{apr}}_{Cn}(X)$, $\underline{\mathbf{apr}}_{Nn}(X)$, $\overline{\mathbf{apr}}_{cn}(X)$, $\overline{\mathbf{apr}}_{Cn}(X)$ and $\overline{\mathbf{apr}}_{Nn}(X)$ for any subset X of U cannot be subgroups of U.

Analogously to the concepts in rough set theory, the following proposition can be stated:

Proposition 2.8. *Let U be a group, X and Y be subsets of U. We have the following statements.*

(a) *(i)* $\underline{\mathbf{apr}}_{cn}(X) \subseteq X \subseteq \overline{\mathbf{apr}}_{cn}(X)$.

(ii) $\underline{\mathbf{apr}}_{Cn}(X) \subseteq X \subseteq \overline{\mathbf{apr}}_{Cn}(X)$.

(iii) $\underline{\mathbf{apr}}_{Nn}(X) \subseteq X \subseteq \overline{\mathbf{apr}}_{Nn}(X)$.

(b) *(i)* $\underline{\mathbf{apr}}_{cn}(\emptyset) = \overline{\mathbf{apr}}_{cn}(\emptyset) = \emptyset$, $\underline{\mathbf{apr}}_{cn}(U) = \overline{\mathbf{apr}}_{cn}(U) = U$.

(ii) $\underline{\mathbf{apr}}_{Cn}(\emptyset) = \overline{\mathbf{apr}}_{Cn}(\emptyset) = \emptyset$, $\underline{\mathbf{apr}}_{Cn}(U) = \overline{\mathbf{apr}}_{Cn}(U) = U$.

(iii) $\underline{\mathbf{apr}}_{Nn}(\emptyset) = \overline{\mathbf{apr}}_{Nn}(\emptyset) = \emptyset$, $\underline{\mathbf{apr}}_{Nn}(U) = \overline{\mathbf{apr}}_{Nn}(U) = U$.

(c) (i) $\overline{\mathbf{apr}}_{cn}(X \cup Y) = \overline{\mathbf{apr}}_{cn}(X) \cup \overline{\mathbf{apr}}_{cn}(Y)$.

(ii) $\overline{\mathbf{apr}}_{Cn}(X \cup Y) = \overline{\mathbf{apr}}_{Cn}(X) \cup \overline{\mathbf{apr}}_{Cn}(Y)$.

(iii) $\overline{\mathbf{apr}}_{Nn}(X \cup Y) = \overline{\mathbf{apr}}_{Nn}(X) \cup \overline{\mathbf{apr}}_{Nn}(Y)$.

(d) (i) $\underline{\mathbf{apr}}_{cn}(X \cap Y) = \underline{\mathbf{apr}}_{cn}(X) \cap \underline{\mathbf{apr}}_{cn}(Y)$.

(ii) $\underline{\mathbf{apr}}_{Cn}(X \cap Y) = \underline{\mathbf{apr}}_{Cn}(X) \cap \underline{\mathbf{apr}}_{Cn}(Y)$.

(iii) $\underline{\mathbf{apr}}_{Nn}(X \cap Y) = \underline{\mathbf{apr}}_{Nn}(X) \cap \underline{\mathbf{apr}}_{Nn}(Y)$.

(e) (i) *If* $X \subseteq Y$ *then* $\underline{\mathbf{apr}}_{cn}(X) \subseteq \underline{\mathbf{apr}}_{cn}(Y)$ *and* $\overline{\mathbf{apr}}_{cn}(X) \subseteq \overline{\mathbf{apr}}_{cn}(Y)$.

(ii) *If* $X \subseteq Y$ *then* $\underline{\mathbf{apr}}_{Cn}(X) \subseteq \underline{\mathbf{apr}}_{Cn}(Y)$ *and* $\overline{\mathbf{apr}}_{Cn}(X) \subseteq \overline{\mathbf{apr}}_{Cn}(Y)$.

(iii) *If* $X \subseteq Y$ *then* $\underline{\mathbf{apr}}_{Nn}(X) \subseteq \underline{\mathbf{apr}}_{Nn}(Y)$ *and* $\overline{\mathbf{apr}}_{Nn}(X) \subseteq \overline{\mathbf{apr}}_{Nn}(Y)$.

Proof. Here, we establish the first condition of (b)(i). The remaining proofs are straightforward.

(b)(i) For any $x \in U$, the subgroup $\langle x \rangle$ is the cyclic subgroup generated by x. The subgroup $\langle x \rangle$ must contain at least the identity element e. The empty set ($\emptyset$) contains no elements; therefore, $e \notin \emptyset$. Consequently, the condition $\langle x \rangle \subseteq \emptyset$ cannot be satisfied for any $x \in U$. Thus, $\underline{\mathbf{apr}}_{cn}(\emptyset) = \emptyset$.

Moreover, it is evident that $\langle x \rangle \cap \emptyset = \emptyset$ for any $x \in U$. Consequently, $\overline{\mathbf{apr}}_{cn}(\emptyset) = \emptyset$. □

Proposition 2.9. *Let* U *be a group and* X *be a subgroup of* U. *Then,* $\underline{\mathbf{apr}}_{cn}(X) = X$.

Proof. It is clear that $\underline{\mathbf{apr}}_{cn}(X) \subseteq X$ from Proposition 2.8 (a)(i).

On the other hand, assume that $x \in X$. Then, since X is a subgroup of U and therefore a group, the group generated by x is obtained as a subgroup of X. By the definition of a subgroup, $\langle x \rangle$ is a subset of X. Consequently, $x \in \underline{\mathbf{apr}}_{cn}(X)$ is obtained.

Hence, $\underline{\mathbf{apr}}_{cn}(X) = X$. □

It is important to note that the converse of Proposition 2.9 does not hold. That is, if $\underline{\mathbf{apr}}_{cn}(X) = X$, then X is not necessarily a subgroup of the group U. To illustrate this, consider the additive group $U = \mathbb{Z} \times \mathbb{Z}$. Let's examine the subset $X = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \mathbb{Z}) \subseteq \mathbb{Z} \times \mathbb{Z}$. Choose elements $x = (1, 0) \in X$ and $y = (0, 1) \in X$. Clearly, $\langle x \rangle = \{(n, 0) | n \in \mathbb{Z}\} \subseteq X$ and $\langle y \rangle = \{(0, n) | n \in \mathbb{Z}\} \subseteq X$. However, $x + y = (1, 0) + (0, 1) = (1, 1) \notin X$, and furthermore, $\langle x + y \rangle = \{(n, n) | n \in \mathbb{Z}\} \not\subseteq X$. Consequently, X is not a subgroup of U since it fails to satisfy the closure property.

Nevertheless, since $X = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \mathbb{Z})$, the elements of X are either of the form $(n, 0)$ or $(0, m)$, where $n, m \in \mathbb{Z}$. For elements of the form $x = (n, 0)$, $\langle x \rangle = \{(k \cdot n, 0) | k \in \mathbb{Z}\}$. This set is a subset of $\mathbb{Z} \times \{0\}$, and therefore is contained within X. Hence, $x = (n, 0) \in \underline{\mathbf{apr}}_{cn}(X)$. For elements of the form $x = (0, m)$, $\langle x \rangle = \{(0, k \cdot m) | k \in \mathbb{Z}\}$. This set is a subset of $\{0\} \times \mathbb{Z}$, and therefore is contained within X. Hence, $x = (0, m) \in \underline{\mathbf{apr}}_{cn}(X)$. Elements outside of X are of the form (a, b), where $a, b \neq 0$. For elements of the form $x = (a, b)$ (where $a, b \neq 0$), $\langle x \rangle = \{(k \cdot a, k \cdot b) | k \in \mathbb{Z}\}$. This set is not contained within X, because for the elements $(k \cdot a, k \cdot b)$ to be in X, either $k \cdot a = 0$ or $k \cdot b = 0$ must hold. However, since $a, b \neq 0$, this is not possible. Therefore, $x = (a, b) \notin \underline{\mathbf{apr}}_{cn}(X)$. Consequently, $\underline{\mathbf{apr}}_{cn}(X) = X$.

It should be noted that when the identity element e of a group is considered, since $Cn(e) = \{x \in U | ex = xe\} = U$, it is clear that $\underline{\mathbf{apr}}_{Cn}(X) \neq X$ for any subgroup X of U. Moreover, since $Cn(x) \subseteq Nn(x)$, we have $Cn(e) = Nn(e) = U$. Consequently, it is also clear that $\underline{\mathbf{apr}}_{Nn}(X)$ cannot be equal to X for an arbitrary subgroup X of U.

In addition, given an arbitrary subgroup X of U, since every subgroup X contains the identity element e of U and by Proposition 2.6, we have $\overline{\mathbf{apr}}_{cn}(X) = \overline{\mathbf{apr}}_{Cn}(X) = \overline{\mathbf{apr}}_{Nn}(X) = U$. To illustrate this, consider the following example:

Table 2: Operation table of Q_8

	1	i	j	k	-1	-i	-j	-k
1	1	i	j	k	-1	-i	-j	-k
i	i	-1	k	-j	-i	1	-k	j
j	j	-k	-1	i	-j	k	1	-i
k	k	j	-i	-1	-k	-j	i	1
-1	-1	-i	-j	-k	1	i	j	k
-i	-i	1	-k	j	i	-1	k	-j
-j	-j	k	1	-i	j	-k	-1	i
-k	-k	-j	i	1	k	j	-i	-1

Example 2.10. *Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group where the operation table is as in Table 2.10:*

Consider the subgroup $X = \{-1, 1, -j, j\}$ of Q_8. As a result, the followings are achieved:

- $\underline{\mathbf{apr}}_{cn}(X) = \{-1, 1, -j, j\} = X$.
- $\overline{\mathbf{apr}}_{cn}(X) = Q_8$.
- $\underline{\mathbf{apr}}_{Cn}(X) = \{-j, j\} \subseteq X$.
- $\overline{\mathbf{apr}}_{Cn}(X) = Q_8$.
- $\underline{\mathbf{apr}}_{Nn}(X) = \emptyset$.
- $\overline{\mathbf{apr}}_{Nn}(X) = Q_8$.

In this way, the above statement is exemplified.

Proposition 2.11. *Let U be a group and $\{e\}$ be trival subgroup. Then,*

(a) $\underline{\mathbf{apr}}_{cn}(\{e\}) = \{e\}$, $\overline{\mathbf{apr}}_{cn}(\{e\}) = U$,

(b) $\underline{\mathbf{apr}}_{Cn}(\{e\}) = \emptyset$, $\overline{\mathbf{apr}}_{Cn}(\{e\}) = U$,

(c) $\underline{\mathbf{apr}}_{Nn}(\{e\}) = \emptyset$, $\overline{\mathbf{apr}}_{Nn}(\{e\}) = U$.

Proof. It is clear that for an arbitrary element $x \in U$, the condition $\langle x \rangle \subseteq \{e\}$ holds only when $x = e$. However, since $\langle x \rangle$ is a subgroup of U for every $x \in U$, it contains the identity element e, and thus, the intersection $\langle x \rangle \cap \{e\}$ is non-empty for every $x \in U$. Consequently, the conditions given in (a) are satisfied.

Since both $Cn(x)$ and $Nn(x)$ are subgroups for an arbitrary element x in U, they contain both the identity element e and the element x. Therefore, for any arbitrary $x \in U$, $Cn(x), Nn(x) \subseteq \{e\}$ cannot be true. Thus, we obtain $\underline{\mathbf{apr}}_{Cn}(\{e\}) = \underline{\mathbf{apr}}_{Nn}(\{e\}) = \emptyset$. Additionally, due to Proposition 2.6, we have $\overline{\mathbf{apr}}_{Cn}(\{e\}) = \overline{\mathbf{apr}}_{Nn}(\{e\}) = U$. □

Since the concept of the boundary of a subset in rough set theory is defined as the difference between the upper approximation set and the lower approximation set, the boundary of a subset in a group is defined similarly as follows:

Definition 2.12. *Let U be a group and X be a subset of U.*

(a) The subset $\mathbf{bnd}_{cn}(X) = \overline{\mathbf{apr}}_{cn}(X) - \underline{\mathbf{apr}}_{cn}(X)$ *of U is called the cyclical boundary of X.*

(b) The subset $\mathbf{bnd}_{Cn}(X) = \overline{\mathbf{apr}}_{Cn}(X) - \underline{\mathbf{apr}}_{Cn}(X)$ *of U is called the centralizing boundary of X.*

(c) *The subset* $\mathbf{bnd}_{Nn}(X) = \overline{\mathbf{apr}}_{Nn}(X) - \underline{\mathbf{apr}}_{Nn}(X)$ *of* U *is called the normalizing boundary of* X.

Example 2.13. *From Example 2.7, we have that*

- $\mathbf{bnd}_{cn}(X) = S_3 - \{I, f_c, f_1, f_2\} = \{f_a, f_b\}$.
- $\mathbf{bnd}_{Cn}(X) = S_3 - \{f_c, f_1, f_2\} = \{I, f_a, f_b\}$.
- $\mathbf{bnd}_{Nn}(X) = S_3 - \{f_c\} = \{I, f_a, f_b, f_1, f_2\}$.

Definition 2.14. *Let* U *be a group and* X *be a subset of* U.

(a) *The set* X *is called cyclical rough if its cyclical boundary is non-empty.*

(b) *The set* X *is called centralizing rough if its centralizing boundary is non-empty.*

(c) *The set* X *is called normalizing rough if its normalizing boundary is non-empty.*

A subset X *is called a crisp set if its boundaries are the empty set with respect to all neighborhoods.*

In [11], four classes of uncertainty are defined for sets according to the classical approximation spaces. These are given in the following form:

Definition 2.15. *Let* (U, R) *be an approximation space and* X *be a non-empty subset of* U. *Then,*

(a) X *is called roughly R-definable, iff* $R^{\downarrow}(X) \neq \emptyset$ *and* $R^{\uparrow}(X) \neq U$.

(b) X *is called internally R-undefinable, iff* $R^{\downarrow}(X) = \emptyset$ *and* $R^{\uparrow}(X) \neq U$.

(c) X *is called externally R-undefinable, iff* $R^{\downarrow}(X) \neq \emptyset$ *and* $R^{\uparrow}(X) = U$.

(d) X *is called totaly R-undefinable, iff* $R^{\downarrow}(X) = \emptyset$ *and* $R^{\uparrow}(X) = U$.

In parallel with Definition 2.15, definitions of the following uncertainty classes can be given for a subset of a group using the *cn*, *Cn*, and *Nn* operators.

Definition 2.16. *Let* U *be a group and* X *be non-empty subset of* U.

(a) (i) X *is roughly cn-definable, iff* $\underline{\mathbf{apr}}_{cn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{cn}(X) \neq U$.

(ii) X *is called internally cn-undefinable, iff* $\underline{\mathbf{apr}}_{cn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{cn}(X) \neq U$.

(iii) X *is called externally cn-undefinable, iff* $\underline{\mathbf{apr}}_{cn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{cn}(X) = U$.

(iv) X *is called totaly cn-undefinable, iff* $\underline{\mathbf{apr}}_{cn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{cn}(X) = U$.

(b) (i) X *is roughly Cn-definable, iff* $\underline{\mathbf{apr}}_{Cn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{Cn}(X) \neq U$.

(ii) X *is called internally Cn-undefinable, iff* $\underline{\mathbf{apr}}_{Cn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{Cn}(X) \neq U$.

(iii) X *is called externally Cn-undefinable, iff* $\underline{\mathbf{apr}}_{Cn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{Cn}(X) = U$.

(iv) X *is called totaly Cn-undefinable, iff* $\underline{\mathbf{apr}}_{Cn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{Cn}(X) = U$.

(c) (i) X *is roughly Nn-definable, iff* $\underline{\mathbf{apr}}_{Nn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{Nn}(X) \neq U$.

(ii) X *is called internally Nn-undefinable, iff* $\underline{\mathbf{apr}}_{Nn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{Nn}(X) \neq U$.

(iii) X *is called externally Nn-undefinable, iff* $\underline{\mathbf{apr}}_{Nn}(X) \neq \emptyset$ *and* $\overline{\mathbf{apr}}_{Nn}(X) = U$.

(iv) X *is called totaly Nn-undefinable, iff* $\underline{\mathbf{apr}}_{Nn}(X) = \emptyset$ *and* $\overline{\mathbf{apr}}_{Nn}(X) = U$.

Proposition 2.17. *Let U be a group and $X \subseteq G$. If $e \notin X$, then X is internally cn-undefinable.*

Proof. From Proposition 2.3, we have that $\underline{\mathbf{apr}}_{cn}(X) = \emptyset$. Moreover, since X does not contain a unit element, $\overline{\mathbf{apr}}_{cn}(X)$ cannot be equal to U. Consequently, X is an internally undefinable set according to Definition 2.16(a)(ii). □

It follows directly from Proposition 2.5, Proposition 2.6 and Proposition 2.17 that the following result holds.

Corollary 2.18. *Let U be a group and $X \subseteq G$. If $e \notin X$, then X is internally Cn-undefinable and internally Nn-undefinable.*

Proposition 2.19. *Let U be a group and X be a proper subset of U. If U is abelian, then $\underline{\mathbf{apr}}_{Cn}(X) = \underline{\mathbf{apr}}_{Nn}(X) = \emptyset$ and $\overline{\mathbf{apr}}_{Cn}(X) = \overline{\mathbf{apr}}_{Nn}(X) = U$.*

Proof. Since U is an abelian group, it is easy to see that $Cn(x) = \{y \in U | xy = yx\} = U$ for any arbitrary $x \in U$. Thus, we obtain

$$\begin{aligned} \underline{\mathbf{apr}}_{Cn}(X) &= \{x \in U | Cn(x) \subseteq X\} \\ &= \{x \in U | U \subseteq X\} = \emptyset. \end{aligned}$$

However, from Proposition 2.5, we obtain $\underline{\mathbf{apr}}_{Nn}(X) = \emptyset$.

The result $\overline{\mathbf{apr}}_{Cn}(X) = \overline{\mathbf{apr}}_{Nn}(X) = U$ can also be readily observed. □

As a direct consequence of Proposition 2.19 and Definition 2.16, we obtain the following result.

Corollary 2.20. *If U is an abelian group and X is an arbitrary subset of U, then X is both totally Cn-undefinable and totally Nn-undefinable set.*

Let U be a group acting on a set X, i.e. X be an U-set. Then, from the definition of the orbit of an element $x \in X$, a partition of X is induced, denoted by

$$\mathcal{P} = \{U \cdot x | x \in X\},$$

and an equivalence relation is naturally defined on X using this partition, given by

$$x \sim y :\Leftrightarrow U \cdot x = U \cdot y.$$

Equivalence relations induced by group actions on sets naturally give rise to approximation operators in the sense of Pawlak. In addition, if the operator $n_{on} : U \to \mathcal{P}(U)$ is defined as $n_{on}(x) = U \cdot x$, then $n_{on}(x)$ is called the orbital neighborhood of the element x. Thus, using Equations 1 and 2, lower and upper approximation sets are obtained for arbitrary subsets of X. These approximation sets are defined, for $S \subseteq X$, as

$$\underline{\mathbf{apr}}_{on}(S) = \{x \in X | n_{on}(x) \subseteq S\} \tag{4}$$

and

$$\overline{\mathbf{apr}}_{on}(S) = \{x \in X | n_{on}(x) \cap S \neq \emptyset\}, \tag{5}$$

and are called the orbital lower approximation set and orbital upper approximation set of S, respectively.

Example 2.21. *Let $U = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$ be a subgroup of the symmetric group S_5. Let U act on the set $X = \{1, 2, 3, 4, 5\}$.*

Since $n_{on}(x) = \mathbf{Orb}(x) = U \cdot x$, it is clear that $\mathbf{Orb}(1) = \mathbf{Orb}(2) = \mathbf{Orb}(3) = \{1, 2, 3\}$ and $\mathbf{Orb}(4) = \mathbf{Orb}(5) = \{4, 5\}$.

Now, let us consider the subset $S = \{3, 4, 5\}$ of X. Then, from Equations 4 and 5, we obtain

$$\underline{\mathbf{apr}}_{on}(S) = \{4, 5\}$$

and

$$\overline{\mathbf{apr}}_{on}(S) = \{1, 2, 3, 4, 5\} = X.$$

Similarly to Definition 2.12, the orbital boundary of a set, denoted by $\mathbf{bnd}_{on}(S)$, can be defined using the orbital approximations of the set as follows:

$$\mathbf{bnd}_{on}(S) = \overline{\mathbf{apr}}_{on}(S) - \underline{\mathbf{apr}}_{on}(S).$$

Therefore, using the arguments in Example 2.21, the orbital boundary of the subset $S \subseteq X$ is found to be:

$$\mathbf{bnd}_{on}(S) = \overline{\mathbf{apr}}_{on}(S) - \underline{\mathbf{apr}}_{on}(S) = X - \{4, 5\} = \{1, 2, 3\}.$$

Proposition 2.22. *Let a group U acting on a set X and $S \subseteq X$. Then,*

(a) $\underline{\mathbf{apr}}_{on}(S) \subseteq S \subseteq \overline{\mathbf{apr}}_{on}(S)$.

In Group Theory, there are various definitions for group actions. Some of these will be presented here. Then, a group action is called transitive if for any $x, y \in X$, there exists an element $g \in U$ such that $g \cdot x = y$. In other words, a group action is called transitive if it has only one orbit. In addition to that, a group action U on a set X is called primitive if $\mathbf{Orb}(x) = X$ holds for every $x \in X$. It is worth noting that all primitive actions are transitive.

Proposition 2.23. *Let X be an U-set. If the group action is transitive, then $\underline{\mathbf{apr}}_{on}(S) = \emptyset$ for arbitrary proper subset S of X, and $\overline{\mathbf{apr}}_{on}(S) = X$ for each non-empty subset S of X.*

Proof. Since the group action transitive if and only if it has exactly one orbit. The proof is straightforward. □

As a consequence of Proposition 2.23, if a group action is primitive, then it is clear that $\underline{\mathbf{apr}}_{on}(S) = \emptyset$ for every proper subset S of X, and $\overline{\mathbf{apr}}_{on}(S) = X$ for every non-empty subset S of X.

Proposition 2.24. *Let U act on itself. Then, $\underline{\mathbf{apr}}_{on}(S) = \emptyset$ for any proper subset S of U and $\overline{\mathbf{apr}}_{on}(S) = U$ for any non-empty subset S of U.*

Proof. If U acts on itself, then the desired results are easily obtained since $\mathbf{Orb}(x) = U$ for every $x \in U$. □

Analogously to Definition 2.16, the rough definability of sets using lower and upper orbital approximations can be given as follows:

Definition 2.25. *Let U be a group, X be non-empty set, U acting on X and S be subset of X.*

(i) S is roughly on-definable, iff $\underline{\mathbf{apr}}_{on}(S) \neq \emptyset$ and $\overline{\mathbf{apr}}_{on}(S) \neq X$.

(ii) S is called internally on-undefinable, iff $\underline{\mathbf{apr}}_{on}(S) = \emptyset$ and $\overline{\mathbf{apr}}_{on}(S) \neq X$.

(iii) S is called externally on-undefinable, iff $\underline{\mathbf{apr}}_{on}(S) \neq \emptyset$ and $\overline{\mathbf{apr}}_{on}(S) = X$.

(iv) S is called totaly on-undefinable, iff $\underline{\mathbf{apr}}_{on}(S) = \emptyset$ and $\overline{\mathbf{apr}}_{on}(S) = X$.

Let U be a group acting on a set X. For any $x \in X$, the stabilizer of x, denoted by $\mathbf{Stab}(x)$, is the subset of U consisting of elements that fix x. Inspired by the notion of a group action and the concept of $\mathbf{Stab}(x)$, we can define a neighborhood operator on U itself. Suppose that U acts on itself. Then, the stabilizer neighborhood operator on U, denoted by $n_{sn} : U \to \mathcal{P}(U)$, is defined for each $g \in U$ as $n_{sn}(g) = \mathbf{Stab}(g)$. Then, using the operator n_{sn}, we can define approximation sets for any subset S of U analogously to the above. These are defined for any $S \subseteq U$ as

$$\underline{\mathbf{apr}}_{sn}(S) = \{g \in U \mid n_{sn}(g) \subseteq S\} \tag{6}$$

and

$$\overline{\mathbf{apr}}_{sn}(S) = \{g \in U \mid n_{sn}(g) \cap S \neq \emptyset\} \tag{7}$$

and are called the lower stabilizer approximation of S and the upper stabilizer approximation of S, respectively. Furthermore, the expression

$$\mathbf{bnd}_{sn}(S) = \overline{\mathbf{apr}}_{sn}(S) - \underline{\mathbf{apr}}_{sn}(S)$$

is also called the stabilizer boundary of S for any $S \subseteq U$.

The following statements are directly obtained from the definitions of the lower and upper stabilizer approximations of a set.

Proposition 2.26. *Let U be a group acting on itself. For $S \subseteq U$, the following properties hold:*

(a) For every $g \in U$, g belongs to the lower stabilizer approximation of S if and only if the stabilizer neighborhood of g is equal to S, i.e. $n_{sn}(g) = \mathbf{Stab}(g) = S$.

(b) For every $g \in U$, g belongs to the upper stabilizer approximation of S if and only if the intersection of the stabilizer neighborhood of g and S is nonempty.

Proposition 2.27. *Let U be acting on itself, and S be an arbitrary subset of U.*

(a) $\underline{\mathbf{apr}}_{sn}(S) \subseteq S \subseteq \overline{\mathbf{apr}}_{sn}(S)$.

(b) Let $h \in U$ be any element. Then,

- *(i)* $\underline{\mathbf{apr}}_{sn}(hS) = h\underline{\mathbf{apr}}_{sn}(S)$.
- *(ii)* $\overline{\mathbf{apr}}_{sn}(hS) = h\overline{\mathbf{apr}}_{sn}(S)$.

Proof. The proof of (a) is straightforward.

We now proceed to prove (b)(i). Suppose that $g \in \underline{\mathbf{apr}}_{sn}(hS)$. This means $n_{sn}(g) \subseteq hS$. Consider an element $t \in n_{sn}(g) = \mathbf{Stab}(g)$. This implies that $tg = g$. Applying the group action by h on both sides: $htg = hg$. Since $t \in n_{sn}(g)$, we know t is also in U. Therefore, $ht \in n_{sn}(hg)$. Because t is arbitrary in $n_{sn}(g)$, this shows $n_{sn}(hg) \subseteq hS$. Hence, $hg \in \underline{\mathbf{apr}}_{sn}(S)$.

On the other hand, suppose that $hg \in \underline{\mathbf{apr}}_{sn}(S)$. This means $n_{sn}(hg) \subseteq S$. Let $t \in n_{sn}(g)$. This means $tg = g$. Since $h \in U$, we can consider the element $h^{-1}t \in U$. We want to show that $h^{-1}t$ leaves hg unchanged under the group action. Applying the group acytion by h on both sides of $tg = g$, we get $htg = gh$. Substituting $h^{-1}t$ for h, we have $(h^{-1}t)g = hg$. Rearranging the equation, we get $h^{-1}t(g) = hg$. Since $h^{-1}t \in U$, this implies that $h^{-1}t \in n_{sn}(hg)$. From the given assuption, we know $n_{sn}(hg) \subseteq S$. Since $h^{-1}t \in n_{sn}(hg)$, this implies $h^{-1}t \in S$. Thus, $t \in hS$. Hence, $n_{sn}(g) \subseteq hS$.

The proof for the upper approximation follows a similar logic to the lower approximation. We can show that $g \in \overline{\mathbf{apr}}_{sn}(hS) \Leftrightarrow hg \in \overline{\mathbf{apr}}_{sn}(S)$, demonstrating that applying the group action by h on both the set S and its upper approximation results in the corresponding transformed upper approximation. $\square$

3. Leveraging Group Approach Operators for Image Processing Tasks

Image processing is the field of analyzing, enhancing, and interpreting digital images by applying mathematical operations to them. These operations involve modifying the properties of the image, such as its brightness, color, texture, and shape. Image processing draws upon various disciplines, including computer science, electrical engineering, and mathematics, and has a wide range of applications in medicine, engineering, aviation, security, and many other fields [5, 6, 24, 28].

Image processing is built upon mathematical foundations, and mathematical concepts and techniques play a crucial role in the development and application of image processing algorithms. Therefore, the applicability and even implementation of abstract mathematical structures directly impacts the future of image processing. In this section, some fundamental examples will be presented as an application of the above-mentioned approach operators derived from a group and its action on a set.

Image processing consists of fundamental elements such as pixels, matrices, vectors, functions, transformations, filters, segmentation, feature extraction, and pattern recognition. These elements are intricately related and work together to perform various operations on images. The smallest units that constitute images are pixels. Each pixel represents the brightness or color value at a specific point in the image. Pixels are typically arranged in a rectangular grid. Digital images are commonly represented using mathematical structures known as matrices. Each pixel corresponds to a cell in a matrix. Matrices can store image properties such as brightness, color, and others. The representation of individual pixels' spatial coordinates and intensity values is frequently accomplished through the utilization of vectors, which are mathematical structures. A vector can be a two-dimensional vector with x and y components. Mathematical operations known as functions are employed to transform the values of pixels within an image. For instance, a function can be used to increase or decrease the brightness of an image. For the purpose of executing operations such as rotation, scaling, and other modifications, images are mapped from one domain to another using transformations. Mathematical operations known as filters are employed to eliminate undesired noise or other distortions within an image. Filters can be applied in the frequency domain or spatial domain of an image. The process of partitioning an image into distinct objects or regions is referred to as segmentation. Segmentation is often performed using techniques such as thresholding, region growing, or split-and-merge. Feature extraction is the term used to describe the process of extracting salient information such as edges, corners, and objects from an image. Feature extraction is often performed using techniques such as edge detectors, texture analyzers, or shape descriptors. The process of identifying and classifying objects within an image is referred to as pattern recognition. Pattern recognition is often performed using techniques such as K-nearest neighbors, support vector machines, or neural networks.

Image processing is a powerful tool for performing various operations on digital images using a complex combination of these fundamental elements and their relationships.

One of the primary goals of this research is to make group-theoretic concepts practical. The presented concepts are thought to have potential applications in image processing such as face recognition, image matching, and noise reduction.

The fundamental concept underlying group action is to comprehend how the elements within this group transform the elements of the set. Rotations, translations, and symmetries constitute notable instances of group action and find widespread applications across diverse fields. A rotation entails rotating an object around a fixed point by a specific angle, a translation involves moving an object in a fixed direction by a specific distance, and a symmetry is the phenomenon where an object becomes congruent to itself after a transformation. For instance, the symmetry group of a square consists of transformations that can rotate, translate, and reflect the square. These transformations aid in comprehending the symmetry properties of the square.

Let U be a group representing a set of transformations. These transformations could encompass rotations, translations, or any other operations pertinent to the specific group action under consideration. Let X be a set representing the collection of geometric shapes. We define a subset $S \subseteq X$ containing solely square shapes. This subset serves as the focal point for our rough set analysis.

Orbital Lower Approximation: This subset encompasses elements of X that can definitively be classified as belonging to S based on the group action. In our context, $\underline{\mathbf{apr}}_{on}(S)$ would comprise shapes in X that can never be transformed into non-square shapes through any element of U (the group of transformations). Analyzing this subset might reveal specific rotated squares that retain their squareness under all possible group actions (rotations in our example).

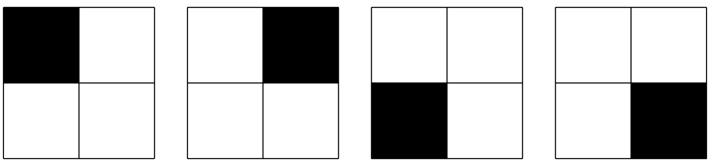
Orbital Upper Approximation: This broader subset encompasses elements of X that could potentially belong to S. It includes shapes in X that can be transformed into elements of S through some element of U. Analyzing $\overline{\mathbf{apr}}_{on}(S)$ might reveal shapes like rectangles that can be transformed into squares via specific rotations within the group action.

This methodology can be utilized for image processing operations on a set of digital images.

Let's consider a group action of rotations and translations on a set of digital images X. Each image in X can be represented by a matrix of pixel values. The group action would involve applying rotations or translations to the pixels of an image, resulting in a new image. The orbital neighborhood of an image $I \in X$ in this context would represent all possible images obtained by applying rotations and translations to I. This includes images that are identical to I (no change), rotated versions of I, and translated versions of I. $\underline{\mathbf{apr}}_{on}(S)$ would consist of images in X for which their entire orbital neighborhood (all possible rotated and translated versions) is contained within a subset S of X. This means that no matter how we rotate or translate an image in this set, we will always end up with an image that is still considered "similar" or belonging to the subset S. $\overline{\mathbf{apr}}_{on}(S)$ would consist of images in X for which their orbital neighborhood (all possible rotated and translated versions) has at least one image that belongs to S. This means that there exists some combination of rotations and translations that can be applied to an image in this set to transform it into an image in the subset S. $\underline{\mathbf{apr}}_{on}(S)$ can help identify images that are very similar to each other, even under rotations and translations. This could be useful for tasks like image matching or object recognition. Thus, it contributes to the identification of similar images. $\overline{\mathbf{apr}}_{on}(S)$ can help analyze the range of variations that an image can undergo under rotations and translations. This could be useful for tasks like image registration or image transformation. By analyzing the orbital neighborhoods of images, we can potentially identify and remove noise or artifacts that are not consistent with the true image content. This could be done by selectively removing elements from the orbital neighborhoods that do not fit the expected characteristics of the image.

Let us illustrate this with a simpler, more explanatory example. Let us consider the group U as the dihedral group D_1, that is, let U be the group of rotations by $0°$ and $180°$. Assume that the set X consists of all 2×2 binary images where each pixel is either black (1) or white (0). Here, we will focus on images with only one black pixel. Accordingly, there are four such images, such as

- I_1: Black pixel at position $(1, 1)$ (top-left).
- I_2: Black pixel at position $(1, 2)$ (top-right).
- I_3: Black pixel at position $(2, 1)$ (bottom-left).
- I_4: Black pixel at position $(2, 2)$ (bottom-right).



I_1 I_2 I_3 I_4

Thus, under the group action of U, each image is rotated by $0°$ and $180°$. Clearly, rotations of $0°$ result in the image itself. Under $180°$ rotations, I_1 becomes I_4, and I_2 becomes I_3. Consequently, the orbit partition of X yields the orbit of I_1 as $\{I_1, I_4\}$ and the orbit of I_2 as $\{I_2, I_3\}$. Considering $S = \{I_1\} \subseteq X$ as a set, the lower orbital approximation of S, clearly, is $\underline{\mathbf{apr}}_{on}(S) = \emptyset$. However, the intersection of S with the orbit of I_1 and I_4 is I_1, and the intersection of S with the orbit of I_2 and I_3 is the empty set. Therefore, $\overline{\mathbf{apr}}_{on}(S) = \{I_1, I_4\}$ is obtained. As a result,

$$\mathbf{bnd}_{on}(S) = \overline{\mathbf{apr}}_{on}(S) - \underline{\mathbf{apr}}_{on}(S) = \{I_1, I_4\} - \emptyset = \{I_1, I_4\}$$

is obtained. In conclusion, this represents the images that can be transformed into S but are not 'exactly similar' to S.

More specifically, consider a set of grayscale images X representing faces of people. The group action involves rotations and translations of the images. Define a subset S of X containing images of a specific person (e.g., Albert Einstein) from different angles and positions. The objective here is to identify images in X that resemble Albert Einstein's face even under rotations and translations.

Step-by step Algorithm:

1. **Define the Group Action:** Specify the specific rotations and translations allowed for the group action. This could involve defining the range of rotation angles and translation distances.
2. **Represent Images:** Convert each image in X into a numerical representation, such as a matrix of pixel intensities or a feature vector extracted using a suitable feature extraction technique.
3. **Compute Orbital Neighborhoods:** For each image I in X, calculate its orbital neighborhood by applying all possible rotations and translations (within the defined range) to the image's numerical representation. This results in a set of transformed images representing the possible variations of I under the group action.
4. **Construct Orbital Lower Approximation:** Identify images I in X for which their entire orbital neighborhood is contained within the subset S of images representing Albert Einstein. This can be done by comparing each transformed image in the orbital neighborhood to the images in S.
5. **Construct Upper Approximation:** Identify images I in X for which their orbital neighborhood has at least one image that belongs to the subset S. This can be done by checking if any transformed image in the orbital neighborhood matches an image in S.
6. **Analyze Results:** Interpret the identified images in the lower approximation as highly similar to Albert Einstein's face, even under rotations and translations. Analyze the images in the upper approximation to understand the range of variations that images of Albert Einstein can undergo.

Nevertheless, by employing the concept of the orbital boundary of the set, it represents the "transition zone" between images that are completely similar and those that are not at all similar.

As a reminder, the orbital boundary of a set S was defined as

$$\mathbf{bnd}_{on}(S) = \overline{\mathbf{apr}}_{on}(S) - \underline{\mathbf{apr}}_{on}(S).$$

It represents the set of elements in X that are "on the edge" of the subset S. These elements belong to the upper approximation (have at least one element in their orbital neighborhood in S) but not to the lower approximation (do not have their entire orbital neighborhood in S).

In the example of identifying Albert Einstein's face, the boundary set $\mathbf{bnd}_{on}(S)$ would consist of images in X that can be transformed into images in S (representing Albert Einstein) through some combination of rotations and translations, but do not have their entire orbital neighborhood contained within S. The boundary set provides insights into the range of variations that images of Albert Einstein can undergo while still being considered similar enough to be included in the upper approximation. It represents the "transition zone" between images that are clearly similar to Albert Einstein (lower approximation) and those that are more distant in appearance (not in the upper approximation). The boundary set can help us understand the range of facial expressions, angles, and positions that Albert Einstein's face can exhibit while still being recognized. This could be useful for developing more robust face recognition algorithms that can handle variations in appearance. Images in the boundary set might represent cases where an image classification algorithm might struggle to determine whether the image belongs to Albert Einstein or not. Analyzing these images could help identify potential weaknesses in the classification algorithm and improve its performance.

Note that the interpretation and significance of the boundary set will depend on the specific characteristics of the group action (rotations and translations), the definition of the subset S (criteria for similarity), and the nature of the images in X.

Moreover, the orbital approximation framework and the concept of the boundary set are powerful tools for analyzing the relationships between images under group actions. By carefully considering the specific details of the problem and applying these concepts appropriately, we can gain valuable insights into image similarity, variation, and classification challenges.

4. Conclusions

Algebra and group theory form the bedrock of mathematics and have numerous applications across diverse fields. Algebra studies the properties of structures, relations, and operations, while group theory delves into the symmetries and transformations of these structures. These theories are not only crucial for pure mathematics but also hold immense significance in physics, chemistry, cryptography, and computer science. Group theory finds applications ranging from explaining the behavior of subatomic particles to ensuring the security of cryptographic systems. Therefore, understanding algebra and group theory plays a pivotal role in solving many complex problems in science and engineering.

Therewithal, rough set theory emerges as a powerful tool for effectively dealing with imprecise and incomplete information. It provides a robust methodology for managing uncertainties and imperfections within information systems. By incorporating rough set theory into data analysis, one can gain a deeper understanding and better handle imprecise information, leading to more accurate and reliable outcomes. This work seamlessly integrates these two crucial mathematical materials, rendering the theoretical concepts of mathematics more applicable. Toward the conclusion of the study, an application of the proposed method to image processing is discussed.

This study highlights the significant role of group-theoretic structures in refining approximation operators, offering a novel mathematical framework for uncertainty modeling. By integrating concepts such as cyclic subgroups, centralizers, and normalizers, this approach extends traditional rough set theory, making it more adaptable to structured and transformation-invariant data. The proposed framework enhances the theoretical understanding of approximation methods and demonstrates practical applications in image processing, particularly in feature extraction and noise reduction. These findings bridge abstract mathematical theory with real-world computational challenges, providing a versatile tool for diverse scientific and engineering domains. Future research can further expand on these ideas by exploring additional group actions and their implications in artificial intelligence, pattern recognition, and data classification.

Although the study maintains its integrity both theoretically and in terms of its application method, it has the following possible limitations. By expanding these limitations, the applicability of the study can be increased.

- While this study successfully integrates group-theoretic structures with rough set theory, the proposed framework primarily focuses on fundamental group properties such as cyclic subgroups, centralizers, and normalizers. A more extensive exploration of additional algebraic structures, such as Lie groups or higher-order group actions, could further enrich the theoretical foundation.

- The application of group-based approximation operators in image processing introduces a new approach to handling uncertainty. However, the computational efficiency of these methods, especially when applied to large-scale datasets or real-time image analysis, remains an open question. Future research could focus on optimizing these operations for practical implementations.

- Although this study provides a mathematical framework and theoretical justifications, the practical performance of the proposed approach has not been extensively validated on diverse real-world datasets. Conducting experiments on different image processing tasks, such as object detection or medical imaging, would help assess its robustness and applicability.

- The proposed method relies on specific group actions and their corresponding approximation operators. However, its adaptability to more complex or irregular transformation groups, such as those encountered in deep learning-based feature extraction, remains unexplored. Extending this framework to accommodate more general transformation groups could broaden its applicability.

- While the study establishes a novel link between group theory and rough sets, a direct comparison with existing uncertainty modeling techniques, such as fuzzy sets or probabilistic models, could provide deeper insights into its advantages and limitations. Future work could incorporate benchmarking studies to evaluate the method's effectiveness relative to other approaches.

The author hopes that this article sheds light on the way of scientists that is working in this area.

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Compliance with ethical standards

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