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Pseudo-Ricci-Bourguignon solitons on real hypersurfaces in the complex hyperbolic space

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Abstract. In this paper, we give a complete classification of pseudo-Ricci-Bourguignon soliton on real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n = SU_{1,n}/S(U_1U_n)$. Next as an application we give a complete classification of gradient pseudo-Ricci-Bourguignon soliton on Hopf real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$.

1. Introduction

In the class of Hermitian symmetric spaces of non-compact type with rank 1, we highlight the example of complex hyperbolic space $\mathbb{C}H^n = SU_{1,n}/S(U_1U_n)$, which is geometrically distinct from rank 2 cases. It has a Kähler structure J such that $\bar{\nabla}J = 0$, and is equipped with a Bergmann metric g that has constant holomorphic sectional curvature -4 (see Romero [28, 29], Smyth [30], Suh [33, 34], and Hwang-Suh [19]). The complex hyperbolic space $\mathbb{C}H^n$ is a real Grassmann manifold of non-compact type with rank 1 (see Kobayashi-Nomizu [22]).

In the complex hyperbolic space $\mathbb{C}H^n$, we have provided a classification of Ricci-Bourguignon soliton real hypersurfaces (see Suh [34]). In the complex hyperbolic quadric $Q^{n*} = SO^o_{2,n}/SO_2SO_n$, Ricci solitons, Ricci-Bourguignon solitons, and pseudo-Einstein real hypersurfaces have been studied by Kimura-Ortega [21] and Suh [31, 32]. More recently, Chaubey-Lee-Suh [11], Chaubey-De-Suh [10, 13, 15], and Wang [36, 37] have investigated Yamabe solitons and Ricci solitons on almost co-Kähler manifolds, three dimensional $N(k)$-contact manifolds, and complex quadrics Q^m. The study of the Yamabe flow was initially introduced

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by Hamilton [17], Morgan-Tian [26], and Perelman [27], providing a geometric method for constructing Yamabe metrics on Riemannian manifolds.

On the other hand, it is well-known that real hypersurfaces in Hermitian symmetric space of compact type have two focal submanifolds, whereas those in non-compact types, such as complex hyperbolic space $\mathbb{C}H^n$, have only one focal submanifold (see Helgason [18] and Wang [38, 39]). Among these, we examine two types of real hypersurfaces in $\mathbb{C}H^n$: those with isometric Reeb flow and contact real hypersurfaces. Using the Hopf fibration

$$\tilde{\pi} : H_1^{2n+1}(1) \to \mathbb{C}H^n, \quad z \to [z],$$

which is a Riemannian submersion from anti-de Sitter space $H_1^{2n+1}(1)$ to complex hyperbolic space $\mathbb{C}H^n$, Montiel-Romero [25] classified real hypersurfaces with isometric Reeb flow as follows:

Theorem 1.1 ([25]). *Let M be a real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, where $n \geq 3$. Then, the Reeb flow on M is isometric if and only if M is an open part of a tube of radius r around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{0, \cdots, n-1\}$, or a horosphere in the complex hyperbolic space $\mathbb{C}H^n$.*

When a real hypersurface M in $\mathbb{C}H^n$ satisfies the formula

$$A\phi + \phi A = 2\varrho\phi, \quad \varrho \neq 0 \text{ constant},$$

we say that M is a *contact* real hypersurface in $\mathbb{C}H^n$. In works by Blair [3], Vernon [35] and Yano-Kon [40], the classification of contact real hypersurfaces in $\mathbb{C}H^n$ is given as follows:

Theorem 1.2 ([3, 35, 40]). *Let M be a connected orientable real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. Then M is contact if and only if M is congruent to one of the following:*

(i) *a horosphere in $\mathbb{C}H^n$,*

(ii) *a geodesic hypersphere in $\mathbb{C}H^n$,*

(iii) *a tube around an n-dimensional totally geodesic real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$,*

(iv) *a tube around the totally geodesic complex hyperbolic space $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$.*

Motivated by these results, we consider some characterizations of real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ with respect to a geometric flow introduced by Bourguignon, [4] and [5], which generalizes the *Ricci-Bourguignon flow*. This flow is an intrinsic geometric flow on Riemannian manifolds, where its fixed points are solitons. Specifically, a solution to the Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = \Omega g(X, Y),$$

where Ω denotes the Ricci soliton constant, and $\mathcal{L}_V$ is the Lie derivative along the direction of the vector field V (see Chaubey-De-Suh [10], Morgan-Tian [26], Perelman [27], and Wang [36, 37]). A solution (M, V, Ω, g) is called a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω. In the complex two-plane Grassmannian $G_2(\mathbb{C}^{n+2})$, Jeong-Suh [20] classified Ricci solitons for real hypersurfaces.

As a generalization of the Ricci flow concept, the Ricci-Bourguignon flow (see Bourguignon [4, 5] and Catino-Cremaschi-Djadli-Mantegazza-Mazzieri [7]) is defined by

$$\frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric}(g(t)) - \rho\gamma g(t)), \quad g(0) = g_0,$$

where γ represents the scalar curvature and ρ is any constant. When $\rho = 0$, this family of geometric flows reduces to the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t))$, $g(0) = g_0$. Furthermore, by appropriately rescaling time, when ρ is nonpositive, the Ricci-Bourguignon flow can be viewed as an interpolation between the Ricci flow and the Yamabe flow, the latter emerging as a limit when $\rho \to -\infty$ (see Brendle [6], Chaubey-Suh-De [13], De-Chaubey-Shenawy [14], and Ye [41]). In [16], Fischer studied a conformal version of this flow where the scalar curvature is constrained along the flow. Similarly, in [23], Lu-Qing-Zheng established additional results on the conformal Ricci–Bourguignon flow. For results concerning solitons of the Ricci–Bourguignon flow see Catino-Mazzieri [8].

Now, let us introduce the Ricci-Bourguignon soliton $(M, V, \Omega, \rho, \gamma, g)$, which is a solution of the Ricci-Bourguignon flow. It satisfies the following equation:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = (\Omega + \rho\gamma)g(X, Y) \tag{1.1}$$

for any tangent vector fields X and Y on M, where Ω denotes the Ricci soliton constant, and ρ is any constant. When the soliton constant $\Omega > 0$, $\Omega = 0$, and $\Omega < 0$, we refer to the Ricci-Bourguignon soliton as shrinking, steady, and expanding, respectively.

On the other hand, when the Reeb vector field ξ satisfies $A\xi = \alpha\xi$ for the shape operator A on a real hypersurface M in the complex hyperbolic space $\mathbb{C}H^n$, M is said to be *Hopf* hypersurface. Using this concept, it can be easily shown in section 5 that a Hopf Ricci-Bourguignon soliton $(M, \xi, \Omega, \rho, \gamma, g)$ in the complex hyperbolic space $\mathbb{C}H^n$ also satisfies the generalized pseudo-anti-commuting property, i.e., $\mathrm{Ric}\phi + \phi\mathrm{Ric} = \ell\phi$, where $\ell \neq 0$ is constant.

If the Ricci operator Ric of a real hypersurface M in $\mathbb{C}H^n$ satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi \tag{1.2}$$

for smooth functions a, b on M, then M is said to be *pseudo-Einstein*. We now present a complete classification of pseudo-Einstein Hopf real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ due to Montiel [24], as follows:

Theorem 1.3 ([24]). *Let M be a pseudo-Einstein real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. Then M is locally congruent to one of the following:*

(i) *a geodesic hypersphere,*

(ii) *a horosphere,*

(iii) *a tube of arbitrary radius r around a totally geodesic hyperbolic hyperplane $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$.*

As a further generalization of the Ricci-Bourguignon flow, we introduce the pseudo-Ricci-Bourguignon flow, given by

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) + \psi\eta \otimes \eta(g(t)) - \rho\gamma g(t)), \quad g(0) = g_0,$$

where γ denotes the scalar curvature and ρ and ψ are any constants. In this paper, we consider a pseudo-Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \rho, \gamma, g)$ satisfying the following equation:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) + \psi\eta(X)\eta(Y) = (\Omega + \rho\gamma)g(X, Y) \tag{1.3}$$

for any tangent vector fields X and Y on M, where Ω is referred to as the pseudo-Ricci-Bourguignon soliton constant, ρ and ψ are any constants, γ is the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the vector field V. When the function ψ vanishes identically, the pseudo-Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \rho, \gamma, g)$ reduces to a Ricci-Bourguignon soliton $(M, V, \Omega, \rho, \gamma, g)$. This generalizes the Ricci-Bourguignon soliton $(M, V, \Omega, \rho, \gamma, g)$ (see Hamilton [17] and Morgan-Tian [26]). Any solution is then called a pseudo-Ricci-Bourguignon soliton with potential vector field V and pseudo-Ricci-Bourguignon soliton constant Ω. Furthermore, the pseudo-Ricci-Bourguignon soliton is categorized as shrinking, steady, or expanding depending on the pseudo-Ricci-Bourguignon soliton constant function $\Omega > 0$, $\Omega = 0$, and $\Omega < 0$, respectively.

In section 5, we use the concept of the generalized pseudo-anti-commuting property, defined by

$$\mathrm{Ric}\phi + \phi\mathrm{Ric} = \ell\phi, \ \ell \neq 0 : \textit{constant},$$

to show that a Hopf pseudo-Ricci-Bourguignon soliton real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$ satisfies this property. Consequently, we can confirm that a real hypersurface with a pseudo-Ricci-Bourguignon soliton has constant principal curvatures. This fact allows us to present a classification theorem due to Berndt [1]. From this classification, we can determine whether a horosphere, a geodesic hypersphere of type A_1, a real hypersurface of type A_2, or a real hypersurface of type B admit a Ricci-Bourguignon soliton. Building on this, in section 6 we can assert the following:

Theorem 1.4. *Let M be a Hopf pseudo-Ricci-Bourguignon soliton in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. Then M is pseudo-Einstein and locally congruent to one of the following:*

(i) *a geodesic hypersphere satisfying $\Omega + \rho\gamma = -2n + 2(n-1)\coth^2(r)$, and $\psi = -2n$,*

(ii) *a horosphere satisfying $\Omega + \rho\gamma = -2$, and $\psi = -2n$,*

(iii) *a tube of radius r around a totally geodesic hyperbolic hyperplane $\mathbb{C}H^{n-1}$ satisfying $\Omega + \rho\gamma = -2n + 2(n-1)\tanh^2(r)$, and $\psi = -2n$.*

Let Df denote the gradient vector field of the function f on M, which is defined by $g(Df, X) = g(\mathrm{grad} f, X) = X(f)$ for any tangent vector field X on M. We now consider a gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \Omega, \rho, \gamma, g)$ (see Catino-Mazzieri [8], Cernea-Guan [9]), which satisfies the equation

$$\mathrm{Hess}(f) + \mathrm{Ric} + \psi\eta\otimes\eta = (\Omega + \rho\gamma)g,$$

where $\mathrm{Hess}(f)$ is defined as $\mathrm{Hess}(f) = \nabla Df$, for any tangent vector fields X and Y on M, in such a way that

$$\mathrm{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Thus, the gradient pseudo-Ricci-Bourguignon soliton satisfies

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \rho\gamma)X \tag{1.4}$$

for any vector field X tangent to M in $\mathbb{C}H^n$. Using Theorem 1.1, we can state the following theorem for gradient pseudo-Ricci-Bourguignon solitons $(M, Df, \Omega, \rho, \gamma, g)$:

Theorem 1.5. *Let M be a Hopf gradient pseudo-Ricci-Bourguignon soliton with isometric Reeb flow in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. Then M is pseudo-Einstein and locally congruent to one of the following:*

(i) *a geodesic hypersphere satisfying $\Omega + \rho\gamma = -2n + 2(n-1)\coth^2(r)$, and $\psi = -2n$,*

(ii) *a horosphere satisfying $\Omega + \rho\gamma = -2$, and $\psi = -2n$,*

(iii) *a tube of radius r around a totally geodesic hyperbolic hyperplane $\mathbb{C}H^{n-1}$ satisfying $\Omega + \rho\gamma = -2n + 2(n-1)\tanh^2(r)$, and $\psi = -2n$.*

Based on Theorem 1.2, we give another theorem for a gradient pseudo-Ricci-Bourguignon soliton on a contact real hypersurface M in the complex hyperbolic space $\mathbb{C}H^n$ as follows:

Theorem 1.6. *Let M be a Hopf gradient pseudo-Ricci-Bourguignon soliton in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$. If M is contact, then M is pseudo-Einstein and locally congruent to one of the following:*

(i) *a horosphere satisfying $\Omega + \rho\gamma = -2$, and $\psi = -2n$,*

(ii) *a tube of radius r around a totally geodesic hyperbolic hyperplane $\mathbb{C}H^{n-1}$ satisfying $\Omega + \rho\gamma = -2n + 2(n-1)\tanh^2(r)$ for $r \to \infty$, and $\psi = -2n$.*

2. The complex hyperbolic space

This section is due to Berndt and Suh [2]. Let $(\bar{M}, g, J)$ be a Kähler manifold and $\bar{R}$ the Riemannian curvature tensor of $(\bar{M}, g)$. Since $\bar{\nabla} J = 0$, we immediately see that

$$\bar{R}(X,Y)JZ = J\bar{R}(X,Y)Z$$

holds for all $X, Y, Z \in T_x(\bar{M})$, $x \in \bar{M}$. From the curvature identities in Kobayashi and Nomizu [22] we also get

$$g(\bar{R}(X,Y)Z,W) = g(\bar{R}(JX,JY)Z,W) = g(\bar{R}(X,Y)JZ,JW).$$

A Kähler manifold M is said to have *constant holomorphic sectional curvature* if the holomorphic sectional curvature function

$$K(V) = K(X,JX) = g(\bar{R}(X,JX)JX,X)$$

is constant for any holomorphic section $V = \mathrm{Span}\{X, JX\} \in T_x(\bar{M})$. Related to this one, the Riemannian curvature tensor $\bar{R}$ on a Kähler manifold $(\bar{M}, g, J)$ is given by the following

Theorem 2.1. *A Kähler manifold $(\bar{M}, g, J)$ has constant holomorphic sectional curvature $c \in \mathbb{R}$ if and only if its Riemannian curvature tensor $\bar{R}$ is of the form*

$$\bar{R}(X,Y)Z = \frac{c}{4}\Big\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\Big\}$$

for any vector fields X, Y and Z on $\bar{M}$.

The construction for the complex hyperbolic space $\mathbb{C}H^n$ can be given as follows: For any points z, w in complex Minkowski space $\mathbb{C}_1^{n+1}$, let us write

$$F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k$$

and let $< z, w >= \mathrm{Re}\, F(z,w)$. Then the anti-de Sitter space of radius 1 in $\mathbb{C}_1^{n+1}$ can be defined by

$$H_1^{2n+1}(1) = \{z \in \mathbb{C}_1^{n+1} :< z, z >= -1\}.$$

We denote $H_1^{2n+1}(1)$ by $\mathcal{H}$ for short. We use the same identification of $\mathbb{C}_1^{n+1}$ with $\mathbb{R}_2^{2n+2}$ so that

$$< z, w >=< u, v >= -u_0v_0 - u_1v_1 + \sum_{k=2}^{2n+1} u_kv_k.$$

For $z \in \mathcal{H}$,

$$T_z\mathcal{H} = \{\, w \in \mathbb{C}_1^{n+1} :< z, w >= 0\}.$$

Restricting $<,>$ to $\mathcal{H}$ gives a Lorentz metric whose Levi-Civita connection $\tilde{\nabla}$ satisfies

$$D_XY = \tilde{\nabla}_XY + < X, Y > \frac{z}{r^2}$$

for X, Y tangent to $\mathcal{H}$ at z. The Gauss equation takes the form

$$\bar{R}(X,Y) = -X \wedge Y \tag{2.1}$$

where $\bar{R}$ denotes the curvature tensor of $\mathcal{H}$ and $X \wedge Y$ is defined by

$$(X \wedge Y)Z =< Y, Z > X- < X, Z > Y$$

for any vector fields X, Y and Z tangent to $\mathcal{H}$ at z.

Again take $V = Jz = iz$ and we get the analogous orthogonal decomposition

$$T_z\mathcal{H} = \mathrm{Span}\{V\} \oplus V^\perp.$$

Let us denote by $\mathbb{C}H^n$ the image of the canonical projection π to complex hyperbolic space,

$$\pi : \mathcal{H} \to \mathbb{C}H^n \subset \mathbb{C}P^n.$$

Then $\mathbb{C}H^n$ is said to be a *complex hyperbolic space*. Thus, topologically, $\mathbb{C}H^n$ is an open subset of $\mathbb{C}P^n$. However, as Riemannian manifolds, they have quite different structures. Then a complex hyperbolic space $\mathbb{C}H^n$ is a Kähler manifold with negative constant holomorphic sectional curvature.

From Theorem 2.1 above, let us put that the complex hyperbolic space $(\mathbb{C}H^n, J, g)$ is a complex space form with constant holomorphic sectional curvature -4. Then the Riemannian curvature tensor $\bar{R}$ of $\mathbb{C}H^n$ can be given for any vector fields X, Y and Z in $T_z(\mathbb{C}H^n)$, $z \in \mathbb{C}H^n$ as follows:

$$\bar{R}(X, Y)Z = -g(Y, Z)X + g(X, Z)Y - g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ.$$

3. Some general equations

Let M be a real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$ and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. Then the vector field ξ is said to be the *Reeb* vector field on M in $\mathbb{C}H^n$. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

In a different way, the complex hyperbolic space $\mathbb{C}H^n$ is defined by using the fibration

$$\tilde{\pi} : H_1^{2n+1}(1) \to \mathbb{C}H^n, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for a real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$ as follows:

$$\begin{array}{ccc} M' = \tilde{\pi}^{-1}(M) & \xrightarrow{\tilde{i}} & H_1^{2n+1}(1) \subset \mathbb{C}_1^{n+1} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\tilde{\pi}}\big\downarrow \\ M & \xrightarrow{i} & \mathbb{C}H^n \end{array}$$

We now assume that M is a Hopf hypersurface in a complex hyperbolic space $\mathbb{C}H^n$. Then we have

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of M in $\mathbb{C}H^n$ and the smooth function α is defined by $\alpha = g(A\xi, \xi)$ on M. When we consider the transform by the Kähler structure J on $\mathbb{C}H^n$ of any vector field X on M in $\mathbb{C}H^n$, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal vector field N to M.

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in $\mathbb{C}H^n$ induced from the curvature tensor $\bar{R}$ of $\mathbb{C}H^n$ can be described in terms of the almost contact structure tensor ϕ and the shape operator A of M in $\mathbb{C}H^n$ as follows:

$$\begin{aligned} R(X, Y)Z = &-g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z \\ &+ g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{3.1}$$

for any vector fields $X, Y, Z \in T_zM$, $z \in M$. From this, contracting Y and Z on M in $\mathbb{C}H^n$, we get the Ricci tensor of a real hypersurface M in $\mathbb{C}H^n$ as follows:

$$\mathrm{Ric}(X) = -(2n + 1)X + 3\eta(X)\xi + (\mathrm{Tr}A)AX - A^2X. \tag{3.2}$$

Then by contracting the Ricci operator in (3.2) the scalar curvature γ of M in $\mathbb{C}H^n$ is given by

$$\gamma = -4(n^2 - 1) + h^2 - \mathrm{Tr}A^2, \tag{3.3}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}H^n$.

Now let us introduce the equation of Codazzi for a Hopf real hypersurface M in the complex hyperbolic space $\mathbb{C}H^n$ as follows:

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)$$

for any X, Y and Z tangent to M. Putting $Z = \xi$ we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = 2g(\phi X, Y).$$

Since we have assumed that M is Hopf in $\mathbb{C}H^n$, differentiating $A\xi = \alpha\xi$ gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

From this, the left side of the above equation becomes

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, \xi) &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Putting $X = \xi$ in above two equations and using the almost contact structure of (M, g), we have

$$Y\alpha = (\xi\alpha)\eta(Y)$$

for any Y tangent to M. Inserting this formula into two previous equation implies

$$0 = 2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) + 2g(\phi X, Y).$$

By virtue of this equation, we can assert the following

Lemma 3.1. *Let M be a Hopf real hypersurface in $\mathbb{C}H^n$, $n \geq 3$. Then we obtain*

$$2A\phi AX = \alpha(A\phi + \phi A)X - 2\phi X$$

for any tangent vector field X on M.

In the proof of our Theorems 1.4 and 1.5, we want to obtain more information on Hopf hypersurfaces in the complex hyperbolic space. By using the formulas given in this section we want to introduce an important lemma due to Berndt-Suh [2] and Montiel-Romero [25] as follows:

Lemma 3.2. *Let M be a Hopf hypersurface in $\mathbb{C}H^n$. Then the Reeb function α is constant. Moreover, let $X \in \mathcal{C}$ be a principal curvature vector of M with principal curvature λ. Then*

- *If $\alpha^2 - 4 = 0$, then all principal curvatures $\lambda = \pm 1$.*
- *If $\alpha^2 - 4 \neq 0$, then $2\lambda \neq \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda - 2}{2\lambda - \alpha}$, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ on M.*

4. Some important propositions

In Theorem 1.1, we have mentioned that the Reeb flow on M in $\mathbb{C}H^n$ is isometric if and only if M is locally congruent to a horosphere of type (A_0) or a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for $k \in \{0, 1, \cdots, n-1\}$. Then for $k = 0$ we say that M is a geodesic hypersphere of type (A_1) with two distinct principal curvatures.

The tube of radius r around $\mathbb{C}H^0$ has therefore two distinct constant principal curvatures $\alpha = 2\coth(2r)$ and $\lambda = \coth(r)$ with multiplicities 1 and $2(n-1)$, respectively. Now by using (3.2) and (3.3), we introduce an important proposition due to Montiel-Romero [25] as follows:

Proposition 4.1. *Let M be the tube of radius $0 < r < \infty$ around the totally geodesic $\mathbb{C}H^0$ for $k = 0$ in $\mathbb{C}H^n$. That is, a geodesic hypersphere in $\mathbb{C}H^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = \coth(r)$	$\mathcal{C} = (\mathbb{R}JN)^\perp$	$2(n-1)$
$\alpha = 2\coth(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A commutes with the structure tensor field ϕ as*

$$A\phi = \phi A.$$

(4) *The trace h of the shape operator A and its square h^2 becomes respectively*

$$h = (2n-1)\coth(r) + \tanh(r),$$

and

$$h^2 = (2n-1)^2\coth^2(r) + \tanh^2(r) + 2(2n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = (2n-1)\coth^2(r) + \tanh^2(r) + 2.$$

(6) *The scalar curvature γ of the geodesic hypersphere is given by*

$$\begin{aligned}\gamma &= -4(n^2-1) + h^2 - \mathrm{Tr}A^2 \\ &= -4n(n-1) + 2(2n-1)(n-1)\coth^2(r).\end{aligned}$$

Next we want to give a definition of the horosphere $H(t)$ in $\mathbb{C}H^n$. Let us consider a Lorentzian hypersurface in $H_1^{2n+1}(1)$ given by

$$H'(t) = \{z \in H_1^{2n+1}(1) \,|\, |z_0 - z_1|^2 = t\}.$$

Then $H'(t)$ is S^1-invariant, where $S^1 = \{e^{i\theta} \,|\, \theta \in \mathbb{R}\}$. So $H(t) = \pi(H'(t))$ becomes a horosphere in $\mathbb{C}H^n$, and $N = (\pi_*)N'$ is a unit normal vector field to the horosphere $H(t)$. Then it becomes a totally η-umbilical hypersurface with two distinct constant principal curvatures 2 and 1 of multiplicities 1 and $2n-2$ respectively. Of course, the horosphere $H(t)$ becomes a pseudo-Einstein real hypersurface in $\mathbb{C}H^n$ with $a = -2$ and $b = 2n$.

By taking the radius $r \to \infty$ in Proposition 4.1, we can assert $\alpha = 2$ and $\lambda = 1$ as follows:

Proposition 4.2. *Let M be a horosphere in the complex hyperbolic space $\mathbb{C}H^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = 1$	$C = (\mathbb{R}JN)^{\perp}$	$2(n-1)$
$\alpha = 2$	$\mathbb{R}JN$	1

(3) *The shape operator A commutes with the structure tensor field ϕ as $A\phi = \phi A$, and also satisfies the contact condition as $A\phi + \phi A = 2\phi$.*

(4) *The trace h of the shape operator A and its square h^2 becomes respectively*

$$h = 2n,$$

and

$$h^2 = 4n^2.$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = 2(n+1).$$

(6) *The scalar curvature γ of the horosphere is given by*

$$\gamma = -2(n-1).$$

A proposition concerned with another kind of geodesic hypersphere in $\mathbb{C}H^n$ can be introduced as follows:

Proposition 4.3. *Let M be a tube of radius r around the totally geodesic $\mathbb{C}H^{n-1}$ in the complex hyperbolic space $\mathbb{C}H^n$. Then the following statements hold:*

(1) *M is a Hopf real hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\mu = \tanh(r)$	$C = (\mathbb{R}JN)^{\perp}$	$2(n-1)$
$\alpha = 2\coth(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A commutes with the structure tensor field ϕ as*

$$A\phi = \phi A.$$

(4) *The trace h of the shape operator A and its square h^2 becomes respectively*

$$h = (2n-1)\tanh(r) + \coth(r),$$

and

$$h^2 = (2n-1)^2\tanh^2(r) + \coth^2(r) + 2(2n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = (2n-1)\tanh^2(r) + \coth^2(r) + 2.$$

(6) *The scalar curvature γ of the horosphere is given by*

$$\gamma = -4n(n-1) + 2(2n-1)(n-1)\tanh^2(r).$$

Let us introduce some examples due to Montiel-Romero [25] as follows:

Let k, ℓ be natural numbers such that $k+\ell=n-1$ and $t\in\mathbb{R}$ with $0<t<1$. Then we can define the Lorentzian hypersurface $M'_{k,\ell}(t)$ in anti-de Sitter space $H_1^{2n+1}(1)$ by the equations

$$M'_{k,\ell}(t)=\Big\{z\in H_1^{2n+1}(1)\,\Big|\,t(-|z_0|+\sum\nolimits_{j=1}^{k}|z_j|^2)=-\sum\nolimits_{j=k+1}^{n}|z_j|^2\Big\}.$$

Then $M'_{k,\ell}(t)$ is isometric to $H_1^{2k+1}(\frac{1}{t-1})\times S^{2\ell+1}(\frac{t}{1-t})$, where $\frac{1}{t-1}$ and $\frac{t}{1-t}$ denotes the square of the respective radii. From this, if we put

$$M_{k,\ell}(t)=\pi(M'_{k,\ell}(t))$$

for a compatible fibration π from the fibration $\tilde{\pi}$: $H_1^{2n+1}(1)\to\mathbb{C}H^n$, then a unit normal vector field N of $M_{k,\ell}(t)$ is defined by $N_{\pi(z)}=(\pi_*)_zN'_z$ for a unit normal N' on $M'_{k,\ell}(t)$, which is S^1-invariant.

Accordingly, the space $M_{k,\ell}(t)$ has three constant principal curvatures $\tanh(r)=\sqrt{t}$, $\coth(r)=\frac{1}{\sqrt{t}}$, and $2\coth(2r)=\sqrt{t}+\frac{1}{\sqrt{t}}$ with multiplicities $2k$, 2ℓ and 1 respectively. Moreover, Montiel-Romero [25] assert that $M_{k,\ell}(t)=\pi(M'_{k,\ell}(t))$ is a tube over a totally geodesic complex submanifold $\mathbb{C}H^k$ in the complex hyperbolic space $\mathbb{C}H^n$.

When the integer $k=0$, the hypersurface $M_{0,\ell}(t)=\pi(H_1^1\times S^{2n+1})$ becomes a geodesic hypersphere in $\mathbb{C}H^n$. In fact it becomes a pseudo-Einstein real hypersurface in $\mathbb{C}H^n$ with $a=-2n+(2n-2)\coth^2(r)$ and $b=2n$.

When $\ell=0$, the hypersurface $M_{k,0}(t)$ is a tube of radius r over a complex hyperplane $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$. In this case, it becomes also pseudo-Einstein in $\mathbb{C}H^n$ with $a=-2n+(2n-2)\tanh^2(r)$ and $b=2n$.

For $k\in\{1,\cdots,n-2\}$, M is locally congruent to a tube over $\mathbb{C}H^k$ in $\mathbb{C}H^n$ and said to be of type (A_2) with three distinct constant principal curvatures, respectively. By using (3.2) and (3.3), together with the result due to Montiel-Romero [25], we introduce another important proposition as follows:

Proposition 4.4. *Let M be a tube of radius $0<r<\infty$ around the totally geodesic $\mathbb{C}H^k$, $k\in\{1,\cdots,n-2\}$ in the complex hyperbolic space $\mathbb{C}H^n$. Then the following statements hold:*

(1) *M is a Hopf real hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda=\coth(r)$	T_λ	2ℓ
$\mu=\tanh(r)$	T_μ	$2k$
$\alpha=2\coth(2r)$	$T_\alpha=\mathbb{R}JN$	1

where $\ell=n-k-1$.

(3) *The shape operator A commutes with the structure tensor field ϕ as*

$$A\phi=\phi A.$$

(4) *The trace h of the shape operator A and its square h^2 becomes the following respectively*

$$h=(2\ell+1)\coth(r)+(2k+1)\tanh(r),$$

and

$$h^2=(2\ell+1)^2\coth^2(r)+(2k+1)^2\tanh^2(r)+2(2\ell+1)(2k+1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2=(2\ell+1)\coth^2(r)+(2k+1)\tanh^2(r)+2.$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = -4(n-1)n + 8k\ell + 2(2\ell+1)\ell\coth^2(r) + 2(2k+1)k\tanh^2(r).$$

Finally we introduce an example due to Montiel [24]. Then we can define the Lorentzian hypersurface $M'(t)$ in an anti-de Sitter space $H_1^{2n+1}(1)$ by the equations

$$M'(t) = \left\{ z \in H_1^{2n+1}(1) \,\Big|\, |-z_0^2 + \sum\nolimits_{j=1}^{n} z_j^2|^2 = t \right\}.$$

Then $M'(t)$ is S^1-invariant. From this, if we put $M(t) = \pi(M'(t))$ for a compatible fibration π from the fibration $\tilde{\pi}$: $H_1^{2n+1} \to \mathbb{C}H^n$, the unit normal vector field N of $M(t)$ is defined by $N_{\pi(z)} = (\pi_*)_z N'_z$ for a unit normal N' on M', because $M'(t)$ is S^1-invariant.

Accordingly, when $t \neq 4$, the space $M(t)$ has constant principal curvatures $\tanh(r) = \frac{\sqrt{t}+1}{\sqrt{t-1}}$, $\coth(r) = \frac{\sqrt{t}-1}{\sqrt{t-1}}$, and $2\tanh(2r) = 2\frac{\sqrt{t-1}}{\sqrt{t}}$ with multiplicities $n-1$, $n-1$ and 1 respectively. Moreover, Montiel [24] assert that $M(t) = \pi(M'(t))$ is a tube of radius r over a totally geodesic totally real hyperbolic space $\mathbb{R}H^n$ in the complex hyperbolic space $\mathbb{C}H^n$. It is neither *totally η-umbilical* nor *pseudo-Einstein*.

When $t = 4$, the hypersurface $M(4)$ is a tube of radius $r = \ln \frac{1+\sqrt{3}}{\sqrt{2}}$ over the real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$. It has two distinct constant principal curvatures and is not totally η-umbilic.

For the type (B) in the complex hyperbolic space $\mathbb{C}H^n$, let us introduce some results on contact hypersurfaces due to Berndt-Suh [2], Montiel [24], and Montiel-Romero [25], as follows:

Proposition 4.5. *Let M be a tube of radius $0 < r < \infty$ around the totally geodesic and totally real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$. Then the following statements hold:*

(1) *M is a Hopf real hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = \coth(r)$	T_λ	$n-1$
$\mu = \tanh(r)$	T_μ	$n-1$
$\alpha = 2\tanh(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A and the structure tensor field ϕ satisfy*

$$A\phi + \phi A = k\phi, \quad k = 2\rho \neq 0 : \textit{constant}.$$

(4) *The trace h of the shape operator A and its square h^2 becomes the following respectively*

$$h = 2\tanh(2r) + 2(n-1)\coth(2r),$$

and

$$h^2 = 4\tanh^2(2r) + 4(n-1)^2\coth^2(2r) + 8(n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = 4\tanh^2(2r) + 4(n-1)\coth^2(2r) - 2(n-1).$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = -2(n-1)(2n-3) + 4(n-1)(n-2)\coth^2(2r).$$

5. Hopf Pseudo-Ricci-Bourguignon solitons in $\mathbb{C}H^n$

Now let us introduce pseudo-Ricci-Bourguignon solitons $(M, V, \Omega, \rho, \gamma, g)$ which are solutions of the pseudo-Ricci-Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) + \psi\eta(X)\eta(Y) = (\Omega + \rho\gamma)g(X, Y),$$

for any tangent vector fields X and Y on M, where Ω is a Ricci-Bourguignon soliton constant, ρ any constant and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Siddiqi-Prakasha [12], and Morgan-Tian [26]). Then let us consider the Reeb vector field ξ as the pseudo-Ricci-Bourguignon soliton vector field V as follows:

$$\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) + \mathrm{Ric}(X, Y) + \psi\eta(X)\eta(Y) = (\Omega + \rho\gamma)g(X, Y) \tag{5.1}$$

for any X, Y tangent to M. Then by virtue of the Lie derivative $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)$, the formula (5.1) can be given by

$$\mathrm{Ric}(X) = \frac{1}{2}(A\phi - \phi A)X - \psi\eta(X)\xi + (\Omega + \rho\gamma)X \tag{5.2}$$

for any X tangent to M. From this, by applying the structure tensor ϕ to both sides, we get the following two formulas

$$\mathrm{Ric}(\phi X) = \frac{1}{2}(A\phi^2 - \phi A\phi)X - \psi\eta(\phi X)\xi + (\Omega + \rho\gamma)\phi X,$$

and

$$\phi\mathrm{Ric}(X) = \frac{1}{2}(\phi A\phi - \phi^2 A)X - \psi\eta(X)\phi\xi + (\Omega + \rho\gamma)\phi X.$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo-anti-commuting property* holds as follows:

$$\mathrm{Ric}(\phi X) + \phi\mathrm{Ric}(X) = 2(\Omega + \rho\gamma)\phi X. \tag{5.3}$$

Then by Lemmas 3.1 and 3.2, if $X \in T_\lambda$, then $\phi X \in T_\mu$, where $\mu = \frac{\alpha\lambda-2}{2\lambda-\alpha}$ if $2\lambda - \alpha \neq 0$. If $2\lambda - \alpha = 0$ in Lemma 3.1, then $\alpha = \pm 2$ and $\lambda = \pm 1$. Now let us consider the case $2\lambda - \alpha \neq 0$. Then (5.3) becomes for $X \in T_\lambda$

$$\lambda^2 + \mu^2 - h(\lambda + \mu) = k, \tag{5.4}$$

where the function k is given by $k = -2(\nu + \rho\gamma) - 2(2n + 1)$. Then substituting $\mu = \frac{\alpha\lambda-2}{2\lambda-\alpha}$ into (5.4), it gives the following

$$4\lambda^4 - 4(\alpha + h)\lambda^3 + 2(\alpha^2 + \alpha h - 2k)\lambda^2 - 4(\alpha - h - \alpha k)\lambda + 4 - 2\alpha h - k\alpha^2 = 0. \tag{5.5}$$

Let λ_1, λ_2, λ_3 and λ_4 be the roots of the above biquadric equation. Then from the relations of the roots and coefficients of the equation (5.4) it follows that

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \alpha + h\\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = (\alpha^2 + \alpha h - 2k)/2\\
&\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = \alpha - h - \alpha k\\
&\lambda_1\lambda_2\lambda_3\lambda_4 = (4 - 2\alpha h - k\alpha^2)/4.
\end{aligned} \tag{5.6}$$

Here we consider the trace h of the shape operator A of M in complex hyperbolic space $\mathbb{C}H^n$. Then it is defined by

$$h = \alpha + m_1\lambda_1 + m_2\lambda_2 + m_1\frac{\alpha\lambda_1 - 2}{2\lambda_1 - \alpha} + m_2\frac{\alpha\lambda_2 - 2}{2\lambda_2 - \alpha}.$$

From this, together with (5.6) and the fact that the principal curvature α is constant, and that the scalar curvature γ in (3.3) is

$$\gamma = -4(n^2-1) + h^2 - \mathrm{Tr}A^2,$$

where $\mathrm{Tr}A^2$ is given by

$$\mathrm{Tr}A^2 = \alpha^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_1\Big(\frac{\alpha\lambda_1-2}{2\lambda_1-\alpha}\Big)^2 + m_2\Big(\frac{\alpha\lambda_2-2}{2\lambda_2-\alpha}\Big)^2.$$

Substituting h and k into (5.6), we can see that (5.6) consists of four linearly independent equations with constant multiplicities m_1 and m_2 of the principal curvatures λ_1 and λ_2 respectively. Consequently, it can be asserted that M has at most 5 distinct constant principal curvatures in $\mathbb{C}H^n$. Then we can introduce a well known result due to Berndt [1] as follows:

Theorem 5.1. *Let M be a connected Hopf real hypersurfaces in complex hyperbolic space $\mathbb{C}H^n$ with constant principal curvatures. Then M is holomorphic congruent to an open part of the following hypersurfaces in $\mathbb{C}H^n$:*

(i) *a horosphere,*

(ii) *a tube of arbitrary radius r around a totally geodesic embedded submanifold $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{0, \cdots, n-1\}$,*

(iii) *a tube of arbitrary radius r around a totally geodesic embedded n-dimensional real hyperbolic space $\mathbb{R}H^n$.*

Now a horosphere, a geodesic hypersphere ($k = 0, k = n-1$) and a tube of radius r over $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{1, \cdots, n-2\}$, belong to the classes of tubes of the first and second type in Theorem 5.1, respectively. That is, all of these 4 kinds of tubes mentioned in Propositions 4.1, 4.2, 4.3 and 4.4 are included in the above classes of type (i) and (ii), respectively. So by Theorem 1.1, they are characterized by having commuting shape operator. That is, $A\phi = \phi A$. Accordingly, from the notion of pseudo-Ricci-Bourguignon soliton $(M, \xi, \Omega, \rho, \gamma, g)$ of M, (5.2) becomes

$$\mathrm{Ric} = (\Omega + \rho\gamma)g - \psi\eta\otimes\eta. \tag{5.7}$$

This means that those hypersurfaces are pseudo-Einstein. Then, bearing in mind Theorem 1.3 we have:

For any $X \in T_\lambda$, $\lambda = \coth(r)$ with multiplicity $2(n-1)$ in Proposition 4.1, from (3.2) we get

$$\begin{aligned}\mathrm{Ric}(X) &= -(2n+1)X + \{(2n-1)\coth^2(r) + 1\}X - \coth^2(r)X \\ &= -2nX + 2(n-1)\coth^2(r)X\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\mathrm{Ric}(\xi) &= -2(n-1)\xi + \alpha(h-\alpha)\xi = 2(n-1)(\alpha\lambda - 1)\xi \\ &= 2(n-1)\Big\{2\coth(2r)\coth(r) - 1\Big\}\xi \\ &= 2(n-1)\coth^2(r)\xi.\end{aligned} \tag{5.9}$$

Then by the pseudo-Einstein property, $\mathrm{Ric}(X) = (\Omega + \rho\gamma)X - \psi\eta(X)\xi$ for any $X \in T_xM$, $x \in M$, (5.8) and (5.9) give respectively that for any $X \in T_\lambda$, $\lambda = \coth(r)$

$$(\Omega + \rho\gamma)X = \{-2n + 2(n-1)\coth^2(r)\}X$$

and

$$\mathrm{Ric}(\xi) = 2(n-1)\coth^2(r)\xi.$$

Then from (5.7), it gives the following

$$\Omega + \rho\gamma = -2n + 2(n-1)\coth^2(r) \quad \text{and} \quad \psi = -2n.$$

For a horosphere it could be enough for us to consider $r \to \infty$. Then we get $\Omega + \rho\gamma = -2$ and $\psi = -2n$. Similarly, by Proposition 4.3, we get the following

$$\Omega + \rho\gamma = -2n + 2(n-1)\tanh^2(r) \quad \text{and} \quad \psi = -2n.$$

Then by virtue of Theorem 1.3 in the introduction, we give a complete proof of our Theorem 1.4 for the first and second cases (i) and (ii) in Theorem 5.1.

Next, in the remained case let us check that real hypersurfaces of type (B) in Proposition 4.5, that is, the third case (iii) in Theorem 5.1 satisfy our condition. It is characterized by $A\phi + \phi A = \ell\phi$, where $\ell \neq 0$ is constant. Moreover, by Proposition 4.5, the principal curvatures are given by $\lambda = \coth(r)$, $\mu = \tanh(r)$ and $\alpha = 2\tanh(2r)$. So $\ell = \frac{4}{\alpha}$. For any $X \in T_\lambda$ the vector field $\phi X \in T_\mu$. So (5.2) gives

$$\begin{aligned} \mathrm{Ric}(X) &= (\mu - \lambda)\phi X - \psi\eta(X)\xi + (\Omega + \rho\gamma)X \\ &= (\mu - \lambda)\phi X + (\Omega + \rho\gamma)X \end{aligned} \tag{5.10}$$

for $X \in T_\lambda$.

On the other hand, from (3.2) the left side of (5.8) becomes the following for any $X \in T_\lambda$

$$\mathrm{Ric}(X) = \{-(2n+1) + (h - \lambda)\lambda\}X, \tag{5.11}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}H^n$. By virtue of (5.11), the first term in the right side of (5.10) is skew-symmetric and the other terms are symmetric. Accordingly, if we take the inner product of (5.10) with any $\phi X \in T_\mu$ for $X \in T_\lambda$ and use (5.8), naturally we get $\lambda = \mu$. This means that $\coth(r) = 1$. That is the radius r becomes ∞. So this gives a contradiction in Proposition 4.5 for the radius $0 < r < \infty$.

Consequently, summing up all the facts mentioned above, we give a complete proof of our Theorem 1.4 in the introduction.

6. Gradient Pseudo-Ricci-Bourguignon solitons with isometric Reeb flow in $\mathbb{C}H^n$

In this section, let us assume that M admits a gradient pseudo-Ricci-Bourguignon soliton $(M, W, \eta, \rho, \gamma, g)$. Then we could consider the soliton vector field W as $W = Df$ for a smooth function on M. Then the gradient pseudo-Ricci-Bourguignon soliton equation becomes

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \rho\gamma)X.$$

Here, by Theorem 1.1 we want to consider only a tube over $\mathbb{C}H^k$, $k \in \{1, \cdots, n-1\}$, or a horosphere. Then the shape operator of A in the complex hyperbolic space $\mathbb{C}H^n$ with isometric Reeb flow can be expressed as

$$A = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \coth(r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \coth(r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \tanh(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \tanh(r) \end{bmatrix}$$

with three constant principal curvatures α, $\coth(r)$ and $\tanh(r)$ with multiplicities 1, 2ℓ and $2k$ respectively, where $\ell = n - k - 1$.

Then, by putting $X = \xi$ in (3.2), and using $A\xi = \alpha\xi$, we have the following

$$\begin{aligned}\mathrm{Ric}(\xi) &= -(2n+1)\xi + 3\xi + hA\xi - A^2\xi \\ &= -2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ &= \kappa\xi,\end{aligned}$$

where we have put $\kappa = 2(n-1) + h\alpha - \alpha^2$. So by Proposition 4.4, the constant κ is given by

$$\begin{aligned}\kappa &= -2(n-1) + (h\alpha - \alpha^2) \\ &= -2(n-1) + \{(2\ell+1)\coth(r) + (2k+1)\tanh(r)\}2\coth(2r) - (2\coth(2r))^2 \\ &= -2(n-1) + 2\{\ell\coth^2(r) + k\tanh^2(r) + (k+\ell)\} \\ &= 2\ell\coth^2(r) + 2k\tanh^2(r).\end{aligned}$$

Then by taking the covariant derivative we get the following two formulas

$$(\nabla_X \mathrm{Ric})\xi = \kappa\phi AX - \mathrm{Ric}(\phi AX),$$

and

$$(\nabla_\xi \mathrm{Ric})X = h(\nabla_\xi A)X - (\nabla_\xi A^2)X.$$

Since M admits a gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \xi, \Omega, \rho, \gamma, g)$, we could consider the soliton vector field W as $W = Df$ for any smooth function f on M. In the introduction we have noted that $\mathrm{Hess}(f)$ is defined by $\mathrm{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\mathrm{Hess}(f)(X,Y) = g(\nabla_X Df, Y).$$

Then the gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \xi, \Omega, \rho, \gamma, g)$ can be given by

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \rho\gamma)X.$$

for any tangent vector field X on M. Then by covariant differentiation, it gives

$$\begin{aligned}&\nabla_X\nabla_Y Df + (\nabla_X \mathrm{Ric})(Y) + \mathrm{Ric}(\nabla_X Y) + \psi(\nabla_X\eta)(Y) + \psi\eta(\nabla_X Y)\xi + \psi\eta(Y)\nabla_X\xi \\ &= X(\Omega + \rho\gamma)Y + (\Omega + \rho\gamma)\nabla_X Y\end{aligned}$$

for any vector fields X and Y tangent to M in $\mathbb{C}H^n$. From this, together with the above two formulas $(\nabla_X \mathrm{Ric})\xi$ and $(\nabla_\xi \mathrm{Ric})X$, it follows that

$$\begin{aligned}R(\xi, Y)Df &= \nabla_\xi\nabla_Y Df - \nabla_Y\nabla_\xi Df - \nabla_{[\xi,Y]}Df \\ &= (\nabla_Y \mathrm{Ric})\xi - (\nabla_\xi \mathrm{Ric})Y + \psi\phi AY \\ &= (k+\psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.\end{aligned} \tag{6.1}$$

Moreover, we have the following for a real hypersurface M in $\mathbb{C}H^n$ with isometric Reeb flow

$$R(\xi, Y)Df = -g(Y, Df)\xi + g(\xi, Df)Y + g(AY, Df)A\xi - g(A\xi, Df)AY. \tag{6.2}$$

From this, let us take a vector field $Y \in T_\lambda$, $\lambda = \coth(r)$. Moreover, we can decompose the tangent space $T\mathbb{C}H^n$ as

$$T\mathbb{C}H^n = T_\lambda \oplus T_\mu \oplus \mathbb{R}\xi \oplus \mathbb{R}N,$$

where $\lambda = \coth(r)$ and $\mu = \tanh(r)$. If M is of type (A_1), that is, a geodesic hypersphere in $\mathbb{C}H^n$, it can be decomposed as

$$T\mathbb{C}H^n = T_\lambda \oplus \mathbb{R}\xi \oplus \mathbb{R}N,$$

or otherwise

$$T\mathbb{C}H^n = T_\mu \oplus \mathbb{R}\xi \oplus \mathbb{R}N.$$

Then for any $Y \in T_\lambda$ (3.1) gives

$$\begin{aligned} R(\xi, Y)Df &= -g(Y, Df)\xi + g(\xi, Df)Y + \alpha\lambda g(Y, Df)\xi - \alpha\lambda g(\xi, Df)Y \\ &= (-1 + \alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}. \end{aligned} \tag{6.3}$$

Then by taking the inner product of (6.2) with the Reeb vector field ξ and using the fact that $g((\nabla_\xi A)Y, \xi) = 0$ and $g((\nabla_\xi A^2)Y, \xi) = 0$ in (6.1) for any $Y \in T_\lambda$, it follows that $(-1 + \alpha\lambda)g(Y, Df) = \coth^2(r)g(Y, Df) = 0$. But $\coth^2(r) \neq 0$ of M. Then we get

$$g(Y, Df) = 0 \tag{6.4}$$

for any $Y \in T_\lambda$.

Now let us check (6.2) for $Y \in T_\mu$, $\mu = \tanh(r)$. Then (6.2) gives

$$R(\xi, Y)Df = -g(Y, Df)\xi + g(\xi, Df)Y + \alpha\mu g(Y, Df)\xi - \alpha\mu g(\xi, Df)Y. \tag{6.5}$$

Then by taking the inner product of (6.5) with the Reeb vector field ξ and $Y \in T_\mu$ respectively and using (6.1), we get

$$(-1 + \alpha\mu)g(Y, Df) = 0 \quad \text{and} \quad (-1 + \alpha\mu)g(\xi, Df) = 0, \tag{6.6}$$

where $g(R(\xi, Y)Df, \xi) = 0$ and the left side $g(R(\xi, Y)Df, Y) = 0$ is given by virtue of the following formulas

$$\begin{aligned} &g(\phi AY, Y) = \mu g(\phi Y, Y) = 0, \\ &\mathrm{Ric}(\phi AY) = \mu\{(2n + 1) + \mu h - \mu^2\}\phi Y, \end{aligned}$$

and

$$g((\nabla_\xi A)Y, Y) = \mu g(\nabla_\xi Y, Y) - \mu g(\nabla_\xi Y, Y) = 0.$$

Since $-1 + \alpha\mu = -1 + (\coth(r) + \tanh(r))\tanh(r) = \tanh^2(r) \neq 0$ for $r > 0$ as M has isometric Reeb flow, (6.6) implies that

$$g(Y, Df) = 0 \text{ and } g(\xi, Df) = 0 \tag{6.7}$$

for any $Y \in T_\mu$, $\mu = \tanh r$. For a geodesic hypersphere of type (A_1) in $\mathbb{C}H^n$ it holds either $g(Y, Df) = 0$ for $Y \in T_\lambda = \mathcal{C}$ or for $Y \in T_\mu = \mathcal{C}$ from the above decomposition, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space TM of M in $\mathbb{C}H^n$. Of course, it also holds $g(\xi, Df) = 0$ for a geodesic hypersphere in $\mathbb{C}H^n$.

Now finally let us check that M is locally congruent to a horosphere. The radius r becomes infinity ∞, and its principal curvatures are $\alpha = 2$, $\lambda = \mu = 1$ with multiplicities 1 and $2(n - 1)$, respectively. Then, (6.4) holds for $\alpha = 2$ and $\lambda = 1$, and (6.7) for $\alpha = 2$ and $\mu = 1$, because $-1 + \alpha\lambda = 1$ and $-1 + \alpha\mu = 1$ respectively. Accordingly, the gradient vector field Df of the smooth function f is vanishing on M.

Summing up (6.4), (6.7) and the above facts, the gradient of the smooth function f is identically vanishing, that is, $Df = 0$ on M in $\mathbb{C}H^n$. Consequently, we can conclude that the gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \xi, \Omega, \rho, \gamma, g)$ is trivial. So it becomes pseudo-Einstein. That is,

$$\mathrm{Ric}(X) = (\Omega + \rho\gamma)X - \psi\eta(X)\xi$$

for any $X \in T_xM$, $x \in M$. Then by Theorem 1.3, we get a complete proof of our Theorem 1.5 in the Introduction.

7. Gradient Pseudo-Ricci-Bourguignon soliton on contact real hypersurfaces in $\mathbb{C}H^n$

In this section let us consider that M is a tube of radius r over the real hyperbolic space $\mathbb{R}H^n$. It is said to be of type (B) and a contact real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$. Here the real hypersurface M in $\mathbb{C}H^n$ is contact if and only if the shape operator A of M in $\mathbb{C}H^n$ satisfies

$$A\phi + \phi A = k\phi, \quad k \neq 0,$$

where the constant k and the Reeb curvature α satisfies $k\alpha = 4$. By Proposition 4.5, it has 3 constant principal curvatures $\alpha = 2\tanh(2r)$, $\lambda = \coth(r)$ and $\mu = \tanh(r)$ with multiplicities 1, $n-1$ and $n-1$ respectively.

Then, by putting $X = \xi$ in (3.2), we have the following

$$\begin{aligned}\mathrm{Ric}(\xi) &= -2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ &= \ell\xi,\end{aligned}$$

where by Proposition 4.5 the above constant ℓ is given by

$$\begin{aligned}\ell &= -2(n-1) + h\alpha - \alpha^2 \\ &= -2(n-1) + \{2\tanh(2r) + 2(n-1)\coth(2r)\}2\tanh(2r) - (2\tanh(2r))^2 \\ &= 2(n-1).\end{aligned}$$

The gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \xi, \Omega, \rho, \gamma, g)$ must satisfy

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \rho\gamma)X. \tag{7.1}$$

By differentiating (7.1), the curvature tensor of M gives

$$\begin{aligned}R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= -(\nabla_X\mathrm{Ric})Y - \mathrm{Ric}(\nabla_X Y) - \psi(\nabla_X\eta)(Y)\xi - \psi\eta(\nabla_X Y)\xi \\ &\quad - \psi\eta(Y)\nabla_X\xi + (\Omega + \theta\gamma)\nabla_X Y \\ &\quad + (\nabla_Y\mathrm{Ric})X + \mathrm{Ric}(\nabla_Y X) + \psi(\nabla_Y\eta)(X)\xi + \psi\eta(\nabla_Y X)\xi \\ &\quad + \psi\eta(X)\nabla_Y\xi - (\Omega + \theta\gamma)\nabla_Y X \\ &\quad + \mathrm{Ric}([X,Y]) - (\Omega + \theta\gamma)[X,Y] + \psi\eta([X,Y])\xi \\ &= (\nabla_Y\mathrm{Ric})X - (\nabla_X\mathrm{Ric})Y - \psi(\nabla_X\eta)(Y)\xi + \psi(\nabla_Y\eta)(X)\xi \\ &\quad - \psi\eta(Y)\nabla_X\xi + \psi\eta(X)\nabla_Y\xi\end{aligned} \tag{7.2}$$

for any vector fields X and Y tangent to M. From this, together with the above two formulas, it follows that

$$\begin{aligned}R(\xi,Y)Df &= (\nabla_Y\mathrm{Ric})\xi - (\nabla_\xi\mathrm{Ric})Y \\ &\quad - \psi(\nabla_\xi\eta)(Y)\xi + \psi(\nabla_Y\eta)(\xi)\xi - \psi\eta(Y)\nabla_\xi\xi + \psi\eta(\xi)\nabla_Y\xi \\ &= (\ell + \psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.\end{aligned} \tag{7.3}$$

Let us suppose that a contact real hypersurface M in $\mathbb{C}H^n$ admits a gradient pseudo-Ricci-Bourguignon soliton. Then by Proposition 4.5, M is Hopf. So the scalar curvature γ is constant. Accordingly, let us take $Y \in T_\lambda$, $\lambda = \coth(r)$, then (7.3) becomes

$$\begin{aligned}R(\xi,Y)Df &= -g(Y,Df)\xi + g(\xi,Df)Y + \alpha g(AY,Df)\xi - \alpha\lambda g(\xi,Df)Y \\ &= (-1 + \alpha\lambda)\Big\{g(Y,Df)\xi - g(\xi,Df)Y\Big\} \\ &= \Big\{-1 + 2\tanh(2r)\coth(r)\Big\}\Big\{g(Y,Df)\xi - g(\xi,Df)Y\Big\}.\end{aligned} \tag{7.4}$$

Here we note that $-1+\alpha\lambda = \coth^2(r) \neq 0$ for the radius $r > 0$. Accordingly, by taking the inner product (7.3) with the Reeb vector field ξ and $Y \in T_\lambda$, $\lambda = \coth(r)$, and use $g(R(\xi,Y)Df,\xi) = 0$ and $g(R(\xi,Y)Df,Y) = 0$ for $Y \in T_\lambda$ we get the following respectively

$$g(\xi, Df) = 0 \text{ and } g(Y, Df) = 0 \tag{7.5}$$

for any $Y \in T_\lambda$. Next, let us consider $Y \in T_\mu$. Then (7.4) with $Y \in T_\mu$ gives

$$\begin{aligned} R(\xi,Y)Df &= -g(Y,Df)\xi + g(\xi,Df)Y + \alpha g(AY,Df)\xi - \alpha\mu g(\xi,Df)Y \\ &= (-1+\alpha\mu)\Big\{g(Y,Df)\xi - g(\xi,Df)Y\Big\}. \end{aligned} \tag{7.6}$$

Finally, let us take the inner product of (7.6) with $Y \in T_\mu$, and use $AY = \mu Y$, $A\phi Y = \lambda\phi Y$ for a contact hypersurface in $\mathbb{C}H^n$ and use (7.3). Then we have

$$\begin{aligned} -(-1+\alpha\mu)g(\xi,Df) &= (\ell+\psi)g(\phi AY,Y) - g(\mathrm{Ric}(\phi AY),Y) - hg((\nabla_\xi A)Y,Y) + g((\nabla_\xi A^2)Y,Y) \\ &= -hg(\nabla_\xi(AY) - A\nabla_\xi Y, Y) + g(\nabla_\xi(A^2Y) - A^2\nabla_\xi Y, Y) \\ &= 0, \end{aligned} \tag{7.7}$$

where in the second equality we have used the following formulas

$$\begin{aligned} \mathrm{Ric}(\phi AY) &= -(2n+1)\phi AY + hA\phi AY - A^2\phi AY \\ &= \mu\{-(2n+1) + \lambda h - \lambda^2\}\phi Y, \end{aligned}$$

$$g(\mathrm{Ric}(\phi AY),Y) = 0,$$

and

$$\begin{aligned} g((\nabla_\xi A)Y,Y) &= g(\nabla_\xi(AY) - A\nabla_\xi Y, Y) \\ &= g(\mu\nabla_\xi Y - A\nabla_\xi Y, Y) = 0. \end{aligned}$$

Here we note that $-1+\alpha\mu = \tanh^2(r) \neq 0$ for the radius $r > 0$. Then (7.7) implies $g(Y,Df) = 0$ for any $Y \in T_\mu$. From this, together with (7.4), we can assert that $Df = 0$. Then from (7.1) M becomes pseudo-Einstein. That is,

$$\mathrm{Ric}(X) = (\Omega + \rho\gamma)X - \psi\eta(X)\xi$$

for any $X \in T_xM$, $x \in M$. Since M is a contact hypersurface, it should satisfy the condition of $A\phi + \phi A = k\phi$, $k \neq 0$ constant. So in order to satisfy the contact condition, the radius r in Propositions 4.1, 4.2 and 4.3 becomes $r \to \infty$. In fact, if a geodesic hypersphere in $\mathbb{C}H^n$ satisfies the contact condition, then $k = \frac{4}{\alpha}$. So this gives $2\coth(r) = 2\tanh(2r)$, which implies $\coth^2(r) = 1$. So we may put $\coth(r) = 1$ for $r \to \infty$. Accordingly, a geodesic hypersphere in Proposition 4.1 should be a horosphere in Proposition 4.2.

Consequently, by Theorems 1.2 and 1.3, we give a complete classification of contact hypersurfaces in complex hyperbolic space $\mathbb{C}H^n$ which admits a gradient pseudo-Ricci-Bourguignon soliton.

Summing up all the discussions mentioned above, we give a complete proof of our Theorem 1.6.

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