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On estimation of Hankel determinants for certain class of starlike functions

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Abstract. In the present study, we consider two subclasses of starlike and convex functions, denoted by $\mathcal{S}^*_{\mathcal{B}}$ and $\mathcal{C}_{\mathcal{B}}$ respectively, associated with a bean-shaped domain. Further, we estimate certain sharp initial coefficients, as well as second, third and fourth-order Hankel determinants for functions belonging to the class $\mathcal{S}^*_{\mathcal{B}}$. Additionally, we compute sharp second and third-order Hankel determinants for functions belonging to the class $\mathcal{C}_{\mathcal{B}}$.

1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and suppose $\mathcal{S}$ be a subclass of $\mathcal{A}$ comprising univalent functions. Consider $\mathcal{P}$ to be the class of analytic functions defined on $\mathbb{D}$ with a positive real part, expressed as $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Suppose h and g are two analytic functions, we say h is subordinate to g, symbolically $h \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| \leq |z|$ such that $h(z) = g(w(z))$. A substantial body of literature exists on coefficient problems, ranging from the seminal Bieberbach's conjecture of 1916 to contemporary research (see [4]). There are two prominent subclasses of $\mathcal{S}$ consisting of starlike and convex functions, respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$. Further, in 1992, Ma and Minda [13] unified various subclasses of $\mathcal{S}^*$ and $\mathcal{C}$ by introducing the following two classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \tag{2}$$

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and

$$C(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\} \tag{3}$$

where φ is analytic and univalent, known as Ma-Minda function satisfying the conditions $\operatorname{Re}\varphi(z) > 0$, $\varphi(\mathbb{D})$ symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$. Recently, many Ma-Minda classes are introduced and studied by several authors by appropriately choosing $\varphi(z)$ in (2). See the various Ma-Minda subclasses of starlike functions listed in the first column with the corresponding choice of $\varphi(z)$ in the second column of Table 1.

Table 1: List of sharp third-order Hankel determinants

Class	$\varphi(z)$	Sharp $\lvert H_3(1)\rvert$	Reference
$\mathcal{S}^*$	$(1+z)/(1-z)$	4/9	[2, 6]
$\mathcal{S}^*_{\wp}$	$1 + ze^z$	1/9	[20]
$\mathcal{SL}^*$	$\sqrt{1+z}$	1/36	[3]
$\mathcal{S}^*_e$	e^z	1/9	[19]
$\mathcal{S}^*_\rho$	$1 + \sinh^{-1}(z)$	1/9	[9]
$\mathcal{S}^*_\tau$	$1 + \arctan z$	1/9	[8]

The concept of Hankel determinants, introduced in 1966 (see [16]), continues to be a topic of significant interest for researchers today. The definition of the qth Hankel determinant $H_q(n)$ of analytic functions $f \in \mathcal{A}$, under the assumption that $a_1 := 1$, is as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad n, q \in \mathbb{N}. \tag{4}$$

The expressions for the second and third-order Hankel determinants for specific values of q and n, are denoted by $H_2(3)$ and $H_3(1)$, respectively, given by

$$H_2(3) := a_3a_5 - a_4^2 \tag{5}$$

and

$$H_3(1) := 2a_2a_3a_4 - a_3^3 - a_4^2 - a_2^2a_5 + a_3a_5. \tag{6}$$

Deriving a sharp bound for Hankel determinants is a formidable challenge, prompting numerous researchers to endeavor to do so for various subclasses of starlike functions, see [1, 3, 5, 14, 15] and some are listed in the third column of Table 1. Recently, Kumar and Yadav [10], by choosing $\varphi(z) = \sqrt{1 + \tanh z}$, introduced and studied the Ma-Minda subclass of starlike functions $\mathcal{S}^*_{\mathcal{B}}$ associated with a bean-shaped domain, given by

$$\mathcal{S}^*_{\mathcal{B}} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1 + \tanh z} \right\}.$$

Motivated by it, we introduce the following convex counterpart of the above class:

$$C_{\mathcal{B}} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \sqrt{1 + \tanh z} \right\}.$$

The authors in [10] have investigated the geometric properties of the univalent function $\sqrt{1 + \tanh z}$, along with some inclusion and sharp radius results involving $\mathcal{S}^*_{\mathcal{B}}$, as well as implications of first-order differential subordination. This class can be further studied to know more about the behavior of the coefficients of functions belonging to this class, second and third-order differential subordination, etc. Thus, studying such subclasses of starlike and convex functions open new avenues in the field of research. Taking this aspect in account, in our current investigation, we focus on coefficient-related problems concerning the aforementioned classes $\mathcal{S}^*_{\mathcal{B}}$ and $\mathcal{C}_{\mathcal{B}}$, which are yet not addressed in the literature. In fact, we are finding the sharp bounds of the initial coefficients, second and third-order Hankel determinants, as well as possible bound of the fourth-order Hankel determinant for functions belonging to the class $\mathcal{S}^*_{\mathcal{B}}$. Additionally, we establish sharp bounds of the second and third-order Hankel determinants for functions belonging to the class $\mathcal{C}_{\mathcal{B}}$.

2. Coefficient related problems for $\mathcal{S}^*_{\mathcal{B}}$

In this section, we start by determining the sharp bounds of the initial coefficients a_i for $(i = 2, 3, 4, 5)$ followed by establishing the sharp bounds of the second and third-order Hankel determinants for functions $f \in S^*_{\mathcal{B}}$. Subsequently, we derive the bounds for a_6 and a_7 to deduce a possible bound of the fourth-order Hankel determinant for functions $f \in S^*_{\mathcal{B}}$.

2.1. *Sharp initial coefficient bounds*

Let $f \in \mathcal{S}^*_{\mathcal{B}}$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{1 + \tanh w(z)}. \tag{7}$$

Suppose that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$ and consider $w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in (7) and then comparing the coefficients, we obtain the expressions of a_i $(i = 2, 3, ..., 7)$ in terms of p_j $(j = 1, 2, ..., 5)$, given as

$$a_2 = \frac{p_1}{4}, \quad a_3 = \frac{1}{64}\Big(8p_2 - 3p_1^2\Big), \quad a_4 = \frac{1}{2304}\Big(23p_1^3 - 168p_1p_2 + 192p_3\Big), \tag{8}$$

$$a_5 = \frac{1}{18432}\Big(-11p_1^4 + 528p_1^2p_2 - 576p_2^2 - 1056p_1p_3 + 1152p_4\Big), \tag{9}$$

$$a_6 = \frac{1}{1843200}\Big(50880p_1p_2^2 - 2367p_1^5 - 8560p_1^3p_2 + 46080p_1^2p_3 - 96000p_2p_3 - 86400p_1p_4 + 92160p_5\Big) \tag{10}$$

and

$$a_7 = \frac{1}{530841600}\Big(601421p_1^6 - 2365320p_1^4p_2 - 4818240p_1^2p_2^2 + 4723200p_2^3 + 26150400p_1p_2p_3 - 2648640p_1^3p_3 - 11980800p_3^2 + 11577600p_1^2p_4 - 23500800p_2p_4 - 21012480p_1p_5\Big). \tag{11}$$

The results stated below are necessary for proving our main result.

Lemma 2.1. *[12] Let $p \in \mathcal{P}$ be of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$|p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4| \le 2 \tag{12}$$

and

$$|p_3 - 2p_1p_2 + p_1^3| \le 2. \tag{13}$$

Lemma 2.2. *[13] Let $p \in \mathcal{P}$ be of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$|p_2 - \beta p_1^2| \leq \begin{cases} 2 - 4\beta, & \beta \leq 0; \\ 2, & 0 \leq \beta \leq 1; \\ 4\beta - 2, & \beta \geq 1 \end{cases}$$

when $\beta < 0$ or $\beta > 1$, the equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. If $0 < \beta < 1$, then the inequality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $\beta = 0$, the equality holds if and only if $p(z) = (1+\eta)(1+z)/(2(1-z)) + (1-\eta)(1-z)/(2(1+z))(0 \leq \eta \leq 1)$ or one of its rotations. If $\beta = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in case of $\beta = 0$. Though the above upper bound is sharp for $0 < \beta < 1$, still it can be improved as follows:

$$|p_2 - \beta p_1^2| + \beta |p_1|^2 \leq 2 \quad (0 < \beta \leq 1/2) \tag{14}$$

and

$$|p_2 - \beta p_1^2| + (1-\beta)|p_1|^2 \leq 2 \quad (1/2 < \beta \leq 1).$$

Also, we recall that

$$\max_{0 \leq t \leq 4}(At^2 + Bt + C) = \begin{cases} C, & B \leq 0, A \leq \frac{-B}{4}; \\ 16A + 4B + C, & B \geq 0, A \geq \frac{-B}{8} \quad \text{or} \quad B \leq 0, A \geq \frac{-B}{4}; \\ \dfrac{4AC - B^2}{4A}, & B > 0, A \leq \frac{-B}{8}. \end{cases} \tag{15}$$

Theorem 2.3. *If $f \in \mathcal{S}^*_{\mathcal{B}}$, then (i) $|a_2| \leq 1/2$, (ii) $|a_3| \leq 1/4$, (iii) $|a_4| \leq 1/6$ and (iv) $|a_5| \leq 847/3216 \simeq 0.263371 \cdots$. These bounds are sharp.*

Proof. (*i*) Since $|p_n| \leq 2$ for $n \geq 1$, therefore, from (8), $|a_2| \leq 1/2$.
(*ii*) For a_3, we use the inequality $|p_2 - \mu p_1^2| \leq 2\max\{1, |2\mu - 1|\}$ given by Ma and Minda [13], which yields $|a_3| \leq 1/4$.
(*iii*) For a_4, (7) is re-written as:

$$zf'(z) = \sqrt{1 + \tanh(w(z))}f(z). \tag{16}$$

On substituting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $w(z) = \sum_{k=1}^{\infty} w_k z^k$ in (16) and comparing the coefficients of z^4, we get

$$6a_4 = \left(w_3 + \frac{1}{4}w_1 w_2 - \frac{13}{48}w_1^3\right).$$

Now using [17, Lemma 2, p.128], we deduce that $|6a_4| \leq 1$ and hence the result follows.
(*iv*) From (9), we get the expression of a_5 as

$$\begin{aligned} a_5 &= \frac{1}{16}\left(-\frac{11}{1152}p_1^4 + \frac{11}{24}p_1^2 p_2 - \frac{p_2^2}{2} - \frac{11}{12}p_1 p_3 + p_4\right) \\ &= \frac{1}{16}\left(-\frac{1}{2}P + \frac{1}{12}p_1 Q - \frac{7}{8}p_1^2 R + \frac{1}{2}p_4\right), \end{aligned}$$

which further gives

$$|a_5| \leq \frac{1}{16}\left(\frac{1}{2}|P| + \frac{1}{12}|p_1||Q| + \frac{7}{8}|p_1|^2|R| + \frac{1}{2}|p_4|\right),$$

where $P = p_1^4 - 3p_1^2 p_2 + p_2^2 + 2p_1 p_3 - p_4$, $Q = p_3 - 2p_1 p_2 + p_1^3$ and $R = p_2 - (67/144)p_1^2$. Moreover, using the bounds of $|P| \leq 2$ from (12), $|Q| \leq 2$ from (13) and $|R| \leq 2$ from (14), respectively, we obtain

$$|a_5| \leq \frac{1}{16}\left(\frac{7}{3} + \frac{7}{4}|p_1|^2 - \frac{469}{1152}|p_1|^4\right).$$

Now, we obtain $|7|p_1|^2/4 - 469|p_1|^4/1152| \leq 126/67$ using (15) by taking $A = -469/1152$, $B = 7/4$ and $C = 0$, which leads to the desired estimate for $|a_5|$. The sharpness of the result can be witnessed when $p_1 = p_4 = 2$, $p_2 = (737 + \sqrt{963326})/402$ and $p_3 = -2$. The function

$$f_n(z) = z \exp\left(\int_0^z \frac{\sqrt{1 + \tanh(t^{n-1})} - 1}{t} dt \right)$$

acts as the extremal function for the initial coefficients a_n for $n = 2, 3$ and 4. □

The formula for p_j $(j = 2, 3, 4)$, which is included in the Lemma 2.4 below, plays a vital role in establishing the sharp bounds for Hankel determinants and forms the foundation for our main results.

Lemma 2.4. *[11, 12] Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then for some γ, η and ρ such that $|\gamma| \leq 1$, $|\eta| \leq 1$ and $|\rho| \leq 1$, we have*

$$2p_2 = p_1^2 + \gamma(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta,$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho).$$

2.2. *Sharp Hankel determinants for $\mathcal{S}^*_{\mathcal{B}}$*

The following theorem presents the sharp bound for $|H_3(1)|$ for functions belonging to the class $\mathcal{S}^*_{\mathcal{B}}$.

Theorem 2.5. *Let $f \in \mathcal{S}^*_{\mathcal{B}}$, then*

$$|H_3(1)| \leq 1/36. \tag{17}$$

This result is sharp.

Proof. Since the class $\mathcal{P}$ is invariant under rotation, we can take $p_1 =: p$ belongs to the interval [0,2]. Substitute the values of a_i $(i = 2, 3, 4, 5)$ in (6) from (8) and (9). We get

$$H_3(1) = \frac{1}{21233664}\Big(-3511p^6 - 5160p^4 p_2 - 14400p^2 p_2^2 - 124416p_2^3 + 56256p^3 p_3 + 216576pp_2p_3 - 147456p_3^2 - 145152p^2 p_4 + 165888p_2p_4 \Big).$$

After simplifying the calculations using Lemma 2.4, we obtain

$$H_3(1) = \frac{1}{21233664}\Big(\beta_1(p,\gamma) + \beta_2(p,\gamma)\eta + \beta_3(p,\gamma)\eta^2 + \beta_4(p,\gamma,\eta)\rho\Big),$$

for $\gamma, \eta, \rho \in \mathbb{D}$. Here

$$\begin{aligned}
\beta_1(p,\gamma) : &= -1099p^6 - 1872\gamma^2 p^2(4-p^2)^2 - 20736\gamma^3(4-p^2)^2 - 5760\gamma^3 p^2(4-p^2)^2 + 1152\gamma^4 p^2(4-p^2)^2 \\
&\quad + 3084\gamma p^4(4-p^2) + 624p^4\gamma^2(4-p^2) - 7776p^4\gamma^3(4-p^2) - 31104\gamma^2 p^2(4-p^2), \\
\beta_2(p,\gamma) : &= 96(1-|\gamma|^2)(4-p^2)(149p^3 + 324\gamma p^3 + 228p\gamma(4-p^2) - 48p\gamma^2(4-p^2)), \\
\beta_3(p,\gamma) : &= 1152(1-|\gamma|^2)(4-p^2)(-32(4-p^2) - 4|\gamma|^2(4-p^2) + 27p^2\bar{\gamma}), \\
\beta_4(p,\gamma,\eta) : &= 10368(1-|\gamma|^2)(4-p^2)(1-|\eta|^2)(4(4-p^2)\gamma - 3p^2).
\end{aligned}$$

By choosing $x = |\gamma|$, $y = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_3(1)| \leq \frac{1}{21233664}\Big(|\beta_1(p,\gamma)| + |\beta_2(p,\gamma)|y + |\beta_3(p,\gamma)|y^2 + |\beta_4(p,\gamma,\eta)|\Big) \leq A(p,x,y),$$

where

$$A(p,x,y) = \frac{1}{21233664}\Big(A_1(p,x) + A_2(p,x)y + A_3(p,x)y^2 + A_4(p,x)(1-y^2)\Big), \tag{18}$$

with

$$\begin{aligned} A_1(p,x) :&= 1099p^6 + 1872x^2p^2(4-p^2)^2 + 20736x^3(4-p^2)^2 + 5760x^3p^2(4-p^2)^2 + 1152x^4p^2(4-p^2)^2 \\ &\quad + 3084xp^4(4-p^2) + 624p^4x^2(4-p^2) + 7776p^4x^3(4-p^2) + 31104x^2p^2(4-p^2), \\ A_2(p,x) :&= 96(1-x^2)(4-p^2)(149p^3 + 324xp^3 + 228px(4-p^2) + 48px^2(4-p^2)), \\ A_3(p,x) :&= 1152(1-x^2)(4-p^2)(32(4-p^2) + 4x^2(4-p^2) + 27p^2x), \\ A_4(p,x) :&= 10368(1-x^2)(4-p^2)(4x(4-p^2) + 3p^2). \end{aligned}$$

In the closed cuboid $Q : [0,2] \times [0,1] \times [0,1]$, we now maximize $A(p,x,y)$, by locating the maximum values in the interior of the six faces, on the twelve edges, and in the interior of Q.

1. We start by taking into account every internal point of Q. Assume that $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. We calculate $\partial A/\partial y$, partially differentiate (18) with respect to y to identify the points of maxima in the interior of Q. We get

$$\frac{\partial A}{\partial y} = \frac{(4-p^2)(1-x^2)}{21233664}\Big(24px(5+2x) + p^3(17+24x-12x^2) + 96(8-9x+x^2)y - 12p^2(25-27x+2x^2)y\Big).$$

Now $\partial A/\partial y = 0$ gives

$$y = \frac{48px(19+4x) + p^3(149+96x-48x^2)}{24(1-x)(16(x-8)+p^2(59-4x))} =: y_0.$$

The existence of critical points requires that y_0 belong to $(0,1)$, which is only possible when

$$300p^2 + 864x + 24p^2x^2 > 17p^3 + 120px + 24p^3x + 48px^2 - 12p^3x^2 + 768 + 864x + 24p^2x^2. \tag{19}$$

Now, we find the solution satisfying the inequality (19) for the existence of critical points using the hit and trial method. If we assume p tends to 0, then there does not exist any $x \in (0,1)$ satisfying (19). But, when p tends to 2, (19) holds for all $x < 175/648$. We also observe that there does not exist any $p \in (0,2)$ when $x \in (174/648, 1)$. Similarly, if we assume x tends to 0, then for all $p > 1.61687$, (19) holds. After calculations, we observe that there does not exist any $x \in (0,1)$ when $p \in (0, 1.61687)$. Thus, the domain for the solution of the equation is $(1.61687, 2) \times (0, 175/648)$. Now, we examine that $\frac{\partial A}{\partial y}|_{y=y_0} \neq 0$ in $(1.61687, 2) \times (0, 175/648)$. So, we conclude that the function A has no critical point in $(0,2) \times (0,1) \times (0,1)$.

2. The interior of each of the cuboid Q's six faces is now being considered.
On the face $p = 0$: We have $x, y \in (0,1)$ and

$$A(0,x,y) = \frac{16y^2 - 14x^2y^2 - 2x^4y^2 - 9x^3(1-2y^2) + 18x(1-y^2)}{576} =: k_1(x,y). \tag{20}$$

Since

$$\frac{\partial k_1}{\partial y} = \frac{(1-x^2)(x+1)(8-x)y}{144} \neq 0, \quad x, y \in (0,1),$$

indicates that k_1 has no critical points in $(0,1)\times(0,1)$.
On the face $p = 2$: We have $x, y \in (0,1)$ and

$$A(2,x,y) = \frac{1099}{331776}. \tag{21}$$

On the face $x = 0$: We have $p \in (0,2)$, $y \in (0,1)$ and

$$A(p,0,y) = \frac{1099p^6 + (4-p^2)(14304p^3y + 36864y^2(4-p^2) + 31104p^2(1-y^2))}{21233664} =: k_2(p,y). \tag{22}$$

To determine the points of maxima, we solve $\partial k_2/\partial p = 0$ and $\partial k_2/\partial y = 0$. After solving $\partial k_2/\partial y = 0$, we get

$$y = \frac{149p^3}{24(59p^2-128)} =: y_p. \tag{23}$$

In order to have $y_p \in (0,1)$ for the given range of y, $p > 1.61687$ is required. Based on calculations, $\partial k_2/\partial p = 0$ gives

$$p(41472 - 20736p^2 + 1099p^4 + 28608py - 11920p^3y - 139776y^2 + 45312p^2y^2) = 0. \tag{24}$$

After substituting (23) in (24), we have

$$75497472p - 107347968p^3 + 50314752p^5 - 8246384p^7 + 133989p^9 = 0. \tag{25}$$

A numerical calculation suggests that $p \approx 1.24748 \in (0,2)$ is the solution of (25). So, we conclude that k_2 does not have any critical point in $(0,2)\times(0,1)$.
On the face $x = 1$: We have $p \in (0,2)$ and

$$A(p,1,y) = \frac{331776 + 99072p^2 - 34704p^4 - 1601p^6}{21233664} =: k_3(p). \tag{26}$$

While computing $\partial k_3/\partial p = 0$, $p \approx 1.14405 =: p_0$ comes out to be the critical point. Undergoing simple calculations, k_3 achieves its maximum value ≈ 0.0187629 at p_0.
On the face $y = 0$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$A(p,x,0) = \frac{1}{21233664}\Big(-1601p^6 - 331776(-1-2x+2x^3) + 2304p^2(97-144x-54x^2+144x^3) - 144p^4(457-288x-216x^2+288x^3)\Big) =: k_4(p,x).$$

After further calculations such as,

$$\frac{\partial k_4}{\partial x} = \frac{(4-p^2)}{1024}\Big(8 - 24x^2 + p^2(-2-3x+6x^2)\Big)$$

and

$$\frac{\partial k_4}{\partial p} = \frac{1}{3538944}\Big(-1601p^5 + 768p(97-144x-54x^2+144x^3) - 96p^3(457-288x-216x^2+288x^3)\Big),$$

we observe that only real solutions (p,x) of the system of equations $\partial k_4/\partial x = 0$ and $\partial k_4/\partial p = 0$ are $(2, 1.74724)$; $(2, -1.74724)$; $(-1.36584, -0.835809)$; $(1.36584, -0.835809)$; $(2, -1.74724)$ and $(-0.854598, 0.524203)$. Thus, no solution exists in $(0,2)\times(0,1)$, resulting in no critical points.

On the face $y = 1$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$A(p,x,1) = \frac{1}{21233664}\Big(1099p^6 + 31104p^2(4-p^2) + 11484p^4(4-p^2) + 20736(4-p^2)^2 + 8784p^2(4-p^2)^2 - 1152(4-p^2)(1-x^2)(-16(8+x^2) + p^2(32-27x+4x^2)) + 96(4-p^2)(1-x^2)(48px(19+4x) + p^3(149+96x-48x^2))\Big) =: k_5(p,x).$$

Simple calculations leads to

$$\frac{\partial k_5}{\partial x} = \frac{(4-p^2)}{110592}\Big(-192x(7+2x^2) + 6p^2(27+56x-81x^2+16x^3) + 24p(19+8x-57x^2-16x^3) + p^3(48-197x-144x^2+96x^3)\Big)$$

and

$$\frac{\partial k_5}{\partial p} = \frac{1}{3538944}\Big(-1601p^5 - 3072x(-19-4x+19x^2+4x^3) + 768p(-85+54x+112x^2-54x^3+16x^4) + 96p^3(15-216x-224x^2+216x^3-32x^4) - 80p^4(149+96x-197x^2-96x^3+48x^4) + 192p^2(149-132x-245x^2+132x^3+96x^4)\Big).$$

We note that the only real solutions (p,x) of the system of equations $\partial k_5/\partial x = 0$ and $\partial k_5/\partial p = 0$ are $(-5.5858, 2.7083)$; $(2, -2.2645)$; $(-2, 0.357662)$; $(-1.98983, 0.350993)$; $(0.932759, -1.56488)$ and $(-1.03049, -0.27789)$. Thus, no solution exists in $(0,2)\times(0,1)$, resulting in no critical points.

3. We next examine the maxima attained by $A(p,x,y)$ on the edges of the cuboid Q.
From (22), we have $A(p,0,0) = (124416p^2 - 31104p^4 + 1099p^6)/21233664 =: e_1(p)$. It is easy to observe that $e_1'(p) = 0$ whenever $p = 0$ and $p = 1.50801 \in [0,2]$ as its points of minima and maxima, respectively. Hence,

$$A(p,0,0) \leq 0.00635802, \quad p \in [0,2].$$

Now considering (22) at $y = 1$, we get $A(p,0,1) = (589824 - 294912p^2 + 57216p^3 + 36864p^4 - 14304p^5 + 1099p^6)/21233664 =: e_2(p)$. It is easy to observe that $e_2'(p) < 0$ in $[0,2]$ and hence $p = 0$ serves as the point of maxima. So,

$$A(p,0,1) \leq \frac{1}{36} \approx 0.0277778, \quad p \in [0,2].$$

Through computations, (22) shows that $A(0,0,y)$ attains its maxima at $y = 1$. This implies that

$$A(0,0,y) \leq \frac{1}{36}, \quad y \in [0,1].$$

Since, (26) does not involve x, we have $A(p,1,1) = A(p,1,0) = (331776 + 99072p^2 - 34704p^4 - 1601p^6)/21233664 =: e_3(p)$. Now, $e_3'(p) = 33024p - 23136p^3 - 1601p^5 = 0$ when $p = 0$ and $p = 1.14405$ in the interval $[0,2]$, acting as the points of minima and maxima, respectively. Hence

$$A(p,1,1) = A(p,1,0) \leq 0.0187629, \quad p \in [0,2].$$

After considering $p = 0$ in (26), we get, $A(0, 1, y) = 1/64$. The equation (21) has no variables. So, on the edges, the maximum value of $M(p, x, y)$ is

$$A(2, 1, y) = A(2, 0, y) = A(2, x, 0) = A(2, x, 1) = \frac{1099}{331776}, \quad x, y \in [0, 1].$$

Using (20), we obtain $A(0, x, 1) = (16 - 14x^2 + 9x^3 - 2x^4)/576 =: e_4(x)$. After calculations, we see that $e_4(x)$ is a decreasing function in $[0, 1]$ and attains its maxima at $x = 0$. Thus

$$A(0, x, 1) \leq \frac{1}{36}, \quad x \in [0, 1].$$

Again utilizing (20), we get $A(0, x, 0) = x(2 - x^2)/64 =: e_5(x)$. On further calculations, we get $e_5'(x) = 0$ when $x = \sqrt{2/3}$, as the point of maxima. Thus

$$A(0, x, 0) \leq 0.0170103, \quad x \in [0, 1].$$

Given all the cases, the inequality (17) holds. Let the function $f_0 \in \mathcal{S}^*_{\mathcal{B}}$, be defined as

$$f_0(z) = z \exp\left(\int_0^z \frac{\sqrt{1 + \tanh t^3} - 1}{t} dt \right) = z + \frac{z^4}{6} - \frac{z^7}{144} + \cdots, \tag{27}$$

with $f_0(0) = 0$ and $f_0'(0) = 1$, acts as an extremal function for the bound of $|H_3(1)|$ for $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/6$. □

Next, we find the sharp bound of $|H_2(3)|$ for functions belonging to the class $S^*_{\mathcal{B}}$, given by

Theorem 2.6. *Let $f \in \mathcal{S}^*_{\mathcal{B}}$, then*

$$|H_2(3)| \leq \frac{1}{36}. \tag{28}$$

This bound is sharp.

Proof. We proceed on the similar lines as in the proof of Theorem 2.5. Assuming $p_1 =: p \in [0, 2]$, we substitute the values of a_i $(i = 3, 4, 5)$ from (8) and (9) into (5), we obtain

$$H_2(3) = \frac{1}{10616832}\Big(-761p^6 + 408p^4p_2 - 2880p^2p_2^2 - 41472p_2^3 + 10848p^3p_3 + 52992pp_2p_3 - 73728p_3^2 - 31104p^2p_4 + 82944p_2p_4 \Big).$$

Using Lemma 2.4, we arrive at

$$H_2(3) = \frac{1}{5308416}\Big(\beta_5(p, \gamma) + \beta_6(p, \gamma)\eta + \beta_7(p, \gamma)\eta^2 + \beta_8(p, \gamma, \eta)\rho\Big),$$

where $\gamma, \eta, \rho \in \mathbb{D}$,

$$\begin{aligned}
\beta_5(p, \gamma) :&= -437p^6 - 5904\gamma^2p^2(4 - p^2)^2 + 1440\gamma^3p^2(4 - p^2)^2 + 576\gamma^4p^2(4 - p^2)^2 \\
&\quad + 5184\gamma^2p^2(4 - p^2) - 852\gamma p^4(4 - p^2) - 4008p^4\gamma^2(4 - p^2) + 1296p^4\gamma^3(4 - p^2), \\
\beta_6(p, \gamma) :&= (1 - |\gamma|^2)(4 - p^2)(5424p^3 - 5184p^3\gamma - 2880p\gamma(4 - p^2) - 2304p\gamma^2(4 - p^2)), \\
\beta_7(p, \gamma) :&= 576(1 - |\gamma|^2)(4 - p^2)(-32(4 - p^2) - 9p^2\bar{\gamma} - 4|\gamma|^2(4 - p^2)) \\
\beta_8(p, \gamma, \eta) :&= 5184(1 - |\gamma|^2)(4 - p^2)(1 - |\eta|^2)(p^2 + 4\gamma(4 - p^2)).
\end{aligned}$$

Additionally, by using the fact that $|\rho| \leq 1$ and taking $x = |\gamma|$, $y = |\eta|$, we obtain

$$|H_2(3)| \leq \frac{1}{10616832}\Big(|\beta_5(p,\gamma)| + |\beta_6(p,\gamma)|y + |\beta_7(p,\gamma)|y^2 + |\beta_8(p,\gamma,\eta)|\Big) \leq B(p,x,y),$$

where

$$B(p,x,y) = \frac{1}{10616832}\Big(B_1(p,x) + B_2(p,x)y + B_3(p,x)y^2 + B_4(p,x)(1-y^2)\Big), \tag{29}$$

with

$$\begin{aligned}
B_1(p,x) :&= 437p^6 + 5904p^2x^2(4-p^2)^2 + 1440p^2x^3(4-p^2)^2 + 576p^2x^4(4-p^2)^2 \\
&\quad + 5184p^2x^2(4-p^2) + 852p^4x(4-p^2) + 4008p^4x^2(4-p^2) + 1296p^4x^3(4-p^2), \\
B_2(p,x) :&= (4-p^2)(1-x^2)(5424p^3 + 5184p^3x + 2880px(4-p^2) + 2304px^2(4-p^2)), \\
B_3(p,x) :&= 576(4-p^2)(1-x^2)(32(4-p^2) + 9p^2x + 4x^2(4-p^2)), \\
B_4(p,x) :&= 5184(4-p^2)(1-x^2)(p^2 + 4x(4-p^2)).
\end{aligned}$$

At this point, we must maximize $B(p,x,y)$ in the closed cuboid $R : [0,2] \times [0,1] \times [0,1]$. By identifying the maximum values on the twelve edges, the interior of R, and the interiors of the six faces, we can prove this.

1. We start by taking into account, every interior point of R. Assume that $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. On differentiating (29) with respect to y, to locate the points of maxima in the interior of R, we obtain

$$\frac{\partial B}{\partial y} = \frac{(1-x^2)(4-p^2)}{221184}\Big(48px(5+4x) + p^3(113 + 48x - 48x^2) + 384(8 - 9x + x^2)y - 24p^2(41 - 45x + 4x^2)y\Big).$$

Now $\partial B/\partial y = 0$ gives

$$y = \frac{48px(5+4x) + p^3(113 + 48x - 48x^2)}{24(1-x)(16(x-8) + p^2(41-4x))} =: y_1.$$

Since y_1 must be a member of $(0,1)$ for critical points to exist, this is only possible if

$$48px(5+4x) + p^3(113 + 48x - 48x^2) < 24(1-x)(16(x-8) + p^2(41-4x)). \tag{30}$$

Now, we find the solutions satisfying the inequality (30) for the existence of critical points using hit and trial method. If p tends to 0 and 2, then no $x \in (0,1)$ exists satisfying (30). Similarly, if take x tending to 0 and 1, then there does not exist any $p \in (0,2)$ such that (30) holds. So, we conclude that the function B has no critical point in $(0,2) \times (0,1) \times (0,1)$.

2. Now, we study the interior of six faces of the cuboid R.
<u>On the face $p = 0$</u>: We have $x, y \in (0,1)$ and

$$B(0,x,y) = \frac{(1-x^2)(8y^2 + x^2y^2 + 9x(1-y^2))}{288} =: l_1(x,y). \tag{31}$$

We note that, in $(0,1) \times (0,1)$, l_1 does not have any critical point. As

$$\frac{\partial l_1}{\partial y} = \frac{y(1-x^2)(x-1)(x-8)}{144} \neq 0 \quad x, y \in (0,1).$$

On the face $p = 2$: We have $x, y \in (0,1)$ and

$$B(2, x, y) = \frac{437}{165888}. \tag{32}$$

On the face $x = 0$: We have $p \in (0,2)$, $y \in (0,1)$ and

$$B(p, 0, y) = \frac{437p^6 + (4-p^2)(5424p^3 y + 18432(4-p^2)y^2 + 5184p^2(1-y^2))}{10616832} =: l_2(p, y). \tag{33}$$

We solve $\partial l_2/\partial p$ and $\partial l_2/\partial y$ to locate the points of maxima. On solving $\partial l_2/\partial y = 0$, we obtain

$$y = \frac{113p^3}{24(41p^2 - 128)} =: y_p. \tag{34}$$

For the given range of y, we should have $y_p \in (0,1)$ but no such $p \in (0,2)$ exists.

On the face $x = 1$: We have $p \in (0,2)$ and

$$B(p, 1, y) = \frac{(147456p^2 - 43920p^4 + 2201p^6)}{10616832} =: l_3(p). \tag{35}$$

When we compute $\partial l_3/\partial p = 0$, $p = 1.40378 =: p_0$ turns out to be the critical point. According to elementary calculations, l_3 reaches its maximum value ≈ 0.0128915 at p_0.

On the face $y = 0$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$\begin{aligned} B(p, x, 0) = \frac{1}{10616832}\Big(&331776x(1-x^2) + 2304p^2(9 - 72x + 41x^2 + 82x^3 + 4x^4) \\ &- 48p^4(108 - 503x + 650x^2 + 564x^3 + 96x^4) \\ &+ p^6(437 - 852x + 1896x^2 + 144x^3 + 576x^4)\Big) =: l_4(p, x). \end{aligned}$$

On computations, we obtain

$$\frac{\partial l_4}{\partial x} = \frac{(4-p^2)}{884736}\Big(6912(1-3x^2) + 96p^2(-18 + 41x + 69x^2 + 8x^3) - p^4(-71 + 316x + 36x^2 + 192x^3)\Big)$$

and

$$\begin{aligned} \frac{\partial l_4}{\partial p} = \frac{1}{1769472}\Big(&768p(9 - 72x + 41x^2 + 82x^3 + 4x^4) - 32p^3(108 - 503x + 650x^2 \\ &+ 564x^3 + 96x^4) + p^5(437 - 852x + 1896x^2 + 144x^3 + 576x^4)\Big). \end{aligned}$$

The common real solutions (p, x) of the system of equations, $\partial l_4/\partial x = 0$ and $\partial l_4/\partial p = 0$ are $(-2, -2.86143)$; $(-2.6516, -0.571214)$; $(-2, -0.247504)$; $(-2, -2.86143)$; $(2, 0.0163426)$; $(-2, 0.0163426)$; $(2.6516, -0.571214)$; $(2, -0.247504)$; $(0, 0.57735)$; $(0, -0.57735)$ and $(-0.914024, 0.721238)$. Thus, no solution exists in $(0,2) \times (0,1)$, resulting in no critical points.

On the face $y = 1$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$\begin{aligned} B(p, x, 1) = \frac{1}{10616832}\Big(&9216px(5 + 4x - 5x^2 - 4x^3) + 36864(8 - 7x^2 - x^4) - 2304p^2(64 - 9x - 106x^2 \\ &- x^3 - 12x^4) - 48p^5(113 + 48x - 161x^2 - 48x^3 + 48x^4) + 192p^3(113 - 12x \\ &- 209x^2 + 12x^3 + 96x^4) + 48p^4(384 - 37x - 1094x^2 - 24x^3 - 144x^4) \\ &+ p^6(437 - 852x + 1896x^2 + 144x^3 + 576x^4)\Big) =: l_5(p, x). \end{aligned}$$

On computations, we get

$$\frac{\partial l_5}{\partial x} = \frac{(4-p^2)}{884736}\Big(-1536x(7+2x^2) + 192p(5+8x-15x^2-16x^3) + 48p^2(9+156x+3x^2+32x^3) + 8p^3(24-161x-72x^2+96x^3) + p^4(71-316x-36x^2-192x^3)\Big)$$

and

$$\frac{\partial l_5}{\partial p} = \frac{1}{1769472}\Big(1536x(5+4x-5x^2-4x^3) - 768p(64-9x-106x^2-x^3-12x^4) - 40p^4(113+48x-161x^2-48x^3+48x^4) + 96p^2(113-12x-209x^2 + 12x^3+96x^4) + 32p^3(384-37x-1094x^2-24x^3-144x^4) + p^5(437-852x+1896x^2+144x^3+576x^4)\Big).$$

The common real solutions (p, x) of the system of equations, $\partial l_5/\partial x = 0$ and $\partial l_5/\partial p = 0$ are $(10.3578, 0.237179)$; $(2.61706, 3.83978)$; $(-2, -2.97479)$; $(2, 2.03127)$; $(-2.52708, -1.62061)$; $(-2, -0.783621)$; $(-1.37805, -1.02453)$; $(-1.77448, 0.0488452)$; $(2.00019, -0.515602)$; $(0, 0)$ and $(1.35192, -1.00909)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

3. Now, we calculate the maximum values achieved by $B(p, x, y)$ on the edges of the cuboid R. From (33), we have $B(p, 0, 0) = (20736p^2 - 5184p^4 + 437p^6)/10616832 =: f_1(p)$. It is easy to observe that $f_1'(p) = 0$ when $p = 0$ in the interval $[0, 2]$. Thus, the maximum value of $f_1(p)$ is 0.
Now considering (33) at $y = 1$, we get $B(p, 0, 1) = (294912 - 147456p^2 + 21696p^3 + 18432p^4 - 5424p^5 + 437p^6)/10616832 =: f_2(p)$. It is easy to observe that $f_2'(p)$ is a decreasing function in $[0, 2]$ and hence $p = 0$ acts as its point of maxima. Thus

$$B(p, 0, 1) \leq \frac{1}{36}, \quad p \in [0, 2].$$

Through computations, (33) shows that $B(0, 0, y) = y^2/36$, attains its maximum value at $y = 1$. This implies that

$$B(0, 0, y) \leq \frac{1}{36}, \quad y \in [0, 1].$$

Since, (35) is independent of x, we have $B(p, 1, 1) = B(p, 1, 0) = (147456p^2 - 43920p^4 + 2201p^6)/10616832 =: f_3(p)$. Now, $f_3'(p) = 294912p - 175680p^3 + 13206p^5 = 0$ when $p = 0$ and $p = 1.40378$ in the interval $[0, 2]$, acting as points of minima and maxima, respectively. Hence

$$B(p, 1, 1) = B(p, 1, 0) \leq 0.0128915, \quad p \in [0, 2].$$

On substituting $p = 0$ in (35), we get, $B(0, 1, y) = 0$. The equation (32) is independent of the all the variables namely p, x and y. Thus the maximum value of $B(p, x, y)$ on the edges $p = 2, x = 1; p = 2, x = 0; p = 2, y = 0$ and $p = 2, y = 1$, respectively, is given by

$$B(2, 1, y) = B(2, 0, y) = B(2, x, 0) = B(2, x, 1) = \frac{437}{165888}, \quad x, y \in [0, 1].$$

From (33), we obtain $B(0, 0, y) = y^2/36$. A simple calculation shows that

$$B(0, 0, y) \leq \frac{1}{36}, \quad y \in [0, 1].$$

Using (31), we obtain $B(0,x,1) = (8-7x^2-x^4)/288 =: f_4(x)$. After calculations, we see that f_4 is a decreasing function in $[0,1]$ and hence attains its maximum value at $x=0$. Thus

$$B(0,x,1) \leq \frac{1}{36}, \quad x \in [0,1].$$

On again using (31), we get $B(0,x,0) = 9x(1-x^2)/288 =: f_5(x)$. On further calculations, we get $f_5'(x)=0$ when $x = 1/\sqrt{3}$, acting as its point of maxima. Thus

$$B(0,x,0) \leq 0.0120281, \quad x \in [0,1].$$

In view of all the cases, the inequality (28) holds. The function specified in (27) acts as an extremal function for the bounds of $|H_2(3)|$ having values $a_3 = a_5 = 0$ and $a_4 = 1/6$. □

2.3. Fourth Hankel determinant

Given that sharp bounds for third-order Hankel determinants have been attained for various subclasses of starlike functions, as shown in Table 1, determining bounds for fourth-order Hankel determinants proves to be considerably challenging, necessitating extensive computations. Consequently, there have been relatively few efforts in this direction in the existing literature, for recent advancements, we refer [7, 9, 19, 20]. Subsequently, we introduce a lemma which is required in forthcoming results.

Lemma 2.7. *[7, 18] Let $p = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$. Then*

$$|p_n| \leq 2, \quad n \geq 1,$$

$$|p_{n+k} - \nu p_n p_k| \leq \begin{cases} 2, & 0 \leq \nu \leq 1; \\ 2|2\nu - 1|, & \textit{otherwise}, \end{cases}$$

and

$$|p_1^3 - \nu p_3| \leq \begin{cases} 2|\nu - 4|, & \nu \leq 4/3; \\ 2\nu\sqrt{\dfrac{\nu}{\nu-1}}, & 4/3 < \nu. \end{cases}$$

Now, we try to estimate possible bounds of sixth and seventh coefficient of function $f \in \mathcal{S}^*_{\mathcal{B}}$ as follows:

Lemma 2.8. *Let $f \in \mathcal{S}^*_{\mathcal{B}}$, then $|a_6| \leq 0.611233$ and $|a_7| \leq 0.690994$.*

Proof. From (10), we have

$$1843200a_6 = -2367p_1^5 - 8560p_1^3p_2 + 50880p_1p_2^2 - 86400p_1p_4 + 92160p_5 - 96000p_2p_3 + 46080p_1^2p_3$$

or

$$1843200|a_6| \leq |p_1^3(-2367p_1^2 - 8560p_2)| + |p_1(50880p_2^2 - 86400p_4)| + |92160p_5 - 96000p_2p_3| + |46080p_1^2p_3|.$$

Using Lemma 2.7 and the triangle inequality, we arrive at

$$|a_6| \leq \frac{35207}{57600} \approx 0.611233. \tag{36}$$

Similarly,

$$530841600a_7 = 601421p_1^6 - 2365320p_1^4p_2 + 4723200p_2^3 - 4818240p_1^2p_2^2 - 2648640p_1^3p_3 - 11980800p_3^2 \\ + 26150400p_1p_2p_3 - 23500800p_2p_4 + 11577600p_1^2p_4 - 21012480p_1p_5$$

or

$$530841600|a_7| \leq |p_1^4(601421p_1^2 - 2365320p_2)| + |p_2^2(4723200p_2 - 4818240p_1^2)| + |p_3(2648640p_1^3 - 11980800p_3)| + |p_2(26150400p_1p_3 - 23500800p_4)| + |p_1(11577600p_1^2p_4 - 21012480p_5)|.$$

By employing Lemma 2.7 and the triangle inequality, we obtain

$$|a_7| \leq \frac{31841}{46080} \approx 0.690994. \tag{37}$$

□

We derive the expression of the fourth Hankel determinant, upon substituting $q = 4$ and $n = 1$ in (4) as follows:

$$H_4(1) = a_7H_3(1) - a_6B_1 + a_5B_2 - a_4B_3, \tag{38}$$

where

$$B_1 := a_6(a_3 - a_2^2) + a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4), \tag{39}$$

$$B_2 := a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \tag{40}$$

and

$$B_3 := a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3). \tag{41}$$

Next, we determine the bound of these B_i for $i = 1, 2, 3$. By substituting the values from (8)-(10) in (39), we get

$$\begin{aligned}707788800B_1 = {}& 110689p_1^7 - 299496p_1^5p_2 - 855360p_1^4p3 + 269760p_1^3p_2^2 + 944640p_1p_2^3 - 1128960p_1^2p_2p_3 \\ & - 3686400p_2^2p_3 + 4608000p_1p_3^2 + 2668800p_1^3p_4 + 460800p_1p_2p_4 - 3686400p_3p_4 \\ & - 3870720p_1^2p_5 + 4423680p_2p_5\end{aligned}$$

or

$$\begin{aligned}707788800|B_1| \leq {}& |p_1^4(110689p_1^7 - 855360p_3)| + |p_2^2(269760p_1^3 - 3686400p_3)| + |4608000p_1p_3^2| \\ & + |p_1p_2(944640p_2^2 + 460800p_4)| + |p_1^2p_2(-299496p_1^3 - 1128960p_3)| \\ & + |p_4(2668800p_1^3 - 3686400p_3)| + |p_5(4423680p_2 - 3870720p_1^2)|\end{aligned}$$

Using Lemma 2.7 and the triangle inequality, we get

$$\begin{aligned}|B_1| \leq {}& \frac{1}{707788800}\left(110590464 + 255027456\sqrt{\frac{165}{744671}} + 471859200\sqrt{\frac{15}{3559}} + 117964800\sqrt{\frac{3}{53}}\right) \\ & \approx 0.244544.\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}5662310400B_2 = {}& 927648p_1^5p_3 - 149225p_1^8 + 785840p_1^6p_2 + 3939840p_1^3p_2p_3 - 2073600p_1^2p_2^3 - 8294400p_2^4 \\ & + 23316480p_1p_2^2p_3 - 2025600p_1^4p_2^2 - 11704320p_1^2p_3^2 - 29491200p_2p_3^2 + 25804800p_1p_3p_4 \\ & - 3676800p_1^4p_4 - 3225600p_1^2p_2p_4 - 29491200p_1p_2p_5 + 27648000p_2^2p_4 - 22118400p_4^2 \\ & + 6144000p_1^3p_5 + 23592960p_3p_5\end{aligned}$$

or

$$\begin{aligned}5662310400|B_2| \leq{}& |p_1^5(927648p_3 - 149225p_1^3)| + |p_1^3p_2(785840p_1^3 + 3939840p_3)| \\ &+ |p_2^3(-2073600p_1^2 - 8294400p_2)| + |p_1p_2^2(23316480p_3 - 2025600p_1^3)| \\ &+ |p_3^2(-11704320p_1^2 - 29491200p_2)| + |p_1p_4(25804800p_3 - 3676800p_1^3)| \\ &+ |p_1p_2(-3225600p_1p_4 - 29491200p_5)| + |p_4(27648000p_2^2 - 22118400p_4)| \\ &+ |p_5(6144000p_1^3 + 23592960p_3|.\end{aligned}$$

With the aid of Lemma 2.7 and the triangle inequality, we obtain

$$\begin{aligned}|B_2| &\leq \frac{1}{5662310400}\left(1982371840 + \frac{22263552}{149225}\sqrt{\frac{6442}{778423}} + \frac{43008}{383}\sqrt{\frac{42}{2305}} + \frac{97152}{1055}\sqrt{\frac{759}{11089}}\right) \\ &\approx 0.350099. \qquad (43)\end{aligned}$$

and

$$\begin{aligned}305764761600B_3 ={}& 133166592p_1^5p_3 - 8521200p_1^8 + 126489600p_1^4p_2p_3 - 621000p_1^7p_2 + 331084800p_1^4p_2^2 \\ &+ 2362245120p_1p_2^2p_3 + 6566400p_1^3p_2^3 - 199065600p_2^3p_3 + 99532800p_1p_2^4 \\ &- 49766400p_1p_2^2p_4 - 1260195840p_1^3p_2p_3 - 1327104000p_2p_3^2 - 103680000p_1^2p_2^2p_3 \\ &- 20260800p_1^5p_2^2 - 177638400p_1^3p_3^2 - 176947200p_3^3 + 398131200p_1^2p_3p_4 \\ &- 12182400p_1^5p_4 + 331776000p_1^3p_5 + 1274019840p_3p_5 + 398131200p_2p_3p_4 \\ &- 311040000p_1^4p_4 - 1194393600p_1p_3p_4 + 1492992000p_1^2p_2p_4 - 1592524800p_1p_2p_5 \\ &+ 637009920p_1^2p_3^2 - 160625p_1^9 - 879206400p_1^2p_2^3 + 99532800p_1p_2p_3^2 - 298598400p_1p_4^2\end{aligned}$$

or

$$\begin{aligned}305764761600|B_3| \leq{}& |p_1^5(133166592p_3 - 8521200p_1^3)| + |p_1^4p_2(126489600p_3 - 621000p_1^3)| \\ &+ |p_1p_2^2(331084800p_1^3 + 2362245120p_3)| + |p_2^3(6566400p_1^3 - 199065600p_3)| \\ &+ |p_1p_2^2(99532800p_2^2 - 49766400p_4)| + |p_2p_3(-1260195840p_1^3 - 1327104000p_3)| \\ &+ |p_1^2p_2^2(-103680000p_3 - 20260800p_1^3)| + |p_3^2(-177638400p_1^3 - 176947200p_3)| \\ &+ |p_1^2p_4(398131200p_3 - 12182400p_1^3)| + |p_5(331776000p_1^3 + 1274019840p_3)| \\ &+ |p_2p_4(398131200p_3 - 95385600p_1^3)| + |p_1p_4(-311040000p_1^3 - 1194393600p_3)| \\ &+ |p_1p_2(1492992000p_1p_4 - 1592524800p_5)| + |637009920p_1^2p_3^2| + |160625p_1^9| \\ &+ |879206400p_1^2p_2^3| + |99532800p_1p_2p_3^2| + |298598400p_1p_4^2|.\end{aligned}$$

Through Lemma 2.7 and the triangle inequality, we have

$$\begin{aligned}|B_3| \leq{}& \frac{1}{305764761600}\left(210371822080 + 147726139392\sqrt{\frac{114}{32059}} + 101921587200\sqrt{\frac{6}{1489}}\right. \\ &\left. + 64762675200\sqrt{\frac{366}{23309}} + 12740198400\sqrt{\frac{6}{73}} + \frac{76441190400}{\sqrt{557}}\right) \\ \approx{}& 0.787068. \qquad (44)\end{aligned}$$

Upon substituting values from (36), (37), (2.5), (42)-(44), and Theorem 2.3 in (38), we obtain the following result, given by

Theorem 2.9. *Let $f \in \mathcal{S}^*_{\mathcal{B}}$, then $|H_4(1)| \leq 0.169251$.*

3. Sharp Hankel Determinants for $\mathcal{C}_{\mathcal{B}}$

In this section, we determine the sharp bounds for the second and third-order Hankel determinants for functions $f \in \mathcal{C}_{\mathcal{B}}$. Below, we provide the expressions for the initial coefficients of functions $f \in \mathcal{C}_{\mathcal{B}}$ in terms of Carathéodory coefficients, serving as a foundation for subsequent calculations.

Let $f \in \mathcal{C}_{\mathcal{B}}$, then there exists a Schwarz function $w(z)$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \sqrt{1 + \tanh w(z)}. \tag{45}$$

Suppose that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$ and consider $w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in (45) and then comparing the coefficients, we obtain the expressions of a_i $(i = 2, 3, ..., 7)$ in terms of p_j $(j = 1, 2, ..., 5)$, given by

$$a_2 = \frac{1}{8}p_1, \quad a_3 = \frac{1}{192}\Big(8p_2 - 3p_1^2\Big), \quad a_4 = \frac{1}{9216}\Big(23p_1^3 - 168p_1p_2 + 192p_3\Big) \tag{46}$$

and

$$a_5 = \frac{1}{92160}\Big(-11p_1^4 + 528p_1^2p_2 - 576p_2^2 - 1056p_1p_3 + 1152p_4\Big). \tag{47}$$

The following theorem presents the sharp bound for $|H_3(1)|$ for functions belonging to the class $\mathcal{C}_{\mathcal{B}}$.

Theorem 3.1. *Let $f \in \mathcal{C}_{\mathcal{B}}$, then*

$$|H_3(1)| \leq 1/576. \tag{48}$$

This result is sharp.

Proof. Since the class $\mathcal{P}$ is invariant under rotation, the value of p_1 belongs to the interval [0,2]. Let $p_1 =: p$ and then substitute the values of a_i $(i = 2, 3, 4, 5)$ in (6) from (46) and (47). We get

$$H_3(1) = \frac{1}{424673280}\Big(-3581p^6 - 11184p^4p_2 - 2880p^2p_2^2 - 141312p_2^3 + 73344p^3p_3 + 211968pp_2p_3 - 184320p_3^2 - 165888p^2p_4 + 221184p_2p_4\Big).$$

After simplifying the calculations through Lemma 2.4, we obtain

$$H_3(1) = \frac{1}{424673280}\Big(\beta_9(p,\gamma) + \beta_{10}(p,\gamma)\eta + \beta_{11}(p,\gamma)\eta^2 + \beta_{12}(p,\gamma,\eta)\rho\Big),$$

for $\gamma, \eta, \rho \in \mathbb{D}$. Here

$$\begin{aligned}
\beta_9(p,\gamma) :&= -1157p^6 - 5328\gamma^2p^2(4-p^2)^2 - 15360\gamma^3(4-p^2)^2 - 4224\gamma^3p^2(4-p^2)^2 + 2304\gamma^4p^2(4-p^2)^2 \\
&\quad + 3144\gamma p^4(4-p^2) - 1056p^4\gamma^2(4-p^2) - 6912p^4\gamma^3(4-p^2) - 27648\gamma^2p^2(4-p^2), \\
\beta_{10}(p,\gamma) :&= 192(1-|\gamma|^2)(4-p^2)(83p^3 + 144\gamma p^3 + 84p\gamma(4-p^2) - 48p\gamma^2(4-p^2)), \\
\beta_{11}(p,\gamma) :&= 9216(1-|\gamma|^2)(4-p^2)(-5(4-p^2) - |\gamma|^2(4-p^2) + 3p^2\bar{\gamma}), \\
\beta_{12}(p,\gamma,\eta) :&= 27648(1-|\gamma|^2)(4-p^2)(1-|\eta|^2)(2(4-p^2)\gamma - p^2).
\end{aligned}$$

By choosing $x = |\gamma|$, $y = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_3(1)| \leq \frac{1}{424673280}\Big(|\beta_9(p,\gamma)| + |\beta_{10}(p,\gamma)|y + |\beta_{11}(p,\gamma)|y^2 + |\beta_{12}(p,\gamma,\eta)|\Big) \leq C(p,x,y),$$

where

$$C(p,x,y) = \frac{1}{424673280}\Big(C_1(p,x) + C_2(p,x)y + C_3(p,x)y^2 + C_4(p,x)(1-y^2)\Big), \tag{49}$$

with

$$\begin{aligned} C_1(p,x) : &= 1157p^6 + 5328x^2p^2(4-p^2)^2 + 15360x^3(4-p^2)^2 + 4224x^3p^2(4-p^2)^2 + 2304x^4p^2(4-p^2)^2 \\ &\quad + 3144xp^4(4-p^2) + 1056p^4x^2(4-p^2) + 6912p^4x^3(4-p^2) + 27648x^2p^2(4-p^2), \\ C_2(p,x) : &= 192(1-x^2)(4-p^2)(83p^3 + 144xp^3 + 84px(4-p^2) + 48px^2(4-p^2)), \\ C_3(p,x) : &= 9216(1-x^2)(4-p^2)(5(4-p^2) + x^2(4-p^2) + 3p^2x), \\ C_4(p,x) : &= 27648(1-x^2)(4-p^2)(2x(4-p^2) + p^2). \end{aligned}$$

In the closed cuboid $S : [0,2] \times [0,1] \times [0,1]$, we now maximize $C(p,x,y)$, by locating the maximum values in the interior of the six faces, on the twelve edges, and in the interior of S.

1. We start by taking into account every internal point of S. Assume that $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. We calculate $\partial C/\partial y$, partially differentiate (49) with respect to y to identify the points of maxima in the interior of U. We get

$$\frac{\partial C}{\partial y} = \frac{(4-p^2)(1-x^2)}{2211840}\Big(48px(7+4x) + p^3(83+60x-48x^2) - 96p^2(8-9x+x^2)y + 384(5-6x+x^2)y\Big).$$

Now $\partial C/\partial y = 0$ gives

$$y = \frac{48px(7+4x) + p^3(83+60x-48x^2)}{96(1-x)(p^2(8-x) - 4(5-x))} =: y_0.$$

The existence of critical points require that y_0 belong to $(0,1)$, which is only possible when

$$83p^3 + 336px + 60p^3x + 192px^2 - 48p^3x^2 < 96(-1+x)(20 - 8p^2 - 4x + p^2x). \tag{50}$$

Now, we find the solution satisfying the inequality (50) for the existence of critical points using the hit and trial method. If we assume p tends to 0, then there does not exist any $x \in (0,1)$ satisfying (50). But, when p tends to 2, (50) holds only when $x < 61/288$. Similarly, if we assume x tends to 0, then for all $p > 1.75665$, (50) holds. After calculations, we observe that there does not exist any $x \in (0,1)$ when $p \in (0, 1.75665)$. Thus, the domain for the solution is $(1.75665, 2) \times (0, 61/288)$. Now, we examine that $\frac{\partial C}{\partial y}|_{y=y_0} \neq 0$ in $(1.75665, 2) \times (0, 61/288)$. So, we conclude that the function M has no critical point in $(0,2) \times (0,1) \times (0,1)$.

2. The interior of each of the cuboid S's six faces is now being considered.
<u>On the face $p = 0$:</u> We have $x, y \in (0,1)$ and

$$C(0,x,y) = \frac{20x^3 + 12(1-x^2)(5+x^2)y^2 + 72x(1-x^2)(1-y^2)}{34560} =: m_1(x,y). \tag{51}$$

Since

$$\frac{\partial m_1}{\partial y} = \frac{(1-x)^2(x+1)(5-x)y}{1440} \neq 0, \quad x, y \in (0,1),$$

indicates that m_1 has no critical points in $(0,1) \times (0,1)$.
<u>On the face $p = 2$:</u> We get

$$C(2,x,y) = \frac{1157}{6635520} \approx 0.000174365, \quad x, y \in (0,1). \tag{52}$$

On the face $x = 0$: We have $p \in (0, 2)$, $y \in (0, 1)$ and

$$C(p, 0, y) = \frac{1157p^6 + (4 - p^2)(15936p^3 y + 46080(4 - p^2)y^2 + 27648p^2(1 - y^2))}{424673280} =: m_2(p, y). \tag{53}$$

To determine the points of maxima, we solve $\partial m_2/\partial p = 0$ and $\partial m_2/\partial y = 0$. After solving $\partial m_2/\partial y = 0$, we get

$$y = \frac{83p^3}{384(2p^2 - 5)} =: y_p. \tag{54}$$

In order to have $y_p \in (0, 1)$ for the given range of y, $p > 1.75665$ is required. Based on calculations, $\partial m_2/\partial p = 0$ gives

$$36864p - 18432p^3 + 1157p^5 + 31872p^2 y - 13280p^4 y - 159744py^2 + 49152p^3 y^2 = 0. \tag{55}$$

On substituting (54) into (55), we have

$$1843200p - 2396160p^3 + 1021152p^5 - 152402p^7 + 2367p^9 = 0. \tag{56}$$

A numerical calculation suggests that $p \approx 1.28894 \in (0, 2)$ is the solution of (56). So, we conclude that m_2 does not have any critical point in $(0, 2) \times (0, 1)$.
On the face $x = 1$: We have $p \in (0, 2)$ and

$$C(p, 1, y) = \frac{245760 + 177408p^2 - 62688p^4 + 1901p^6}{424673280} =: m_3(p, y). \tag{57}$$

While computing $\partial m_3/\partial p = 0$, $p \approx 1.23293 =: p_0$ comes out to be the critical point. After simple calculations, m_3 achieves its maximum value ≈ 0.000888358 at p_0.
On the face $y = 0$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$\begin{aligned} C(p, x, 0) = \frac{1}{424673280}\Big(&49152x(18 - 13x^2) + 2304p^2(48 - 192x + 37x^2 + 168x^3 \\ &+ 16x^4) - 96p^4(288 - 707x + 400x^2 + 480x^3 + 192x^4) \\ &+ p^6(1157 - 3144x + 4272x^2 - 2688x^3 + 2304x^4)\Big) =: m_4(p, x). \end{aligned}$$

After further calculations such as

$$\frac{\partial m_4}{\partial x} = \frac{(4 - p^2)}{17694720}\Big(1536(6 - 13x^2) - 48p^2(48 - 37x - 148x^2 - 32x^3) + p^4(131 - 356x + 336x^2 - 384x^3)\Big)$$

and

$$\begin{aligned} \frac{\partial m_4}{\partial p} = \frac{1}{70778880}\Big(&768p(48 - 192x + 37x^2 + 168x^3 + 16x^4) - 64p^3(288 - 707x + 400x^2 \\ &+ 480x^3 + 192x^4) + p^5(1157 - 3144x + 4272x^2 - 2688x^3 + 2304x^4)\Big), \end{aligned}$$

we observe that only real solutions (p, x) of the system of equations $\partial m_4/\partial x = 0$ and $\partial m_4/\partial p = 0$ are $(3.85748, 0.257377)$; $(3.76933, 0.0851082)$; $(-3.76933, 0.0851082)$; $(2, -0.644779)$; $(-3.85748, 0.257377)$; $(-1.57205, -1.03976)$; $(-2, -0.644779)$; $(0, -0.679366)$; $(0, 0.679366)$; $(-1.12904, 0.941715)$ and $(1.57205, -1.03976)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

On the face $y = 1$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$C(p, x, 1) = \frac{1}{424673280}\Big(1157p^6 + 3144p^4(4-p^2)x + 27648p^2(4-p^2)x^2 + 1056p^4(4-p^2)x^2 + 5328p^2(4-p^2)^2x^2 + 6912p^4(4-p^2)x^3 + 15360(4-p^2)^2x^3 + 4224p^2(4-p^2)^2x^3 + 2304p^2(4-p^2)^2x^4 - 9216(4-p^2)(1-x^2)(-4(5+x^2) + p^2(5-3x+x^2)) + 192(4-p^2)(1-x^2)(48px(7+4x) + p^3(83+60x-48x^2))\Big) =: m_5(p, x).$$

After calculations,

$$\frac{\partial m_5}{\partial p} = \frac{1}{70778880}\Big(6144x(7+4x-7x^2-4x^3) - 768p(160-48x-213x^2+72x^3-48x^4) - 160p^4(83+60x-131x^2-60x^3+48x^4) + 384p^2(83-24x-179x^2 + 24x^3+96x^4) + 64p^3(480-157x-1072x^2+384x^3-288x^4) + p^5(1157-3144x+4272x^2-2688x^3+2304x^4)\Big)$$

and

$$\frac{\partial m_5}{\partial x} = \frac{(4-p^2)}{17694720}\Big(1536x(5x-8-4x^2) + 384p(7+8x-21x^2-16x^3) + 48p^2(24 + 149x - 68x^2 + 64x^3) + 16p^3(30-131x-90x^2+96x^3) + p^4(131-356x+336x^2-384x^3)\Big),$$

we observe that only real solutions (p, x) of the system of equations $\partial m_5/\partial x = 0$ and $\partial m_5/\partial p = 0$ are $(27.1136, 0.413453)$; $(-1.50007, -6.38485)$; $(2.68178, 5.28944)$; $(2, 1.39993)$;$(2, -0.192442)$; $(0, 0)$; $(-4.98296, 1.51367)$; $(-2.89665, 3.0885)$; $(-1.80124, -0.775344)$; $(-1.74974, 0.159528)$; $(-2, 0.737618)$; $(-0.776187, 0.905316)$; $(1.6981, -0.807967)$; $(2, -1.20749)$ and $(1.5135, -0.953552)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

3. We next examine the maxima attained by $C(p, x, y)$ on the edges of the cuboid S.
From (53), we have $M(p, 0, 0) = (110592p^2 - 27648p^4 + 1157p^6)/424673280 =: g_1(p)$. It is easy to observe that $g_1'(p) = 0$ whenever $p = 0$ and $p = 1.53142 \in [0, 2]$ as its points of minima and maxima, respectively. Hence,

$$C(p, 0, 0) \le 0.0002878, \quad p \in [0, 2].$$

Now consider (53) at $y = 1$, we get $C(p, 0, 1) = (737280 - 368640p^2 + 63744p^3 + 46080p^4 - 15936p^5 + 1157p^6)/424673280 =: g_2(p)$. It is easy to observe that $g_2'(p) < 0$ in $[0, 2]$ and hence $p = 0$ serves as the point of maxima. So,

$$C(p, 0, 1) \le \frac{1}{576} \approx 0.00173611, \quad p \in [0, 2].$$

Through computations, (53) shows that $C(0, 0, y) = y^2/576$ attains its maxima at $y = 1$. This implies that

$$C(0, 0, y) \le \frac{1}{36}, \quad y \in [0, 1].$$

Since, (57) does not involve x, we have $C(p, 1, 1) = C(p, 1, 0) = (245760 + 177408p^2 - 62688p^4 + 1901p^6)/424673280 =: g_3(p)$. Now, $g_3'(p) = 354816p - 250752p^3 + 11406p^5 = 0$ when $p = 0$ and $p = 1.23293$ in the interval $[0, 2]$ acting as the points of minima and maxima, respectively. Hence

$$C(p, 1, 1) = C(p, 1, 0) \le 0.000888358, \quad p \in [0, 2].$$

After considering $p = 0$ in (57), we get, $C(0, 1, y) = 1/1728 \approx 0.000578704$.
The equation (52) has no variables. So, on the edges, the maximum value of $C(p, x, y)$ is

$$C(2, 1, y) = C(2, 0, y) = C(2, x, 0) = C(2, x, 1) = \frac{1157}{6635520}, \quad x, y \in [0, 1].$$

Using (51), we obtain $C(0, x, 1) = (15 - 12x^2 + 5x^3 - 3x^4)/8640 =: g_4(x)$. After calculations, we see that $g_4(x)$ is a decreasing function in $[0, 1]$ and attains its maxima at $x = 0$. Hence

$$C(0, x, 1) \leq \frac{1}{576}, \quad x \in [0, 1].$$

Again utilizing (51), we get $C(0, x, 0) = x(18 - 13x^2)/8640 =: g_5(x)$. On further calculations, we get $g_5'(x) = 0$ when $x = \sqrt{6/13}$, acting as its point of maxima. Thus

$$C(0, x, 0) \leq 0.000943564, \quad x \in [0, 1].$$

Given all the cases, the inequality (48) holds. Let the function $f_1 \in \mathcal{C}_{\mathcal{B}}$, be defined as

$$f_1(z) = \int_0^z \left(\exp\left(\int_0^t \frac{\sqrt{1 + \tanh u^3} - 1}{u} du \right) \right) dt = z + \frac{z^4}{24} - \frac{z^7}{1008} - \cdots, \tag{58}$$

with $f_1(0) = 0$ and $f_1'(0) = 1$, acts as an extremal function for the bound of $|H_3(1)|$ for $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/24$. □

Theorem 3.2. *Let $f \in \mathcal{C}_{\mathcal{B}}$, then*

$$|H_2(3)| \leq \frac{1}{576}. \tag{59}$$

This bound is sharp.

Proof. We proceed on the similar lines as in the proof of Theorem 2.5. Assuming $p_1 =: p \in [0, 2]$, we substitute the values of a_i $(i = 3, 4, 5)$ from (46) and (47) in (5), we obtain

$$H_2(3) = \frac{1}{424673280}\Big(- 1853p^6 - 1488p^4p_2 + 1728p^2p_2^2 - 110592p_2^3 + 31872p^3p_3 + 119808pp_2p_3 - 184320p_3^2 - 82944p^2p_4 + 221184p_2p_4\Big).$$

Using Lemma 2.4 for simplification, we arrive at

$$H_2(3) = \frac{1}{424673280}\Big(\beta_{13}(p, \gamma) + \beta_{14}(p, \gamma)\eta + \beta_{15}(p, \gamma)\eta^2 + \beta_{16}(p, \gamma, \eta)\rho\Big),$$

where $\gamma, \eta, \rho \in \mathbb{D}$,

$$\begin{aligned}
\beta_{13}(p, \gamma) : &= -1109p^6 - 15696\gamma^2p^2(4 - p^2)^2 + 3456\gamma^3p^2(4 - p^2)^2 + 2304\gamma^4p^2(4 - p^2)^2 \\
&\quad + 13824\gamma^2p^2(4 - p^2) - 2376\gamma p^4(4 - p^2) - 10272p^4\gamma^2(4 - p^2) + 3456p^4\gamma^3(4 - p^2), \\
\beta_{14}(p, \gamma) : &= 192(1 - |\gamma|^2)(4 - p^2)(71p^3 - 72p^3\gamma - 36p\gamma(4 - p^2) - 48p\gamma^2(4 - p^2)), \\
\beta_{15}(p, \gamma) : &= 4608(1 - |\gamma|^2)(4 - p^2)(-10(4 - p^2) - 3p^2\bar{\gamma} - 2|\gamma|^2(4 - p^2)) \\
\beta_{16}(p, \gamma, \eta) : &= 13824(1 - |\gamma|^2)(4 - p^2)(1 - |\eta|^2)(p^2 + 4\gamma(4 - p^2)).
\end{aligned}$$

Additionally, by using the fact that $|\rho| \leq 1$, and taking $x = |\gamma|$, $y = |\eta|$, we have

$$|H_2(3)| \leq \frac{1}{424673280}\Big(|\beta_{13}(p, \gamma)| + |\beta_{14}(p, \gamma)|y + |\beta_{15}(p, \gamma)|y^2 + |\beta_{16}(p, \gamma, \eta)|\Big) \leq D(p, x, y),$$

where

$$D(p,x,y) = \frac{1}{424673280}\Big(D_1(p,x) + D_2(p,x)y + D_3(p,x)y^2 + D_4(p,x)(1-y^2)\Big), \tag{60}$$

with

$$\begin{aligned}
D_1(p,x) :&= 1109p^6 + 15696p^2x^2(4-p^2)^2 + 3456p^2x^3(4-p^2)^2 + 2304p^2x^4(4-p^2)^2 \\
&\quad + 13824p^2x^2(4-p^2) + 2376p^4x(4-p^2) + 10272p^4x^2(4-p^2) + 3456p^4x^3(4-p^2), \\
D_2(p,x) :&= 192(4-p^2)(1-x^2)(71p^3 + 72p^3x + 36px(4-p^2) + 48px^2(4-p^2)), \\
D_3(p,x) :&= 4608(4-p^2)(1-x^2)(10(4-p^2) + 3p^2x + 2x^2(4-p^2)), \\
D_4(p,x) :&= 13824(4-p^2)(1-x^2)(p^2 + 4x(4-p^2)).
\end{aligned}$$

Now, we must maximize $D(p,x,y)$ in the closed cuboid $T : [0,2]\times[0,1]\times[0,1]$. By identifying the maximum values on the twelve edges, the interior of T, and the interiors of the six faces, we can prove this.

1. We start by taking into account, every interior point of T. Assume that $(p,x,y) \in (0,2)\times(0,1)\times(0,1)$. We partially differentiate (60) with respect to y in order to locate the points of maxima in the interior of T. We obtain

$$\frac{\partial D}{\partial y} = \frac{(1-x^2)(4-p^2)}{2211840}\Big(48px(3+4x) + p^3(71+36x-48x^2) + 384(5-6x+x^2)y - 48p^2(13-15x+2x^2)y\Big).$$

Now $\partial D/\partial y = 0$ gives

$$y = \frac{48px(3+4x) + p^3(71+36x-48x^2)}{48(1-x)(8(x-5)+p^2(13-2x))} =: y_1.$$

Since y_1 must be a member of $(0,1)$ for critical points to exist, this is only possible if

$$71p^3 + px(144 + 36p^2 + 192x - 48p^2x) < 48(-40 + 13p^2 + x(48 - 15p^2 - 8x + 2p^2x)). \tag{61}$$

Now, we find the solutions satisfying the inequality (61) for the existence of critical points using hit and trial method. If p tends to 2, then (61) holds whenever $x < 1/144$. Also, no such $x \in (0,1)$, satisfying (61) when p tends to 0. Similarly, if take x tending to 0, then (61) holds for $p > 1.99513$ only, whereas there does not exist any $p \in (0,2)$ such that equation (61) holds when x tends to 1. Thus, the domain for the solution of the equation is $(1.99513,2)\times(0,1/144)$. Now, we examine that $\frac{\partial D}{\partial y}|_{y=y_0} \neq 0$ in $(1.99513,2)\times(0,1/144)$. So, we conclude that the function M has no critical point in $(0,2)\times(0,1)\times(0,1)$.

2. Now, we study the interior of each of six face of the cuboid T.
On the face $p = 0$: We have $x, y \in (0,1)$ and

$$D(0,x,y) = \frac{(1-x^2)(5y^2 + x^2y^2 + 6x(1-y^2))}{2880} =: n_1(x,y). \tag{62}$$

We note that, in $(0,1)\times(0,1)$, $n_1(x,y)$ does not have any critical point. As

$$\frac{\partial n_1}{\partial y} = \frac{(1-x^2)(x-1)(x-5)y}{1440} \neq 0 \quad x, y \in (0,1).$$

On the face $p = 2$: We have

$$D(2,x,y) = \frac{1109}{6635520} \approx 0.000167131, \quad x, y \in (0,1). \tag{63}$$

On the face $x = 0$: We have $p \in (0, 2)$, $y \in (0, 1)$ and

$$D(p, 0, y) = \frac{1109p^6 + (4 - p^2)(13632p^3 y + 46080(4 - p^2)y^2 + 13824p^2(1 - y^2))}{42467328} =: n_2(p, y). \tag{64}$$

We solve $\partial n_2/\partial p$ and $\partial n_2/\partial y$ to locate the points of maxima. On solving $\partial n_2/\partial y = 0$, we obtain

$$y = \frac{71p^3}{48(13p^2 - 40)} =: \epsilon. \tag{65}$$

For the given range of y, we should have $\epsilon \in (0, 1)$ which is possible only when $p > 1.99513$ exists. On computations, $\partial n_2/\partial p = 0$ gives

$$18432p - 9216p^3 + 1109p^5 + 27264p^2 y - 11360p^4 y - 141312py^2 + 39936p^3 y^2 = 0. \tag{66}$$

On substituting (65) in (66), we get,

$$88473600p - 101744640p^3 + 38582784p^5 - 5470944p^7 + 169065p^9 = 0. \tag{67}$$

The solution of (67) in the interval $(0, 2)$ is $p \approx 1.34821$, according to a numerical calculation. In $(0, 2) \times (0, 1)$, n_2 does not have a critical point.
On the face $x = 1$ We have $p \in (0, 2)$ and

$$D(p, 1, y) = \frac{398592p^2 - 121056p^4 + 6461p^6}{424673280} =: n_3(p). \tag{68}$$

And when computing $\partial n_3/\partial p = 0$, $p = 1.39681 =: p_0$ turns out to be the critical point. According to elementary calculations, n_3 reaches its maximum value ≈ 0.000859123 at p_0.
On the face $y = 0$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$\begin{aligned} D(p, x, 0) = \frac{1}{424673280}\Big(&884736x(1 - x^2) + 2304p^2(24 - 192x + 109x^2 + 216x^3 + 16x^4) \\ &- 96p^4(144 - 675x + 880x^2 + 720x^3 + 192x^4) \\ &+ p^6(1109 - 2376x + 5424x^2 + 2304x^4)\Big) =: n_4(p, x). \end{aligned}$$

On computations,

$$\frac{\partial n_4}{\partial x} = \frac{(4 - p^2)}{17694720}\Big(9216(1 - 3x^2) - 48p^2(48 - 109x - 180x^2 - 32x^3) + p^4(99 - 452x - 384x^3)\Big)$$

and

$$\begin{aligned} \frac{\partial n_4}{\partial p} = \frac{1}{70778880}\Big(&768p(24 - 192x + 109x^2 + 216x^3 + 16x^4) - 64p^3(144 - 675x + 880x^2 + 720x^3 + 192x^4) \\ &+ p^5(1109 - 2376x + 5424x^2 + 2304x^4)\Big), \end{aligned}$$

we observe that only real solutions (p, x) of the system of equations $\partial n_4/\partial x = 0$ and $\partial n_4/\partial p = 0$ are $(-2, -2.72495)$; $(-2.64507, -0.503718)$; $(-2.23927, -0.0103472)$; $(-2, -0.163482)$; $(2.23927, -0.0103472)$; $(-2.00568, -0.120347)$; $(2, -2.72495)$; $(-2, -0.0837889)$; $(0, -0.57735)$; $(2, -0.0837889)$; $(2.00568, -0.120347)$; $(2, -0.163482)$; $(2.64507, -0.503718)$ and $(-0.91207, 0.721981)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

On the face $y = 1$ We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$D(p, x, 1) = \frac{1}{424673280}\Big(36864px(3 + 4x - 3x^2 - 4x^3) + 147456(5 - 4x^2 - x^4) - 2304p^2(160 - 24x - 261x^2 - 48x^4) + 768p^3(71 - 167x^2 + 96x^4) - 192p^5(71 + 36x - 119x^2 - 36x^3 + 48x^4) - 96p^4(-480 + 45x + 1408x^2 + 288x^4) + p^6(1109 - 2376x + 5424x^2 + 2304x^4)\Big) =: n_5(p, x).$$

On computations,

$$\frac{\partial n_5}{\partial x} = \frac{(4 - p^2)}{17694720}\Big(- 6144x(2 + x^2) + 384p(3 + 8x - 9x^2 - 16x^3) + 48p^2(12 + 197x + 64x^3) + 16p^3(18 - 119x - 54x^2 + 96x^3) + p^4(99 - 452x - 384x^3)\Big)$$

and

$$\frac{\partial n_5}{\partial p} = \frac{1}{70778880}\Big(6144x(3 + 4x - 3x^2 - 4x^3) + 768p(-160 + 24x + 261x^2 + 48x^4) - 160p^4(71 + 36x - 119x^2 - 36x^3 + 48x^4) + 384p^2(71 - 167x^2 + 96x^4) + 64p^3(480 - 45x - 1408x^2 - 288x^4) + p^5(1109 - 2376x + 5424x^2 + 2304x^4)\Big),$$

we observe that only real solutions (p, x) of the system of equations $\partial n_5/\partial x = 0$ and $\partial n_5/\partial p = 0$ are $(10.6873, 0.22889)$; $(-2, -2.84595)$; $(2.70664, 2.22228)$; $(2, 2.03047)$; $(-2, -0.783981)$; $(2, -0.385354)$; $(-1.77349, 0.0460014)$; $(2, -0.64512)$; $(1.98564, -0.516895)$; $(0, 0)$; $(2.00568, -0.120347)$; $(2, 0.657704)$; $(-2.33662, -1.5235)$; $(-1.03494, 0.828224)$ and $(1.36475, -1.03084)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

3. Now, we calculate the maximum values achieved by $D(p, x, y)$ on the edges of the cuboid T.
From (64), we have $D(p, 0, 0) = (55296p^2 - 13824p^4 + 1109p^6)/424673280 =: h_1(p)$. It is easy to observe that $h_1'(p) = 0$ when $p = 0$ and $p = 1.83094$, acting as the points of minima and maxima, respectively. Thus

$$D(p, 0, 0) \le 0.000169059.$$

Now considering (64) at $y = 1$, we get $D(p, 0, 1) = (737280 - 368640p^2 + 54528p^3 + 46080p^4 - 13632p^5 + 1109p^6)/424673280 =: h_2(p)$. It is easy to observe that $t_2'(p)$ is a decreasing function in $[0, 2]$ and hence $p = 0$ acts as its point of maxima. Thus

$$D(p, 0, 1) \le \frac{1}{576} = 0.00173611, \quad p \in [0, 2].$$

Through computations, (64) shows that $D(0, 0, y) = y^2/576$, attains its maximum value at $y = 1$. This implies that

$$D(0, 0, y) \le \frac{1}{576}, \quad y \in [0, 1].$$

Since, (68) is independent of x, we have $D(p, 1, 1) = D(p, 1, 0) = (398592p^2-121056p^4+6461p^6)/424673280 =: h_3(p)$. Now, $h_3'(p) = 294912p - 175680p^3 + 13206p^5 = 0$ when $p = 0$ and $p = 1.39681$, acting as the points of minima and maxima, respectively. Hence

$$D(p, 1, 1) = D(p, 1, 0) \le 0.000859123, \quad p \in [0, 2].$$

On substituting $p = 0$ in (68), we get, $D(0, 1, y) = 0$. The equation (63) is independent of the all the variables namely p, x and y. Thus the maximum value of $D(p, x, y)$ on the edges $p = 2, x = 1; p = 2, x = 0; p = 2, y = 0$ and $p = 2, y = 1$, respectively, is given by

$$D(2, 1, y) = D(2, 0, y) = D(2, x, 0) = D(2, x, 1) = \frac{1109}{6635520}, \quad x, y \in [0, 1].$$

From (64), we obtain $D(0, 0, y) = y^2/576$. A simple calculation shows that

$$D(0, 0, y) \leq \frac{1}{576}, \quad y \in [0, 1].$$

Using (62), we obtain $D(0, x, 1) = (5 - 4x^2 - x^4)/2880 =: h_4(x)$. Upon calculations, we see that h_4 is a decreasing function in $[0, 1]$ and hence attains its maximum value at $x = 0$. Thus

$$D(0, x, 1) \leq \frac{1}{576}, \quad x \in [0, 1].$$

On again using (62), we get $D(0, x, 0) = x(1 - x^2)/480 =: h_5(x)$. On further calculations, we get $t_5'(x) = 0$ when $x = 1/\sqrt{3}$, the point of maxima. Thus

$$D(0, x, 0) \leq 0.000801875, \quad x \in [0, 1].$$

In view of all the cases, the inequality (59) holds. The function specified in (58) acts as an extremal function for the bounds of $|H_2(3)|$ having values $a_3 = a_5 = 0$ and $a_4 = 1/24$. □

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