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Well-posedness and general decay of nonlinear coupled waves system with viscoelastic term

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Abstract. This research investigates a coupled wave system where the second equation is influenced by viscoelastic damping and is subject to Dirichlet boundary conditions on the interval $(0, 1)$. For a broad class of relaxation functions and within a general framework, we establish the global existence of solutions using the Faedo-Galerkin method. In addition, we derive general decay estimates by applying Lyapunov's method and employing certain convexity arguments.

1. Introduction

In our study, we examine a coupled system of viscoelastic wave equations:

$$\begin{cases} u_{tt}(x,t) - \{u_x(x,t) + au_{xt}(x,t)\}_x + f(u,v) = 0 & (x,t) \in (0,1) \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) + g(u,v) = -b\int_0^t \phi(t-s)v_{xx}(x,s)ds & (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = v(0,t) = 0, u(1,t) = v(1,t) = 0 & \forall t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \forall x \in (0,1), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) & \forall x \in (0,1), \end{cases} \tag{1}$$

where the constants a and b are assumed to be positive, ϕ is a positive function, f and g represents source terms and (u_0, u_1, v_0, v_1) denotes the initial data.

Wave equations are fundamental in modeling dynamic systems where oscillations propagate through a medium. These oscillations, or vibrations, travel across various parts of the medium, resulting in wave phenomena. One of the key challenges in these systems is suppressing excessive vibrations and ensuring the stability of the model. To address these challenges, several types of internal damping are adopted, including frictional damping (see [3, 7, 25, 28]), Kelvin-Voigt damping (see [14, 16, 18]), and viscoelastic

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damping (see [5, 6, 10]), which is the most commonly used, although it is considered one of the weakest forms of damping. In [23], Mustafa considers a coupled viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + f_1(u,v) = 0, \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + f_2(u,v) = 0, \end{cases} \tag{2}$$

They demonstrated that this system exhibits generalized stability under hypothesis on the the coupling

$$|f_i(u,v)| \leq d(|u|^{p_i} + |v|^{q_i}), \qquad d > 0, \ \ p_i, q_i \geq 1, \ \ i = 1, 2,$$

and

$$g_i'(t) \leq -d_i g_i^{a_i}(t) \qquad a_i \in [1, 3/2[, \ \ i = 1, 2.$$

Notably, Mustafa's work [24] explored the asymptotic stability of the system (2) with

$$g_i(t) \leq -\xi_i(t) g_i(t), \quad \text{for } i = 1, 2.$$

Another closely related result was achieved by Messaoudi and Tatar [21], as well as by Al-Gharabli and Kafini [1], involving coupling functions

$$f_1(u,v) = f_2(v,u) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho} u |v|^{\rho+2}.$$

For a Timoshenko system with nonlinear source terms, Feng [8, 9] demonstrated the existence of exponential attractors under specific hypotheses regarding the coupling functions. Gheraibia and Boumaza [11], as well as Kamache et al. [13], demonstrated stability for a Kirchhoff equation with nonlinearity of the form $|u|^p u$. Al-Mahdi et al. [2] demonstrated exponential energy decay for a plate system with source terms $u \ln |u|$. Similarly, Yüksekkaya [26] obtained a comparable result for the logarithmic Lamé system with source terms $|u|^{p-2} u \ln |u|^k$. Lekdim and Khemmoudj, in [15, 17], investigated the stability of an Euler-Bernoulli system with coupling $f_1(u,v) = f_2(v,u) = a\{(u_x^2 + v_x^2) u_x\}_x$.

The authors in [22], considered a weakly dissipative plate equation

$$u_{tt} + \Delta^2 u + \int_0^t g(t-s) u_{xx}(s)\,ds - u_{xxt} = 0,$$

with mild hypotheses on the kernels g, and showed that the energy exhibits a general decay behavior.

Motivated by previous studies, we analyze the interaction between the wave equation and viscoelastic damping in the presence of nonlinear source terms. Our objective is to explore the existence, uniqueness, and stability of solutions. This research contributes to a deeper understanding of the relationship between the system's behavior and the properties of the relaxation function.

The paper is structured as follows: Section 2 provides preliminary results, Section 3 addresses well-posedness, and Section 4 establishes a general decay result.

2. Preliminaries

We assume that ϕ satisfies:

(H1) Let $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ denote a decreasing function that satisfies

$$\phi(0) > 0, \qquad \int_0^\infty \phi(s) ds = \bar{\phi} < 1.$$

(H2) We assume the existence of a positive upper C^1-function $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is linear or a strictly increasing and strictly convex C^2-function on $(0, l]$ such that $l \leq \phi(0)$, with $\Phi(0) = \Phi'(0) = 0$, and a positive non-increasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$\phi'(t) \leq -\zeta(t)\Phi(\phi(t)), \quad \text{for all } \ t \geq 0.$$

(H3) We assume the existence of a positive function G such that

$$(f(u,v), g(u,v)) = (\frac{\partial G}{\partial u}, \frac{\partial G}{\partial v}),$$
$$G \geq 0, \quad uf(u,v) + vg(u,v) \geq G(u,v),$$
$$\text{and} \quad \left|g(u,v)\right| \leq r(|u|^p + |v|^q).$$

Remark 2.1 ([24]). *There are two positive real numbers d and t_0, such that*

$$\phi'(t) \leq -\zeta(t)\Phi(\phi(t)) \leq -\frac{d}{\phi(0)}\phi(0) \leq -\frac{d}{\phi(0)}\phi(t), \quad \forall t \in [0, t_0]. \tag{3}$$

The energy of problem (1) is represented by

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\|u_x\|^2 + \frac{1 - \int_0^t \phi(s)ds}{2} b\|v_x\|^2 + \frac{b}{2}\phi \circ v_x + \int_0^1 G(u,v)dx, \tag{4}$$

where $\|\cdot\|$ represents the norm in $L^2(0,1)$ and

$$\left(\phi \circ u\right) = \int_0^t \phi(t-s)\|u(t) - u(s)\|^2 ds.$$

Lemma 2.2. *The energy* (4) *satisfies*

$$E'(t) = -\frac{b}{2}\phi(t)\|v_x\|^2 + \frac{b}{2}\phi' \circ v_x - a\|u_{xt}\|^2 \leq 0. \tag{5}$$

Proof. Multiplying the system (1) by (u_t, v_t) and performing integration by parts over $(0,1)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \|v_t\|^2 + \|u_x\|^2 + b\|v_x\|^2\right] &= -a\|u_{xt}\|^2 - \int_0^1 f(u,v)u_t dx - \int_0^1 g(u,v)v_t dx \\ &+ b\int_0^t \phi(s-t)\int_0^1 v_x(s)v_{xt}dxds. \end{aligned}$$

To complete the proof, we utilize condition (H3) and the following identity

$$\int_0^t \phi(s-t)\int_0^1 v_x v_{xt}dxds = -\frac{1}{2}\phi(t)\|v_x\|^2 + \frac{1}{2}\phi' \circ v_x - \frac{1}{2}\frac{d}{dt}\left[(\phi \circ v_x) - \int_0^t \phi(s)ds\|v_x\|^2\right].$$

□

Remark 2.3. *Based on (H1), (H2),(H3) and Lemma 2.2, we can infer energy dissipation. Furthermore,*

$$E(t) \leq E(0) \qquad \forall t > 0.$$

3. Well-posedness

This section focuses on proving the well-posedness for the problem described in (1). To accomplish this, we utilize the Faedo-Galerkin approach.

Theorem 3.1. *Let $V = H_0^1(0,1)$. Suppose (H1) holds, and let $(u_0, u_1), (v_0, v_1) \in V \times L^2(0,1)$. Then, there is a unique weak solution to the problem* (1).

Proof. Let us consider $\{\omega_i\}_{i=1}^{\infty}$ and $\{z_i\}_{i=1}^{\infty}$ as bases of the space V. Then we define the finite-dimensional subspaces $V_m = \text{span}\{\omega_1, \omega_2, ..., \omega_m\}$ and $Z_m = \text{span}\{z_1, z_2, ..., z_m\}$. The initial data is projected on these subspaces V_m and Z_m as follows

$$u_0^m(x) = \sum_{i=1}^{m} a_i\omega_i, \quad u_1^m(x) = \sum_{i=1}^{m} b_i\omega_i \quad \text{and} \quad v_0^m(x) = \sum_{i=1}^{m} c_i z_i, \quad v_1^m(x) = \sum_{i=1}^{m} d_i z_i,$$

where

$$\begin{cases} (u_0^m, v_0^m) \to (u_0, v_0) & \text{in } V \times V. \\ (u_1^m, v_1^m) \to (u_1, v_1) & \text{in } L^2(0,1) \times L^2(0,1). \end{cases}$$

Now, we search the approximate solutions

$$u^m(x,t) = \sum_{i=1}^{m} \theta_i^m(t) w_i(x) \text{ and } v^m(x,t) = \sum_{i=1}^{m} g_i^m(t) z_i(x),$$

to the finite dimensional Cauchy problem

$$\begin{cases} \int_0^1 u_{tt}^m w dx + \int_0^1 u_x^m w_x dx + a\int_0^1 u_{xt}^m w_x dx + \int_0^1 f(u^m, v^m) w dx = 0, \\ \int_0^1 v_{tt}^m z dx + b\int_0^1 v_x^m z_x dx + \int_0^1 g(u^m, v^m) z dx = b\int_0^t \phi(t-s) \int_0^1 v_x^m z_x dx ds, \\ (u^m(0), v^m(0)) = (u_0^m, v_0^m), \ (u_t^m(0), v_t^m(0)) = (u_1^m, v_1^m). \end{cases} \tag{6}$$

According to the theory of ODEs, the problem (6) has solutions $(\theta_i^m(t), g_i^m(t))$ on $[0, t_m]$. The following a priori estimates will prove that $t_m = +\infty$.

Energy estimates: Multiplying the system (6) by $(\theta_i^m(t))'$ and $(g_i^m(t))'$, respectively, and using (H3), we get

$$\frac{d}{dt}E^m(t) + a\left\|u_{xt}^m\right\|^2 = \frac{b}{2}\phi' \circ v_x^m - \frac{b}{2}\phi(t)\left\|v_x^m\right\|^2 \le 0, \tag{7}$$

where

$$E^m(t) = \frac{1}{2}\left(\left\|u_t^m\right\|^2 + \left\|v_t^m\right\|^2\right) + \frac{1}{2}\left\|u_x^m\right\|^2 + \frac{1 - \int_0^t \phi(s)ds}{2} b\left\|v_x^m\right\|^2 + \frac{b}{2}\phi \circ v_x^m + \int_0^1 G(u^m, v^m) \ge 0.$$

Integrating (7) over $(0,t)$, and as (u_0^m, v_0^m) and (u_1^m, v_1^m) are bounded in V and $L^2(0,1)$, it follows that

$$E^m(t) + \int_0^t a\left\|u_{xt}^m\right\|^2 \le E^m(0) = e_0,$$

where e_0 denotes a positive constant, independent of both t and m. Consequently,

$$\begin{cases} (u^m), \ (v^m) & \text{are bounded in } L^\infty(0,\infty; V), \\ (u_t^m) & \text{is bounded in } L^\infty(0,\infty; L^2(0,1)) \cap L^2(0,\infty; V), \\ (v_t^m) & \text{is bounded in } L^\infty(0,\infty; L^2(0,1)). \end{cases}$$

So, there exists a subsequence of (u^m) and (v^m), denoted also by (u^m) and (v^m), where

$$\begin{cases} u^m \rightharpoonup u & \text{weak star in } L^\infty(0,\infty; V), \\ u_t^m \rightharpoonup u_t & \text{weak star in } L^\infty(0,\infty; L^2(0,1)) \cap L^2(0,\infty; V), \\ v^m \rightharpoonup v & \text{weak star in } L^\infty(0,\infty; V), \\ v_t^m \rightharpoonup v_t & \text{weak star in } L^\infty(0,\infty; L^2(0,1)). \end{cases}$$

Consequently,

$$\begin{cases} u^m \rightharpoonup u & \text{weak in } L^2(0,\infty;V), \\ u_t^m \rightharpoonup u_t & \text{weak in } L^2(0,\infty;V), \\ v^m \rightharpoonup v & \text{weak in } L^2(0,\infty;V), \\ v_t^m \rightharpoonup v_t & \text{weak in } L^2(0,\infty;L^2(0,1)). \end{cases}$$

We are now able to take the limit in the approximate problem (6) to obtain a weak solution to the problem (1) (see [19, 27]).

The proof of uniqueness can be performed by combining Visik-Ladyzenskaya method and the arguments used, for example, in [12, 19]. □

4. Decay Result

This section is devoted to presenting and proving our stability result. To accomplish this, we begin by establishing several auxiliary lemmas.

Lemma 4.1. *The functional*

$$I_1(t) = \int_0^1 u_t u dx + \int_0^1 v_t v dx + \frac{a}{2}\|u_x\|^2,$$

under the assumption (H1) and (H3), satisfies

$$I_1'(t) \quad \leq \quad \|u_t\|^2 + \|v_t\|^2 - \|u_x\|^2 - b(1-\bar{\phi}-\frac{\beta_1}{2})\|v_x\|^2 - \int_0^1 G(u,v)dx + \frac{bC_\beta}{2\beta_1}(L\circ v_x),$$

where β_1 is a very small positive number and

$$C_\beta = \int_0^1 \frac{\phi^2(s)}{\beta\phi(s)-\phi'(s)}ds \quad \text{and} \quad L(t) = \beta\phi(t) - \phi'(t).$$

Proof. Direct computation, combined with the use of (1), yields

$$\begin{aligned} I_1'(t) &= \|u_t\|^2 + \|v_t\|^2 - \|u_x\|^2 - b\|v_x\|^2 - \int_0^1 f(u,v)udx - \int_0^1 g(u,v)vdx + b\int_0^1 v_x \int_0^t \phi(t-s)v_x(s)dsdx \\ &\leq \|u_t\|^2 + \|v_t\|^2 - \|u_x\|^2 - b\|v_x\|^2 - \int_0^1 G(u,v)dx + b\int_0^1 v_x \int_0^t \phi(t-s)v_x(s)dsdx. \end{aligned}$$

For the last term, we estimate it as follows for any $\beta_1 > 0$

$$\begin{aligned} \int_0^1 v_x(t)\int_0^t \phi(t-s)v_x(s)dsdx &= \int_0^1 v_x(t)\int_0^t \phi(t-s)(v_x(s)-v_x(t))dsdx + \left(\int_0^t \phi(s)ds\right)\|v_x\|^2 \\ &\leq \frac{1}{2\beta_1}\int_0^1\left(\int_0^t \phi(t-s)|v_x(s)-v_x(t)|ds\right)^2 dx + \frac{\beta_1+2\bar{\phi}}{2}\|v_x\|^2 \end{aligned}$$

and

$$\begin{aligned} &\int_0^1\left(\int_0^t \phi(t-s)|v_x(s)-v_x(t)|ds\right)^2 dx \\ =\; &\int_0^1\left(\int_0^t \frac{\phi(t-s)}{\sqrt{\beta\phi(t-s)-\phi'(t-s)}}\sqrt{\beta\phi(t-s)-\phi'(t-s)}\,|v_x(s)-v_x(t)|ds\right)^2 dx \\ \leq\; &\left(\int_0^t \frac{\phi^2(t-s)}{\beta\phi(t-s)-\phi'(t-s)}ds\right)\int_0^1\int_0^t\left[\beta\phi(t-s)-\phi'(t-s)\right]|v_x(s)-v_x(t)|^2 dsdx \\ \leq\; &C_\beta(L\circ v_x). \end{aligned} \tag{8}$$

Combining the results obtained above leads us to the conclusion of the proof. □

Lemma 4.2. *Assuming (H1) and (H3), the functional*

$$I_2(t) = -\int_0^1 v_t \int_0^t \phi(t-s)(v(t)-v(s))dsdx,$$

satisfies

$$\begin{aligned} I_2'(t) &\leq -\left(\int_0^t \phi(s)\,ds - \frac{\beta_2}{2}\phi(0)\right)\|v_t\|^2 + \frac{r^2\bar{\phi}\beta_3}{2}[2E(0)]^{p-1}\|u_x\|^2 + \left(\frac{r^2\bar{\phi}\beta_3}{2}[\frac{2}{b}E(0)]^{q-1} + \frac{b}{2}\beta_2\right)\|v_x\|^2 \\ &+ \left(\frac{b}{2\beta_2}+b\right)C_\beta L\circ v_x - \frac{1}{2\beta_2}\left(\phi'\circ v_x\right) + \frac{1}{\beta_3}(\phi\circ v_x), \end{aligned} \tag{9}$$

where β_2 is a very small positive number.

Proof. Using (1) into $I_2'(t)$, and applying integration by parts leads to

$$\begin{aligned} I_2'(t) &= -\left(\int_0^t \phi(s)\,ds\right)\|v_t\|^2 - \int_0^1 v_t \int_0^t \phi'(t-s)(v(t)-v(s))\,dsdx + \int_0^1 g(u,v)\int_0^t \phi(t-s)(v(t)-v(s))\,dsdx \\ &+ b\int_0^1\left(v_x - \int_0^t \phi(t-s)v_x(s)\,ds\right)\int_0^t \phi(t-s)(v_x(t)-v_x(s))\,dsdx. \end{aligned}$$

By applying *Young* and *Poincare* inequalities, we find

$$-\int_0^1 v_t \int_0^t \phi'(t-s)(v(t)-v(s))\,dsdx \leq \frac{\beta_2}{2}\phi(0)\|v_t\|^2 - \frac{1}{2\beta_2}\left(\phi'\circ v_x\right),$$

and

$$-\int_0^1 g(u,v)\int_0^t \phi(t-s)(v(t)-v(s))\,dsdx \leq \frac{\beta_3\bar{\phi}}{4}\int_0^1 \left|g(u,v)\right|^2 dx + \frac{1}{\beta_3}(\phi\circ v_x).$$

On the other hand, applying (H3) and *Poincare*'s inequality, we conclude

$$\begin{aligned} \frac{\beta_3\bar{\phi}}{4}\int_0^1 \left|g(u,v)\right|^2 dx &\leq \frac{\beta_3\bar{\phi}r^2}{2}(\int_0^1 (|u|^{2p}+|v|^{2q})dx \\ &\leq \frac{\beta_3\bar{\phi}r^2}{2}\left(\|u_x\|^{2(p-1)}\|u_x\|^2 + \|v_x\|^{2(q-1)}\|v_x\|^2\right) \\ &\leq \frac{\beta_3\bar{\phi}r^2}{2}\left([2E(0)]^{p-1}\|u_x\|^2 + [\frac{2}{b}E(0)]^{q-1}\|v_x\|^2\right). \end{aligned}$$

We now apply *Young*'s inequality and identity (8), we find

$$\begin{aligned} &b\int_0^1\left(v_x - \int_0^t \phi(t-s)v_x(s)\,ds\right)\int_0^t \phi(t-s)(v_x(t)-v_x(s))\,dsdx \\ &= b\left(1-\int_0^t \phi(s)\,ds\right)\int_0^1 v_x\int_0^t \phi(t-s)(v_x(t)-v_x(s))\,dsdx + b\int_0^1\left(\int_0^t \phi(t-s)(v_x(t)-v_x(s))\,ds\right)^2 dx \\ &\leq \frac{b\left(1-\int_0^t \phi(s)\,ds\right)^2}{2}\beta_2\|v_x\|^2 + \left(\frac{b}{2\beta_2}+b\right)\int_0^1\left(\int_0^t \phi(t-s)(v_x(t)-v_x(s))\,ds\right)^2 dx \\ &\leq \frac{b}{2}\beta_2\|v_x\|^2 + \left(\frac{b}{2\beta_2}+b\right)C_\beta\left(L\circ v_x\right). \end{aligned}$$

When all the above estimates are combined, we obtain (9). □

Let us define *Lyapunov* function by

$$M(t) = E(t) + m_1 I_1(t) + m_2 I_2(t),$$

where m_1 and m_2 represent positive constants.

Lemma 4.3. *Assuming (H1), the function*

$$M(t) = E(t) + m_1 I_1(t) + m_2 I_2(t),$$

satisfies

$$M'(t) \quad \leq \quad -kE(t) + k_1 \phi \circ v_x, \tag{10}$$

where k and k_1 are defined as positive constants.

Proof. Let $t_0 > 0$ be fixed and $\phi_0 = \int_0^{t_0} \phi(s) ds > 0$. By combining Lemma 2.2, 4.1, and 4.2, using *Poincare*'s inequality ($\|u_t\|^2 \leq \|u_{xt}\|^2$) and taking $\beta_2 < \dfrac{\phi_0}{\phi(0)}$, $m_1 < \dfrac{a}{2}$, $\beta_1 = 1 - \bar{\phi}$, for $t \geq t_0$, we obtain

$$\begin{aligned} M'(t) \quad \leq \quad & -\frac{a}{2} \|u_t\|^2 - \left[\frac{\phi_0}{2} m_2 - m_1\right] \|v_t\|^2 - \left[m_1 - \frac{r^2 \bar{\phi} \beta_3}{2} [2E(0)]^{p-1} m_2\right] \|u_x\|^2 \\ & - \left[\frac{(1-\bar{\phi})b}{2} m_1 - \left(\frac{r^2 \bar{\phi} \beta_3}{2} [\frac{2}{b} E(0)]^{q-1} + \frac{b}{2} \beta_2\right) m_2\right] \|v_x\|^2 - m_1 \int_0^1 G(u,v) dx \\ & + (\frac{b}{2} - \frac{\phi(0)}{2\phi_0} m_2) \phi' \circ v_x + \frac{1}{\beta_3} m_2 \phi \circ v_x + \left[\frac{b}{2\beta_1} m_1 + \left(\frac{b}{2\beta_2} + b\right) m_2\right] C_\beta L \circ v_x. \end{aligned}$$

If β_2 and β_3 are small enough and $m_2 > \dfrac{2}{\phi_0} m_1$ we find,

$$\left[\frac{b}{2\beta_1} m_1 + \left(\frac{b}{2\beta_2} + b\right) m_2\right] C_\beta \leq \frac{1}{\beta_3} m_2.$$

Then, for some $k > 0$, we obtain (10). □

Theorem 4.4. *Let (H1), (H2) and (H3) be satisfied. Then, positive constants n_1 and n_2 exist such that $E(t)$ satisfies the following inequality:*

$$E(t) \leq n_2 {\Phi_1}^{-1} \left(n_1 \int_{\Phi^{-1}(l)}^{t} \zeta(s) ds\right) \qquad \forall t \geq t_0, \tag{11}$$

where Φ_1 is strictly decreasing and convex on $(0, l]$, and

$$\Phi_1(t) = \int_t^l \frac{1}{s\Phi'(s)} ds.$$

Proof. Using (3) and (5) we find:

$$\int_0^{t_0} \phi(s) \|v_x(t) - v_x(t-s)\|^2 ds \leq -\frac{\phi(0)}{d} \int_0^{t_0} \phi'(s) \|v_x(t) - v_x(t-s)\|^2 ds \leq -\mu E'(t).$$

for all $t \geq t_0$.

Next, we use Lemma 4.3, for some $k, k_1 > 0$ and $t \geq t_0$, to get:

$$\begin{aligned} M'(t) \quad & \leq \quad -kE(t) + k_1 \phi \circ v_x \\ & \leq \quad -kE(t) - \mu E'(t) + k_1 \int_{t_0}^{t} \phi(s) \|v_x(t) - v_x(t-s)\|^2 ds. \end{aligned}$$

By taking $\mathcal{L}(t) = M(t) + \mu E(t)$, we have

$$\mathcal{L}'(t) \leq -kE(t) + k_1 \int_{t_0}^{t} \phi(s) \left\| v_x(t) - v_x(t-s) \right\|^2 ds. \tag{12}$$

According to the properties of Φ, we examine the following two cases:
1) $\Phi(t)$ **is linear:** Using (H2) and (5), we get

$$\begin{aligned} \zeta(t)\mathcal{L}'(t) &\leq -k\zeta(t)E(t) + k_1 \int_{t_0}^{t} \phi(s) \left\| v_x(t) - v_x(t-s) \right\|^2 ds \\ &\leq -k\zeta(t)E(t) + k_1 \int_{t_0}^{t} \zeta(s)\phi(s) \left\| v_x(t) - v_x(t-s) \right\|^2 ds \\ &\leq -k\zeta(t)E(t) - k_2 E'(t). \end{aligned}$$

Since ζ is decreasing, we obtain

$$(\zeta\mathcal{L} + k_2 E)'(t) \leq -k\zeta(t)E(t),$$

and $\zeta\mathcal{L} + k_2 E \sim E$. Therefore, from *Gronwall*'s lemma, we obtain

$$E(t) \leq k_2 e^{-k\int_{t_0}^{t} \zeta(s)ds}.$$

2) $\Phi(t)$ **is nonlinear:** Let $I_v(t) = \frac{\vartheta}{t} \int_{t_0}^{t} \left\| v_x(t) - v_x(t-s) \right\|^2 ds$. Then

$$\begin{aligned} I_v(t) &\leq \frac{\vartheta}{t} \int_0^t \left\| v_x(t) - v_x(t-s) \right\|^2 ds \\ &\leq \frac{2\vartheta}{t} \int_0^t \left\| v_x(t) \right\|^2 + \left\| v_x(t-s) \right\|^2 ds \\ &\leq \frac{4\vartheta}{(1-\bar{\phi})t} \int_0^t \left(E(s) + E(t-s) \right) ds \\ &\leq \frac{8\vartheta}{(1-\bar{\phi})t} \int_0^t E(s)ds \\ &\leq \frac{8\vartheta}{(1-\bar{\phi})t} \int_0^t E(0)ds = \frac{8\vartheta}{(1-\bar{\phi})} E(0) < \infty. \end{aligned}$$

Choosing $0 < \vartheta < 1$, so that

$$I_v(t) < 1. \tag{13}$$

Furthermore, the function $w_v(t)$ is given by

$$w_v(t) = -\int_{t_0}^{t} \phi'(s) \left\| v_x(t) - v_x(t-s) \right\|^2 ds,$$

so we find

$$\omega_v(t) \leq -\mu E'(t).$$

As Φ is strictly convex on $(0,l]$ and $\Phi(0)=0$, then $\Phi(\alpha s)\leq \alpha\Phi(s)$ for $(\alpha,s)\in(0,1)\times(0,l)$. By using *Jensen*'s inequality, (13) and (H2), we arrive at

$$\begin{aligned} w_v(t) &= \frac{1}{\vartheta I_v(t)}\int_{t_0}^{t} I_v(t)\left(-\phi'(s)\vartheta\,\|v_x(t)-v_x(t-s)\|^2\right)ds \\ &\geq \frac{1}{\vartheta I_v(t)}\int_{t_0}^{t} I_v(t)\zeta(s)\Phi(\phi(s))\vartheta\,\|v_x(t)-v_x(t-s)\|^2\,ds \\ &\geq \frac{\zeta(t)}{\vartheta I_v(t)}\int_{t_0}^{t}\Phi(I_v(t)\phi(s)\vartheta\,\|v_x(t)-v_x(t-s)\|^2)ds \\ &\geq \frac{\zeta(t)}{\vartheta}\Phi\left(\vartheta\int_{t_0}^{t}\phi(s)\,\|v_x(t)-v_x(t-s)\|^2\,ds\right) \\ &= \frac{\zeta(t)}{\vartheta}\bar{\Phi}\left(\vartheta\int_{t_0}^{t}\phi(s)\,\|v_x(t)-v_x(t-s)\|^2\,ds\right), \end{aligned}$$

where $\bar{\Phi}$ is an extension of Φ so that $\bar{\Phi}$ is strictly increasing and strictly convex $C^2(0,+\infty)$. Then

$$\int_{t_0}^{t}\phi(s)\,\|v_x(t)-v_x(t-s)\|^2\,ds\leq\frac{1}{\vartheta}\bar{\Phi}^{-1}\left(\frac{\vartheta w_v(t)}{\zeta(t)}\right).$$

Thus, equation (12) becomes

$$\mathcal{L}'(t)\leq -kE(t)+k_3\bar{\Phi}^{-1}\left(\frac{\vartheta w_v(t)}{\zeta(t)}\right),\qquad t\geq t_0. \tag{14}$$

Now, for $\epsilon_0<l$, we define

$$\mathcal{L}_1(t)=\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t)+E(t),$$

The equivalence between $\mathcal{L}_1$ and E can be straightforwardly inferred.
Using (14) and the conditions $E'(t)\leq 0$, $\bar{\Phi}'>0$, and $\bar{\Phi}''>0$, we have

$$\begin{aligned} \mathcal{L}_1'(t) &= \epsilon_0\frac{E'(t)}{E(0)}\bar{\Phi}''\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t)+\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}'(t)+E'(t) \\ &\leq -kE(t)\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)+k_3\bar{\Phi}^{-1}\left(\frac{\vartheta w_v(t)}{\zeta(t)}\right)\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right). \end{aligned}$$

Let $\bar{\Phi}^*$ be defined as the convex conjugate of Φ in the sense of *Young* [4]. Then,

$$\bar{\Phi}^*(s)=s\left(\bar{\Phi}'\right)^{-1}(s)-\bar{\Phi}\left(\left(\bar{\Phi}'\right)^{-1}(s)\right), \tag{15}$$

satisfies *Young*'s inequality, i.e.,

$$XY\leq\bar{\Phi}^*(X)+\bar{\Phi}(Y), \tag{16}$$

with $X=\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)$ and $Y=\bar{\Phi}^{-1}\left(\frac{\vartheta w_v(t)}{\zeta(t)}\right)$.
Using (5) and (15)-(16), we get

$$\begin{aligned} \mathcal{L}_1'(t) &\leq -kE(t)\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)+k_3\bar{\Phi}^*\left(\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right)+k_3\bar{\Phi}\left(\bar{\Phi}^{-1}\left(\frac{\vartheta w_v(t)}{\zeta(t)}\right)\right) \\ &\leq -kE(t)\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)+k_3\epsilon_0\frac{E(t)}{E(0)}\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)-\bar{\Phi}'\left(\epsilon_0\frac{E(t)}{E(0)}\right)+k_3\vartheta\frac{w_v(t)}{\zeta(t)}. \end{aligned} \tag{17}$$

Multiplying (17) by $\zeta(t)$ and using $\epsilon_0 \frac{E(t)}{E(0)} < l$, $\bar{\Phi}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = \Phi'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$ and $w_v(t) \leq -\mu E'(t)$ we obtain:

$$\zeta(t)\mathcal{L}_1'(t) \leq -kE(t)\zeta(t)\bar{\Phi}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + k_4\epsilon_0 \frac{E(t)}{E(0)}\zeta(t)\bar{\Phi}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - k_4E'(t).$$

Letting $\mathcal{L}_2 = \zeta\mathcal{L}_1 + k_4E$, and for some $\lambda_1, \lambda_2 > 0$ satisfies

$$\lambda_1\mathcal{L}_2(t) \leq E(t) \leq \lambda_2\mathcal{L}_2(t). \tag{18}$$

For $n_1 > 0$ and $t \geq t_0$, we obtain

$$\mathcal{L}_2' \leq -n_1\zeta(t)\frac{E(t)}{E(0)}\Phi'\left(\frac{\epsilon_0 E(t)}{E(0)}\right) = -n_1\zeta(t)\Phi_2\left(\frac{E(t)}{E(0)}\right), \tag{19}$$

where $\Phi_2(t) = t\Phi'(\epsilon_0 t)$. Since $\Phi_2'(t) = \Phi'(\epsilon_0 t) + \epsilon_0 t\Phi''(\epsilon_0 t)$, and Φ is strictly convex on $(0, l]$, we observe that $\Phi_2'(t), \Phi_2(t) > 0$ on $(0, l]$. Hence, with

$$Z(t) = \lambda_1 \frac{\mathcal{L}_2(t)}{E_0},$$

by (18)–(19), we can conclude

$$Z(t) \sim E(t). \tag{20}$$

A positive constant n_2 exists, so that

$$Z'(t) \leq -n_2\zeta(t)\Phi_2(Z(t)) \quad \text{for all } \ t > t_0.$$

Then, integrating over (t_0, t) yields

$$\int_{t_0}^{t} -\frac{Z'(s)}{\Phi_2(Z(s))}ds \geq n_2 \int_{t_0}^{t} \zeta(s)ds \quad \Rightarrow \quad \frac{1}{\epsilon_0}\int_{\epsilon_0 Z(t)}^{\epsilon_0 Z(t_0)} \frac{1}{s\Phi'(s)}ds \geq n_2 \int_{t_0}^{t} \zeta(s)ds$$
$$\Rightarrow \quad Z(t) \leq \frac{1}{\epsilon_0}\Phi^{-1}\left(n_1 \int_{t_0}^{t} \zeta(s)ds\right), \tag{21}$$

where $\Phi_1(t) = \int_t^l \frac{1}{s\Phi'(s)}ds$, which is strictly decreasing on $(0, l]$ and $\lim_{t\to 0} \Phi_1(t) = +\infty$.
Finally, combining (21) with (20), we get (11). □

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References

[1] M. M. Al-Gharabli and M. M. Kafini, *A general decay result of a coupled system of nonlinear wave equations*, Rend. Circ. Mat. Palermo (2), **67** (2018), 145–157.

[2] A. M. Al-Mahdi, M. M. Al-Gharabli, M. Noor and J. D. Audu, *Stability results for a weakly dissipative viscoelastic equation with variable-exponent nonlinearity: Theory and numerics*, Math. Comput. Appl., **28** (2023), 5.

[3] A. M. Al-Mahdi, M. M. Al-Gharabli and N. E. Tatar, *On a nonlinear system of plate equations with variable exponent nonlinearity and logarithmic source terms: existence and stability results*, AIMS Math., **8** (2023), 19971–19992.

[4] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Springer, New York, 2013.

[5] N. Boumaza and B. Gheraibia, *General decay and blowup of solutions for a degenerate viscoelastic equation of Kirchhoff type with source term*, J. Math. Anal. Appl., **489** (2020), 124185.

[6] A. Choucha, S. Boulaaras and D. Ouchenane, *Exponential decay and global existence of solutions of a singular nonlocal viscoelastic system with distributed delay and damping terms*, Filomat, **35** (2021), 795–826.
[7] M. Douib, S. Zitouni and A. Djebabla, *Well-posedness and energy decay for full von Kármán system of dynamic thermoplasticity damped with distributed delay and past history terms*, Filomat, **37** (2023), 4195–4210.
[8] B. Feng and X. G. Yang, *Long-time dynamics for a nonlinear Timoshenko system with delay*, Appl. Anal., **96** (2017), 606–625.
[9] B. Feng, *On a semilinear Timoshenko-Coleman-Gurtin system: quasi-stability and attractors*, Discrete Contin. Dyn. Syst., **37** (2017), 4729–4751.
[10] B. Feng and T. A. Apalara, *Optimal decay for a porous elasticity system with memory*, J. Math. Anal. Appl., **470** (2019), 1108–1128.
[11] B. Gheraibia and N. Boumaza, *Initial boundary value problem for a viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term: decay estimates and blow-up result*, Bound. Value Probl., **2023**, 93.
[12] Z. Hajjej, *Existence and general decay of solutions for a weakly coupled system of viscoelastic Kirchhoff plate and wave equations*, Symmetry, **15** (2023), 1917.
[13] H. Kamache, N. Boumaza and B. Gheraibia, *Global existence, asymptotic behavior and blow-up of solutions for a Kirchhoff-type equation with nonlinear boundary delay and source terms*, Turk. J. Math., **47** (2023), 1350–1361.
[14] B. Lekdim and A. Khemmoudj, *General decay of energy to a nonlinear viscoelastic two-dimensional beam*, Appl. Math. Mech., **39** (2018), 1661–1678.
[15] B. Lekdim and A. Khemmoudj, *Uniform decay of a viscoelastic nonlinear beam in two-dimensional space*, Asian J. Math. Comput. Res., **25** (2018), 50–73.
[16] B. Lekdim and A. Khemmoudj, *Existence and energy decay of solution to a nonlinear viscoelastic two-dimensional beam with a delay*, Multidimens. Syst. Signal Process., **32** (2021), 915–931.
[17] B. Lekdim and A. Khemmoudj, *General stability of two-dimensional viscoelastic nonlinear beam with bending couplings*, Proc. 2021 Int. Conf. Recent Adv. Math. Inform. (ICRAMI), IEEE, 2021, 1–4.
[18] B. Lekdim and A. Khemmoudj, *Existence and general decay of solution for nonlinear viscoelastic two-dimensional beam with a nonlinear delay*, Ric. Mat., **73** (2024), 261–282.
[19] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, 1969.
[20] S. A. Messaoudi, *General decay of the solution energy in a viscoelastic equation with a nonlinear source*, Nonlinear Anal. Theory Methods Appl., **69** (2008), 2589–2598.
[21] S. A. Messaoudi and N.-E. Tatar, *Uniform stabilization of solutions of a nonlinear system of viscoelastic equations*, Appl. Anal., **87** (2008), 247–263.
[22] S. A. Messaoudi, S. E. Mukiawa and M. M. Al-Gharabli, *General decay estimate for a weakly dissipative viscoelastic suspension bridge*, Recent Dev. Math., Springer, 2022, 15–26.
[23] M. I. Mustafa, *Well-posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations*, Nonlinear Anal. Real World Appl., **13** (2012), 452–463.
[24] M. I. Mustafa, *Optimal decay rates for the viscoelastic wave equation*, Math. Methods Appl. Sci., **41** (2018), 192–204.
[25] N. Taouaf and B. Lekdim, *Global existence and exponential decay for thermoelastic system with nonlinear distributed delay*, Filomat, **37** (2023), 8897–8908.
[26] H. Yüksekkaya, E. Piskin, M. M. Kafini and A. M. Al-Mahdi, *Well-posedness and exponential stability for the logarithmic Lamé system with a time delay*, Appl. Anal., **103** (2024), 506–518.
[27] S. Zheng, *Nonlinear evolution equations*, Chapman and Hall/CRC, New York, 2004.
[28] S. Zitouni, K. Zennir and L. Bouzettouta, *Uniform decay for a viscoelastic wave equation with density and time-varying delay in* R^n, Filomat, **33** (2019), 961–970.