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Remarks on selective and game versions of special classes of points

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Abstract. This work discusses some classic generalizations of points with countable character, namely W, w, $\widetilde{W}$ and q-points, under the framework of selection principles introduced by Scheepers [12]. The influence of G_δ points is also analysed.

Introduction

First countable spaces and their many generalizations have a long history in General Topology. Among the many properties introduced to extend these spaces, some of them can be regarded as selective properties related to limit points or convergent sequences. In this work we will explore some of these properties within the framework of selection principles, as defined by Scheepers [12], which we briefly recall below.

Given families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a fixed infinite set, we use the notation $S_1(\mathcal{A}, \mathcal{B})$ to express that for every sequence $(A_n)_{n\in\omega}$ of members of $\mathcal{A}$, there exists a set $\{b_n : n \in \omega\} \in \mathcal{B}$ where $b_n \in A_n$ for all n. The game associated with $S_1(\mathcal{A}, \mathcal{B})$ is denoted by $G_1(\mathcal{A}, \mathcal{B})$, which consists of a two-person infinite game, played as follows: at the first inning Player I begins by choosing an element $A_0 \in \mathcal{A}$ and Player II answers with an element $b_0 \in A_0$; at the next inning Player I chooses an element $A_1 \in \mathcal{A}$ and Player II answers with an element $b_1 \in A_1$, and so on; Player II wins a play of this game if the set $\{b_n : n \in \omega\}$ belongs to $\mathcal{B}$.

A **strategy** for a player is a function that determines how that player shall answer to her opponent based on all their previous choices. We say a strategy for a player is **winning** if there is no way for the opponent to defeat it in any legal play according to the strategy. We use the notation $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ to abbreviate the assertion "Player I has a winning strategy", and its negation is denoted by $\mathrm{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$. Similar notations are adopted regarding Player II. In general, the following implications hold:

$$\neg S_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{II} \not\uparrow G_1(\mathcal{A}, \mathcal{B}). \tag{1}$$

The reader interested in a more comprehensive discussion about selection principles can refer to [2]. Here, we will use the notions mentioned above, as well as some natural variations, to analyze q-spaces, W-spaces, $\widetilde{W}$-spaces, and $\widetilde{w}$-spaces under the unifying framework of selection principles.

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This work is organized as follows. In the first section, we recall the definitions of the aforementioned spaces and discuss them in the context of selection principles. Sections 2 and 3 deal with the natural variations of q-points and $\widetilde{W}$-points, respectively, while in the last section we discuss related problems and possible directions of further investigation.

Remark 0.1. *Along this work, (X, τ) stands for an infinite T_1 space. Also:*

$$\begin{aligned}
\tau^* &= \tau \setminus \{\emptyset\},\\
\tau_x &= \{A \in \tau : x \in A\},\\
\Omega_x &= \left\{A \subseteq X : x \in \overline{A \setminus \{x\}}\right\},\\
\Gamma_x &= \{A \in \Omega_x : \forall V \in \tau_x \quad |A \setminus V| < \aleph_0\}, \text{ and}\\
\neg\mathcal{A} &= \{A \subseteq X : A \notin \mathcal{A}\} \text{ for every family } \mathcal{A} \text{ of subsets of } X.
\end{aligned}$$

It is straightforward to check that a sequence $(x_n)_{n\in\omega}$ with infinite image converges to x if and only if its image $\{x_n : n \in \omega\}$ belongs to Γ_x. This will be important later.

1. S_1/G_1 variations

1.1. W and w-points

In [5], Gruenhage introduced the notions of *W-spaces* and *w-spaces* as a way to generalize first countable spaces while maintaining connections with other common generalizations, such as Fréchet-Urysohn spaces and bisequential spaces. Using the terminology presented in the Introduction, Gruenhage's definitions can be translated as follows.

Definition 1.1. *A point x of a topological space is a W-**point** if $\mathrm{I} \uparrow G_1(\tau_x, \neg\Gamma_x)$. A point x is a w-**point** if $\mathrm{II} \not\uparrow G_1(\tau_x, \neg\Gamma_x)$. The space is called a W-space if each of its points is a W-point. w-spaces are defined similarly.*

A typical advantage of the selection principle framework is that, when we place a property within the sequence of implications

$$\neg S_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{II} \not\uparrow G_1(\mathcal{A}, \mathcal{B}),$$

it naturally suggests variations corresponding to the other principles in the chain, creating a comprehensive exploration of related properties.

In the present case, the remaining spot corresponds to the negation of the selection principle $S_1(\mathcal{A}, \mathcal{B})$, which suggests one to take points satisfying $\neg S_1(\tau_x, \neg\Gamma_x)$ as a stronger version of W-points. Such a point could be referred as *strong* W-point. However, in this particular case, they happen to be precisely the first countable points. We emphasize this in the next proposition for ease of future references.

Proposition 1.2. *A point x has a countable local basis if and only if $S_1(\tau_x, \neg\Gamma_x)$ does not hold.*

Proof. Left to the reader. □

1.2. $\widetilde{W}$-points and their variations

In the more recent work of Doležal and Moors [3], W-spaces are generalized with an adaptation of the previous game, denoted by $\widetilde{G}(x)$: first, instead of choosing open sets containing x, Player I may pick any nonempty open set, while Player II follows the previous rules; the second difference is the winning condition for Player I, which in this game just asks for the *sequence* of points selected by Player II to have x as an *accumulation* point. Then, the point x is said to be a $\widetilde{W}$**-point** if Player I has a winning strategy in this game, and X is a $\widetilde{W}$**-space** if every point is a $\widetilde{W}$-point.

The game $\widetilde{G}(x)$ can be replaced by $G_1(\tau^*, \neg\Omega_x)$ if x is non-isolated, but this does not hold in general.

Example 1.3. *Isolated points are clearly $\widetilde{W}$-points. However, if X is a discrete space (therefore, a $\widetilde{W}$-space), then $\neg\Omega_p = \wp(X)$ for every $p \in X$, thus implying that Player I does not have a winning strategy in the game $G_1(\tau^*, \neg\Omega_p)$.*

A simple way to overcome this technical limitation is to adapt Scheepers' definitions. Instead of $S_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{A}, \mathcal{B})$, we consider the following:

- $(S_1)(\mathcal{A}, \mathcal{B})$ express that for every sequence $(A_n)_{n\in\omega}$ of members of $\mathcal{A}$, there exists a *sequence* $(b_n)_{n\in\omega}$ in $\mathcal{B}$ such that $b_n \in A_n$ for every $n \in \omega$;
- the game $(G_1)(\mathcal{A}, \mathcal{B})$ is played like $G_1(\mathcal{A}, \mathcal{B})$, except for the winning condition, which asks for the *sequence* of answers of Player II to be a member of the set $\mathcal{B}$.

As before, it follows easily that

$$\neg(S_1)(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{I} \uparrow (G_1)(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{II} \not\uparrow (G_1)(\mathcal{A}, \mathcal{B}). \tag{2}$$

Under this adapted setting, let L_x be the collection of all *sequences* accumulating at the point x. Now, by taking $\neg L_x$ to be the set of all other sequences in X, it follows that x is a $\widetilde{W}$-point if and only if $\mathrm{I} \uparrow (G_1)(\tau^*, \neg L_x)$. The corresponding position of the $\widetilde{W}$ property in the above chain of implications suggests the following.

Definition 1.4. *Let x be a point of a topological space.*

(i) *We say x is a **$\widetilde{w}$-point** if* $\mathrm{II} \not\uparrow (G_1)(\tau^*, \neg L_x)$.

(ii) *We say x is a **strong $\widetilde{W}$-point** if $(S_1)(\tau^*, \neg L_x)$ does not hold.*

$\widetilde{w}$-spaces and strong $\widetilde{W}$-spaces are defined accordingly.

So, strong $\widetilde{W}$-points are $\widetilde{W}$-points, which are $\widetilde{w}$-points. Since $\tau_x \subseteq \tau^*$ and every sequence converging to x belongs to L_x, it follows that

$$\neg S_1(\tau_x, \neg\Gamma_x) \Rightarrow \neg(S_1)(\tau^*, \neg L_x) \tag{3}$$

$$\mathrm{I} \uparrow G_1(\tau_x, \neg\Gamma_x) \Rightarrow \mathrm{I} \uparrow (G_1)(\tau^*, \neg L_x) \tag{4}$$

$$\mathrm{II} \not\uparrow G_1(\tau_x, \neg\Gamma_x) \Rightarrow \mathrm{II} \not\uparrow (G_1)(\tau^*, \neg L_x) \tag{5}$$

where (4) and (5) can be respectively interpreted as "every W-point is a $\widetilde{W}$-point"[1)] and "every w-point is a $\widetilde{w}$-point". Finally, (3) simply states that points with countable character are strong $\widetilde{W}$-points. This is no coincidence, as $\widetilde{W}$-points are precisely the points with countable π-character, just as in Proposition 1.2.

In the article [3], the authors establish that $\beta\omega$ is a $\widetilde{W}$-space by providing a countable dense set D of $\widetilde{W}$-points, namely, $D = \omega$, which is sufficient because then every point in $\overline{D} = \beta\omega$ is a $\widetilde{W}$-point (cf. [3, Lemma 3]). By the previous discussion, it would be enough to observe that $\beta\omega$ has countable π-character. It happens that $\beta\omega$ also works as counterexample to another implication.

Example 1.5. *There are $\widetilde{w}$-points which are not w-points.*

Proof. Indeed, no point x in $\beta\omega \setminus \omega$ can be a w-point, as Player II has a winning strategy in the game $G_1(\tau_x, \neg\Gamma_x)$. To verify this claim, take a choice function $f\colon \{A \in \tau^* : x \notin A\} \to X$ and define the following strategy: if A_n is the open set picked by Player I at the n-th inning of play in $G_1(\tau_x, \neg\Gamma_x)$, let the answer of Player II be $f(A_n \setminus \{x, a_0, \ldots, a_{n-1}\})$, where $a_0, \ldots, a_{n-1}$ are the previous choices of Player II. In this way, Player II selects an injective sequence $(a_n)_n$, which in turns cannot converge in $\beta\omega$. $\square$

[1)] As already indicated in [3].

In [5], Gruenhage poses the question whether every w-space is a W-space. Similarly, we do not know whether every $\widetilde{w}$-point is a $\widetilde{W}$-space. Certainly, a possible counterexample cannot be a W-space, what suggests one to check for known spaces lacking this property but verifying the $\widetilde{w}$-condition. In this task, the following proposition could be helpful.

Proposition 1.6. *If σ is a winning strategy for Player I in the game $(G_1)(\tau^*, \neg L_x)$, then the image of σ is a π-base for the point x.*

Proof. The argument is similar to the proof of Theorem 3.3 in [5]. Let $\mathrm{Im}(\sigma)$ denote the image of σ, and suppose that it does not form a π-base for the point x. In such a case, there exists an open set $U \in \tau_x$ such that every $V \in \mathrm{Im}(\sigma)$ satisfies $V \not\subseteq U$, which gives a way for Player II to win a play. Indeed, Player II can simply choose points not belonging to U. If $(y_n)_{n\in\omega}$ is the sequence of the choices made by Player II in this manner, then x is not an accumulation point of $(y_n)_{n\in\omega}$, since $y_n \notin U$ for every $n \in \omega$. This defines a winning strategy for Player II. $\square$

$$
\begin{tikzcd}
\mathrm{II} \not\uparrow G_1(\tau_x, \neg\Gamma_x) \arrow[r, equal] & w\text{-point} \arrow[r, Rightarrow] & \widetilde{w}\text{-point} \arrow[r, equal] \arrow[l, Rightarrow, bend right, "\beta\omega\ (\text{Ex. } 1.5)"', "|" marking] \arrow[d, Rightarrow, bend left, "?"] & \mathrm{II} \not\uparrow (G_1)(\tau^*, \neg L_x) \\
\mathrm{I} \uparrow G_1(\tau_x, \neg\Gamma_x) \arrow[r, equal] & W\text{-point} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \widetilde{W}\text{-point} \arrow[r, equal] \arrow[u, Rightarrow] \arrow[l, Rightarrow, bend left, "\beta\omega\ [3]", "|" marking] & \mathrm{I} \uparrow (G_1)(\tau^*, \neg L_x) \\
\neg S_1(\tau_x, \neg\Gamma_x) \arrow[r, equal] \arrow[dr, equal] & \text{strong } W\text{-point} \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, equal] & \text{strong } \widetilde{W}\text{-point} \arrow[r, equal] \arrow[u, Rightarrow] \arrow[d, equal] & \neg(S_1)(\tau^*, \neg L_x) \arrow[dl, equal] \\
 & \chi = \omega & \pi\chi = \omega &
\end{tikzcd}
$$

Figure 1: A summary of the relations between W-w and $\widetilde{W}$-$\widetilde{w}$ points

1.3. q-points and their variations

According to Michael [8, 9], a point x of a topological space X is a q**-point** if it has a sequence of neighborhoods $(V_n)_{n\in\omega}$ such that any *sequence* $(x_n)_{n\in\omega}$ satisfying $x_n \in V_n$ for all n accumulates at some point of X, possibly distinct from x. The space is referred to as a q**-space** if every point in the space is a q-point.

By calling AS the collection of sequences of X having accumulation points, it follows that x is a q-point if and only if $(S_1)(\tau_x, \neg AS)$ does not hold. Therefore, the chain of implications (2) suggests two natural weakenings of q-points, defined below.

Definition 1.7. *Let x be a point of a topological space.*

(i) *We say x is a **weak** q**-point** if* $\mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$.
(ii) *We say x is a **soft** q**-point** if* $\mathrm{II} \not\uparrow (G_1)(\tau_x, \neg AS)$.

Weak q-spaces and soft q-spaces are defined accordingly.

Clearly, first countable spaces and (locally) countably compact spaces are q-spaces and therefore also satisfy both the conditions above.

Problem 1.8. *Are there weak q-points which are not q-points?*

In the next section we shall see how another weakening of first countability, namely the G_δ-point condition[2)], can be used to relate q-points with $\widetilde{W}$-points and their selective variations.

2. The influence of G_δ-points

We begin this section by drawing attention to the following: the open sets played by Player I in the game $(G_1)(\tau_x, \neg AS)$ satisfy the conditions of the game $(G_1)(\tau^*, \neg L_x)$. However, the winning condition for Player I in the first game requires that the points chosen by Player II accumulates at *some point*, while the winning conditions for Player I in the second game explicitly imposes x to be the accumulation point. We shall see that if x is a G_δ-point, then we may force the choices of Player II to accumulate at x.

Remark 2.1. *If the answer to Problem 1.8 turns out to be "no", then the next result would follow from the fact the G_δ-q-points have countable basis. However, in the absence of answers, this is the best we got so far.*

Theorem 2.2. *Let X be a regular space such that $x \in X$ is a G_δ-point. If $\mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$, then $\mathrm{I} \uparrow (G_1)(\tau^*, \neg L_x)$, i.e., G_δ weak q-points are $\widetilde{W}$-points.*

Proof. First of all, note that we may assume x is not isolated; otherwise, x is a $\widetilde{W}$-point regardless of the hypotheses. Let σ be a winning strategy for Player I in $(G_1)(\tau_x, \neg AS)$ and let $\{U_n\}_{n\in\omega}$ be a countable family of open sets such that $\bigcap_{n\in\omega} U_n = \{x\}$. We shall obtain a winning strategy for Player I in the game $(G_1)(\tau^*, \neg L_x)$ in the following way.

Player I starts with $V_0 = \sigma(\emptyset) \cap U_0$, to which Player II responds by picking a point $x_0 \in V_0$. Since X is regular, there is an open set $A_1 \in \tau_x$ such that $x \in A_1 \subset \overline{A_1} \subset V_0$ with $x_0 \notin \overline{A_1}$. At the next inning, Player I chooses $V_1 = \sigma(x_0) \cap A_1 \cap U_1$, to which Player II replies with a point $x_1 \in V_1$, and again we may take an open set $A_2 \in \tau_x$ with $x \in A_2 \subset \overline{A_2} \subset V_1$ and $x_1 \notin \overline{A_2}$.

At the n-th inning, the regularity of X allows us to take an open set A_n such that $x \in A_n \subset \overline{A_n} \subset V_{n-1}$ with $x_{n-1} \notin \overline{A_n}$. So, following the previous pattern, Player I chooses $V_n = \sigma(x_0, ..., x_{n-1}) \cap A_n \cap U_n$. This describe a strategy for Player I in the game $(G_1)(\tau^*, \neg L_x)$.

Now, let $P = (V_0, x_0, V_1, x_1, \dots)$ be a play in this game accordingly to the strategy described above. Notice that:

- $\bigcap_{n\in\omega} V_n = \{x\}$ and for every $n \in \omega$ we have $\overline{V_{n+1}} \subset V_n$, since $V_{n+1} = \sigma(x_0, \dots, x_n) \cap A_{n+1} \cap U_{n+1} \subset \overline{A_{n+1}} \subset V_n$;
- for every $n \in \omega$ we have $x_n \in V_n \setminus \overline{A_n}$ with $x_k \notin V_n \setminus \overline{A_n}$ for all $k > n$; since X is a T_1 space, it follows that x_n is not an accumulation point of the sequence.

By the way Player I chooses their open sets, the points selected by Player II are pairwise distinct, so the sequence $((V_n, x_n))_{n\in\omega}$ is a valid play in the game $(G_1)(\tau_x, \neg AS)$. Since σ is a winning strategy for Player I in the later game, it follows that $(x_n)_{n\in\omega}$ accumulates in X, and the remarks above guarantee that the point has to be x. Indeed, for $y \in X$ such that $y \notin \{x_n : n \in \omega\} \cup \{x\}$, there exists $k \in \omega$ such that $y \notin \overline{V_k}$, implying that $X \setminus \overline{V_k}$ is an open set containing y but only finitely many points of the sequence. Thus the sequence $(x_n)_{n\in\omega}$ can only accumulate at x. □

With a similar reasoning, it is not hard to see that G_δ q-points have countable character, in which case they are trivially $\widetilde{W}$-points. Thus, the previous proposition suggests one to ask for G_δ weak q-points without countable character.

[2)]Recall that a point x in X is a G_δ-point if there is a countable family $\mathcal{G}$ of open sets such that $\{x\} = \bigcap \mathcal{G}$.

Example 2.3. *Without the G_δ-point assumption, the previous proposition may fail in general. For instance, in $X = [0, \omega_1]$ with the order topology, ω_1 is a q-point. On the other hand, Player II does have a winning strategy in the game $(G_1)(\tau^*, \neg L_{\omega_1})$: just set Player II to pick points distinct from ω_1, say α_n in the n-th turn, since in this way the sequence $(\alpha_n)_n$ cannot accumulate at ω_1.*

The pattern becomes complete once we show that with the G_δ-point assumption, II $\not\uparrow (G_1)(\tau_x, \neg AS)$ also implies II $\not\uparrow (G_1)(\tau^*, \neg L_x)$.

Proposition 2.4. *Let X be a regular space with a G_δ-point x. If II $\not\uparrow (G_1)(\tau_x, \neg AS)$, then II $\not\uparrow (G_1)(\tau^*, \neg L_x)$, i.e., every soft q-point is a $\widetilde{w}$-point.*

Proof. Let μ be a strategy for Player II in the game $(G_1)(\tau^*, \neg L_x)$. Since μ knows how to answer to every nonempty open set of X, we can use it to define a strategy for Player II in the game $(G_1)(\tau_x, \neg AS)$, where the hypothesis shall give a play in which Player II loses, meaning that the points selected along the innings accumulate at some point. The G_δ condition will guarantee that this point is x. Let $\{U_n\}_{n\in\omega}$ be a countable family of open sets such that $\{x\} = \bigcap_{n\in\omega} U_n$.

If Player I starts with $V_0 \in \tau_x$, let Player II responds with $x_0 = \mu(V_0 \cap U_0)$. In the next inning, if Player I chooses an open set $V_1 \in \tau_x$, the regularity of X gives an open set $A_0 \in \tau_x$ such that $\overline{A_0} \subseteq V_0$ and $x_0 \notin \overline{A_0}$, which we use to define x_1 as $\mu(V_0 \cap U_0, V_1 \cap A_0 \cap U_1)$. Proceeding like this, we obtain a strategy for Player II in the game $(G_1)(\tau_x, \neg AS)$. Similarly as in the previous propositions, a play in this game lost by Player II induces a sequence which accumulates at x, since Player I can play $V_{n+1} \cap A_n \cap U_{n+1}$, showing that μ is a not a winning strategy. $\square$

3. Duality and countable strong fan tightness

Recall that a topological space has **countable strong fan tightness** at a point $x \in X$ [11] if $S_1(\Omega_x, \Omega_x)$ holds. Since every point with a countable local basis fulfills $S_1(\Omega_x, \Omega_x)$, this property can be viewed as an intermediate property between first countability and countable tightness.

Following the terminology of [1] we say that two games G and G' are dual if

- Player I has a winning strategy in G if and only if Player II has a winning strategy in G'; and
- Player II has a winning strategy in G if and only if Player I has a winning strategy in G'.

Theorem 3.1. *The games $G_1(\Omega_x, \bigcup_{p\in X} \Omega_p)$ and $(G_1)(\tau_x, \neg AS)$ are dual.*

Proof. Let us first analyze how a winning strategy for Player I in one of the games yields a winning strategy for Player II in the other game.

(i) I $\uparrow (G_1)(\tau_x, \neg AS) \Rightarrow$ II $\uparrow G_1(\Omega_x, \bigcup_{p\in X} \Omega_p)$.

Let σ be a winning strategy for Player I in the game $(G_1)(\tau_x, \neg AS)$. Since the choices of Player I in the game $G_1(\Omega_x, \bigcup_{p\in X} \Omega_p)$ intercept every open set in τ_x, one can readily define a winning strategy for Player II in this game by choosing points in the open sets selected by σ. The details are left to the reader.

(ii) I $\uparrow G_1(\Omega_x, \bigcup_{p\in X} \Omega_p) \Rightarrow$ II $\uparrow (G_1)(\tau_x, \neg AS)$.

Let ρ be a winning strategy for Player I in the game $G_1(\Omega_x, \bigcup_{p\in X} \Omega_p)$. If $A_0 \in \tau_x$ is the first move of Player I in the game $(G_1)(\tau_x, \neg AS)$, then Player II may select a point x_0 belonging to $\rho(\emptyset) \cap A_0 \setminus \{x\}$ since $\rho(\emptyset) \in \Omega_x$. If Player I responds with $A_1 \in \tau_x$, then again Player II may select $x_1 \in \rho(x_0) \cap (A_1 \setminus \{x, x_0\})$, and so on. Since the strategy ρ is winning, it follows that by the end of a play $(A_0, x_0, A_1, x_1, \dots)$ we have $\{x_n : n \in \omega\} \notin \Omega_p$ for all $p \in X$ with all x_n being different, implying $(x_n)_n \in \neg AS$.

Now we shall see how winning strategies for Player II in one of the games give winning strategies for Player I in the other game.

(iii) $\mathrm{II} \uparrow (G_1)(\tau_x, \neg AS) \Rightarrow \mathrm{I} \uparrow G_1(\Omega_x, \bigcup_{p \in X} \Omega_p)$.

Let σ be a winning strategy for the Player II in the game $(G_1)(\tau_x, \neg AS)$. Since x is not isolated, there is no loss of generality in assuming $x \notin \mathrm{Im}(\sigma)$. We first show that $\{\sigma(V) : V \in \tau_x\} \in \Omega_x$. If this is not the case, then there is $U \in \tau_x$ such that $U \cap \{\sigma(V) : V \in \tau_x\} = \emptyset$, which is absurd since $\sigma(U) \in U$. A similar argument shows that $\{\sigma(V_0, ..., V_n, V) : V \in \tau_x\} \in \Omega_x$ for every $V_0, ..., V_n \in \tau_x$. Thus Player I may use the strategy σ to choose subsets in Ω_x while keeping track of a valid play in the game $G_1(\Omega_x, \bigcup_{p \in X} \Omega_p)$: Player I starts with $A_0 = \{\sigma(V) : V \in \tau_x\}$, then responds to a Player II's choice, say $\sigma(V_0)$, with $A_1 = \{\sigma(V_0, V) : V \in \tau_x\}$ and so on. It is clear that Player I wins every play of $G_1(\Omega_x, \bigcup_{p \in X} \Omega_p)$ with this strategy.

(iv) $\mathrm{II} \uparrow G_1(\Omega_x, \bigcup_{p \in X} \Omega_p) \Rightarrow \mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$.

Let ρ be a winning strategy for Player II in the game $G_1(\Omega_x, \bigcup_{x \in X} \Omega_x)$. Once again, as x is not isolated, we may assume $x \notin \mathrm{Im}(\rho)$. Now, we note that there is an open set $V_0 \in \tau_x$ such that each point $y \in V_0$ is the first movement of Player II with respect to ρ, i.e., there is an $A \in \Omega_x$ such that $y = \rho(A)$. If this is not the case, then we may obtain a subset $C \in \Omega_x$ such that $\rho(C) \notin C$, which is absurd. As in the previous paragraph, Player I may use this neighborhood V_0 as her first movement, to which Player II responds with a point $x_0 = \rho(A_0)$ for some $A_0 \in \Omega_x$. Proceeding like this, it is easy to see that Player I obtains a winning strategy in the game $(G_1)(\tau_x, \neg AS)$, as desired. □

Since $\Omega_x \subseteq \bigcup_{p \in X} \Omega_p$, both the implications $\mathrm{II} \uparrow G_1(\Omega_x, \Omega_x) \Rightarrow \mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$ and $\mathrm{II} \uparrow (G_1)(\tau_x, \neg AS) \Rightarrow \mathrm{I} \uparrow G_1(\Omega_x, \Omega_x)$ hold, and none of these are reversible, as the space $X = [0, \omega_1]$ shows: as we already showed in Example 2.3, X satisfies $\mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$, and Player I can win every play of the game $G_1(\Omega_{\omega_1}, \Omega_{\omega_1})$ by choosing the subset $[0, \omega_1)$ at every inning. Once again, the G_δ-condition gives one of the converses.

Proposition 3.2. *Let X be a regular space and let $x \in X$ be a G_δ-point. If $\mathrm{I} \uparrow (G_1)(\tau_x, \neg AS)$ then $\mathrm{II} \uparrow G_1(\Omega_x, \Omega_x)$.*

Proof. As in the proof of Theorem 2.2, let us to take a winning strategy σ for Player I in the game $(G_1)(\tau^*, \neg L_x)$ such that $x \in \bigcap \mathrm{Im}(\sigma)$. Now, Player II may use σ to play in the game $G_1(\Omega_x, \Omega_x)$ as follows: if $A_0 \in \Omega_x$ is the first move of Player I in the game $G_1(\Omega_x, \Omega_x)$, then Player II picks a point $x_0 \in \sigma(\emptyset) \cap A_0$, what can be done since $\sigma(\emptyset) \in \tau_x$; at the next inning, Player I chooses $A_1 \in \Omega_x$ and Player II answers with $x_1 \in \sigma(x_0) \cap A_1$. Proceeding like this and making sure that x_n are always different to x, we obtain a winning strategy for Player II in the game $G_1(\Omega_x, \Omega_x)$, since a play $(A_0, x_0, A_1, x_1, \dots)$ in this game, played according with the previous strategy, corresponds to the play $(\sigma(\emptyset), x_0, \sigma(x_0), x_1, \sigma(x_0, x_1), x_2, \dots)$ in the game $(G_1)(\tau^*, \neg L_x)$ according with the winning strategy σ, from which it follows that $x \in \overline{\{x_n : n \in \omega\}}$. □

Problem 3.3. *Let X be a regular space and let $x \in X$ be a G_δ-point. Are the games $(G_1)(\tau_x, \neg AS)$ and $G_1(\Omega_x, \Omega_x)$ dual?*

Notice that by the previous proposition, the above problem depends on the converse of the implication $\mathrm{II} \uparrow (G_1)(\tau_x, \neg AS) \Rightarrow \mathrm{I} \uparrow G_1(\Omega_x, \Omega_x)$.

4. The big picture

$$\begin{tikzcd}
 & & & & S_1(\Omega_x,\Omega_x) \\
w\text{-point} \arrow[r, Rightarrow] & \widetilde{w}\text{-point} \arrow[l, Rightarrow, bend right, "\beta\omega\ (\text{Ex. } 1.5)"', "|"] \arrow[d, Rightarrow, bend left, "?"] & \text{soft } q\text{-point} \arrow[l, Rightarrow, "+G_\delta"'] \arrow[r, Leftrightarrow] & \mathrm{I} \not\uparrow G_1(\Omega_x, \bigcup_{p\in X}\Omega_p) \arrow[r, Rightarrow, bend left, "+G_\delta\ ?"] & \mathrm{I} \not\uparrow G_1(\Omega_x,\Omega_x) \arrow[l, Rightarrow] \arrow[u, Rightarrow] \\
W\text{-point} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \widetilde{W}\text{-point} \arrow[u, Rightarrow] \arrow[l, Rightarrow, bend left, "\beta\omega\,[3]", "|"'] & \text{weak } q\text{-point} \arrow[u, Rightarrow] \arrow[l, Rightarrow, "+G_\delta"'] \arrow[r, Leftrightarrow] & \mathrm{II}\uparrow G_1(\Omega_x, \bigcup_{p\in X}\Omega_p) \arrow[u, Rightarrow] \arrow[r, Rightarrow, bend right, "+G_\delta"'] & \mathrm{II}\uparrow G_1(\Omega_x,\Omega_x) \arrow[u, Rightarrow] \arrow[l, Rightarrow] \\
\text{strong } W\text{-point} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, equal] & \text{strong } \widetilde{W}\text{-point} \arrow[u, Rightarrow] \arrow[d, equal] & q\text{-point} \arrow[u, Rightarrow] \arrow[l, Rightarrow, "+G_\delta"'] \\
\chi=\omega \arrow[urr, Leftrightarrow, bend right, "+G_\delta"'] & \pi\chi=\omega &
\end{tikzcd}$$

The diagram above summarizes the implications discussed along the text.

Once these local properties are established under the selection principles landscape, the typical combinatorial questions apply. For instance, considering the equivalence between the $S_1(\mathcal{A},\mathcal{B})$ principle with I $\not\uparrow G_1(\mathcal{A},\mathcal{B})$ when $\mathcal{A}$ and $\mathcal{B}$ are replaced by the family of all open coverings[3)], one can ask whether something similar happens in the present context. This is the case for q-points under the presence of a countable local π-basis.

There are natural connections with C_p-theory and covering properties as well. Indeed, for a Tychonoff space Y, Theorem 4.4 in [7] establishes that the conditions

- $C_p(Y)$ is first countable,
- $C_p(Y)$ is a q-space, and
- Y is countable

are equivalent.

Example 4.1. *Player II has a winning strategy at the game $(G_1)(\tau_{\underline{0}}, \neg AS)$ played on $C_p(\mathbb{R})$, where $\underline{0}$ indicates the constant zero function.*

Proof. Since the open sets around a continuous function f have the form $(F,\varepsilon)[f] = \{g \in C_p(\mathbb{R}) : |g(x) - f(x)| < \varepsilon$ for all $x \in F\}$ for finite subsets $F \subset \mathbb{R}$ and $\varepsilon > 0$, if Player I picks an open set $(F_n, \varepsilon_n)[\underline{0}]$ around $\underline{0}$ at the n-th turn, Player II can choose an open set U_n around F_n with length less than $\frac{1}{2^n}$, and then pick a function f_n such that $f_n|_{F_n} \equiv 0$ while $f_n \equiv n$ outside U_n. Notice that in the end of a play according to this strategy, any point $x \in X \setminus \bigcup_{n\in\omega} U_n$ is such that $(\{x\}, 1)[f]$ witnesses that f cannot be an accumulation point of $(f_n)_n$, for every $f \in C_p(\mathbb{R})$. $\square$

Problem 4.2. *Is there any uncountable space Y such that $C_p(Y)$ is a weak q-space?*

Since the pseudocharacter of $C_p(Y)$ is the density of the space Y [6], an uncountable separable space Y such that $C_p(Y)$ satisfies $\mathrm{II} \uparrow G_1(\Omega_f, \Omega_f)$ could provide positive answers to some instances of the above problem[4)].

[3)]Pawlikowski [10].

[4)]Thus, by a result of Scheepers [14], Y should be a space such that $\mathrm{II} \uparrow G_1(\Omega(Y), \Omega(Y))$ holds, where $\Omega(Y)$ is the family of all ω-coverings of Y [4].

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