Filomat 39:12 (2025), 3963–3976
https://doi.org/10.2298/FIL2512963W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some characterizations of p-adic mixed central Campanato spaces via commutator of p-adic Hardy type operator

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Abstract. In this paper, we mainly prove the boundedness of commutators generated by Hardy type operator and p-adic mixed central campanato functions in p-adic mixed central Morrey spaces, and give some characterizations of p-adic central mixed campanato spaces.

1. Introduction and main results

Let $x = p^{\gamma}\frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, p is any prime number, a and b are integers coprime with p and a is integers coprime with b,then the p-adic norm is defined by

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is interesting to note that the above p-adic norm satisfy $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, where $|x|_p \neq |y|_p \Rightarrow |x+y|_p = \max\{|x|_p, |y|_p\}$, for precise information regarding the ultrametric inequality, we can see book [1] for example.

We denote by $\mathbb{Q}_p^n$ the n dimensional p-adic vector space. The magnitude of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{Q}_p^n$ ($x_i \in \mathbb{Q}_p$ ($i = 1, ..., n$)) is $|x|_p = \max_{1\leq j\leq n} |x_j|_p$. A p-adic ball centered at $a \in \mathbb{Q}_p^n$ of radius p^{γ} ($\gamma \in \mathbb{Z}$) is denoted by $B_{\gamma}(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \leq p^{\gamma}\}$. A corresponding p-adic sphere is shown as $S_{\gamma}(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^{\gamma}\} = B_{\gamma}(a) \setminus B_{\gamma-1}(a)$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists unique Haar measure dx on $\mathbb{Q}_p^n$. The measure dx by $\int_{B_0(0)} dx = |B_0(0)|_h = 1$, where $|B_0(0)|_h$ is denoted by the Haar measure of p-adic unit ball. By simple analysis, we have $\int_{B_{\gamma}(a)} dx = |B_{\gamma}(a)|_h = p^{n\gamma}$ and $\int_{S_{\gamma}(a)} dx = |S_{\gamma}(a)|_h$.

In recent years, the intersection of harmonic analysis and number theory has attracted the attention of many people, not only due to the generalization of classical operators and space theories [2–4], but also

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 11EE95, 42B35.
Keywords. p-adic field; Hardy type operator; mixed central campanato spaces; mixed central Morrey spaces.
Received: 07 October 2024; Accepted: 03 February 2025
Communicated by Snežana Č. Živković-Zlatanović
Research supported by the Fundamental Research Funds for Education Department of Heilongjiang Province (No.1454YB020, No.SJGYB2024673)
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because of its wide applications in many fields, such as mathematical physics and p-adic wavelet theory [5–7], which continue to promote the development of this field.

Definition 1.1. *(see[8]) Let $0 < \alpha < n$ and $f \in L^1_{loc}\left(\mathbb{Q}_p^n\right)$, $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$.*

$$\mathcal{H}_\alpha^p f(x) = \frac{1}{|x|_p^{n-\alpha}} \int_{B(0,|x|_p)} f(t)dt, \mathcal{H}_\alpha^{p,*} f(x) = \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^{n-\alpha}} dt,\ x \in \mathbb{Q}_p^n \setminus \{0\}.$$

Remark 1.1. *(see [9]) If $\alpha = 0$, then $\mathcal{H}_\alpha^p = \mathcal{H}^p$ and $\mathcal{H}_\alpha^{p,*} = \mathcal{H}^{p,*}$, that is*

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} f(t)dt, \mathcal{H}^{p,*} f(x) = \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^n} dt,\ x \in \mathbb{Q}_p^n \setminus \{0\}.$$

Definition 1.2. *Let $b : \mathbb{Q}_p^n \to \mathbb{R}$ be a locally integrable function, the commutators of fractional p-adic Hardy operator can be defined by*

$$\mathcal{H}_{\alpha,b}^p f(x) = \frac{1}{|x|_p^{n-\alpha}} \int_{B(0,|x|_p)} (b(x) - b(t)) f(t)dt,$$
$$\mathcal{H}_{\alpha,b}^{p,*} f(x) = \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^{n-\alpha}} (b(x) - b(t))dt, x \in \mathbb{Q}_p^n \setminus \{0\}.$$

Remark 1.2. *If $\alpha = 0$, then $\mathcal{H}_{\alpha,b}^p = \mathcal{H}_b^p$ and $\mathcal{H}_{\alpha,b}^{p,*} = \mathcal{H}_b^{p,*}$, that is*

$$\mathcal{H}_b^p f(x) = \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} (b(x) - b(t)) f(t)dt,$$
$$\mathcal{H}_b^{p,*} f(x) = \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^n} (b(x) - b(t))dt,\ x \in \mathbb{Q}_p^n \setminus \{0\}.$$

In 2015, Shi and Lu [10] gave some characterizations of the central Campanato spaces. Next, the similar results of [10] has been extended to number theory in [11]. Lu and Zhou [12] established some characterizations of mixed central Campanato spaces, via the boundedness of the commutator of Hardy type. Recently, some p-adic mixed central function spaces were first introduced in [13], which provides a basis for further study on these spaces. A natural idea is whether we can consider the generalization of the results of [12] to number theory?

In the section 3, we prove the boundedness of commutators generated by $\mathcal{H}^p$, $\mathcal{H}^{p,*}$, $\mathcal{H}_\alpha^p$, $\mathcal{H}_\alpha^{p,*}$ and p-adic central mixed Campanato functions in p-adic mixed central Morrey spaces, and give some characterizations of p-adic central mixed Campanaro spaces.

It is commonly known that function b adheres to the well-established mean value inequality if there exists a positive constant $C > 0$ such that for any ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ with $\gamma \in \mathbb{Z}$,

$$\sup_{B_\gamma(x) \ni y} |b(y) - b_{B_\gamma(x)}| \leq \frac{C}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}| dy, \tag{1.1}$$

where $b_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y)| dy$, a function class that satisfies (1.1) is called a reverse Hölder class. See [11, 14, 15] for more details on the reverse Hölder class and some examples.

Throughout this paper, the letter C always takes place of a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|_h$ will always denote the Haar measure of a measurable set E on $\mathbb{Q}_p^n$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$. Additionally, we represent $\vec{q_i'} = (q'_{i1}, \cdots, q'_{in}) = (\frac{q_{i1}}{q_{i1}-1}, \cdots \frac{q_{in}}{q_{in}-1})$, and $0 < \vec{q_i'} < \infty$ that means $0 < q'_{ij} < \infty$ for all j.When $0 \leq \vec{q_i'} \leq \infty$,we represent $\vec{q'} = (q'_1, \cdots, q'_n)$ such that $\frac{1}{q_j} + \frac{1}{q'_j} = 1$.

Theorem 1.1. *Let* $1 < \vec{q}, \vec{q_2} < \infty, -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i} < \lambda < 0, -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{ji}} < \lambda_j < 0, j = 1,2, 1/\vec{q} = 1/\vec{q_1} + 1/\vec{q_2}, \lambda = \lambda_1 + \lambda_2$ *and suppose b satisfies (1.1). Then the following statements are equivalent:*

(a) $b \in \mathfrak{C}^{\vec{q_1},\lambda_1}\left(\mathbb{Q}_p^n\right)$;

(b) Both $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q_2},\lambda_2}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$.

Remark 1.3. *(1) If* $\vec{q} = q$*, then the above results can be found in [11].*

(2) In the Euclidean space, we can see [12].

Under some stronger conditions on λ and $\vec{p}$, the following result can be obtained if remove the assumption that b satisfies the condition (1.1).

Theorem 1.2. *Let* $1 < \vec{q} < \infty,\ 1/\vec{q} + 1/\vec{q'} = 1, -\min\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i}, \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i'}\right\} < \lambda < 0.$ *Then the following statements are equivalent:*

(a) $b \in \mathfrak{C}^{\max\{\vec{q},\vec{q'}\},\lambda}\left(\mathbb{Q}_p^n\right)$;

(b) Both $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q},2\lambda}\left(\mathbb{Q}_p^n\right)$.

In addition, both $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q'},\lambda}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q'},2\lambda}\left(\mathbb{Q}_p^n\right)$.

Remark 1.4. *(1) If* $\vec{q} = q$*, then the above result can be found in [11].*

(2) In the Euclidean space, we can see [12].

Next, we give some characterizations of $\mathfrak{C}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$ via the boundedness of $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ on p-adic mixed central Morrey spaces.

Theorem 1.3. *Let* $\vec{q}, \lambda, \vec{q_j}, \lambda_j, j = 1,2, b$ *be as in Theorem 1.1,* $0 < \alpha < \min\left\{n(1 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}), n(\lambda_2 + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{2i}})\right\}$*, and* $\beta = \lambda_2 - \alpha/n$*. Then the following statements are equivalent:*

(a) $b \in \mathfrak{C}^{\vec{q_1},\lambda_1}\left(\mathbb{Q}_p^n\right)$;

(b) Both $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q_2},\beta}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$.

Remark 1.5. *(1) If* $\vec{q} = q$*, then the above result can be found in [11].*

(2) In the Euclidean space, we can see [12].

Theorem 1.4. *Let* $1 < \vec{q} < \infty,\ 1/\vec{q} + 1/\vec{q'} = 1, 0 < \alpha < \min\left\{n(1 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}), n(\lambda + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i})\right\}$,
$-\min\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i}, \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i'}\right\} < \lambda < 0$*, and* $\beta = \lambda - \alpha/n$*. Then the following statements are equivalent:*

(a) $b \in \mathfrak{C}^{\max\{\vec{q},\vec{q'}\},\lambda}\left(\mathbb{Q}_p^n\right)$;

(b) Both $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q},\beta}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q},2\lambda}\left(\mathbb{Q}_p^n\right)$.

In addition, both $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ *are bounded operators from* $\mathcal{B}^{\vec{q'},\beta}\left(\mathbb{Q}_p^n\right)$ *to* $\mathcal{B}^{\vec{q'},2\lambda}\left(\mathbb{Q}_p^n\right)$.

Remark 1.6. *(1) If* $\vec{q} = q$*, then the above result can be found in [11].*

(2) In the Euclidean space, we can see [12].

2. *p*-adic function spaces

Recently, Sarfraz, Aslam and Malik [13] introduced the following definitions 2.1,2.2.

Definition 2.1. *Let* $0 < \vec{r} < \infty$ $(\vec{r} = (r_1, r_2 \cdots r_n))$, *given a measurable function* f defined on $\mathbb{Q}_p^n$, *then p-adic mixed Lebesgue spaces is written as*

$$L^{\vec{r}}\left(\mathbb{Q}_p^n\right) = \left\{f \in \mathcal{M}\left(\mathbb{Q}_p^n\right) : \|f\|_{L^{\vec{r}}(\mathbb{Q}_p^n)} < \infty\right\},$$

where

$$\|f\|_{L^{\vec{r}}(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p} \cdots \left(\int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p} \left|f(y_1, y_2, \cdots, y_n)\right|^{r_1} dy_1\right)^{\frac{r_2}{r_1}} dy_2\right)^{\frac{r_3}{r_2}} \cdots dy_n\right)^{\frac{1}{r_n}}.$$

Remark 2.1. *(1) If* $r_j = \infty$ *for all j, then we have to make some suitable modifications.*
(2) If $r_j = r$ *for all j, then* $L^{\vec{r}}\left(\mathbb{Q}_p^n\right) = L^r\left(\mathbb{Q}_p^n\right)$*(p-adic Lebesgue spaces).*
(3) The class $\mathcal{M}\left(\mathbb{Q}_p^n\right)$ *consists of all Lebesgue measurable functions.*

Definition 2.2. ***(p-adic mixed central Morrey spaces)*** *Let* $1 < \vec{q} < \infty$ *and* $\lambda \in \mathbb{R}$. *Define the p-adic mixed central Morrey space* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$ *as follows*

$$\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right) = \{f \in \mathcal{M}(\mathbb{Q}_p^n) : \|f\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \frac{\|f\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^\lambda \|\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}.$$

Remark 2.2. *(1) If* $\lambda = -\frac{1}{r}$, *then* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right) = \mathcal{B}^{\vec{q},-\frac{1}{r}}\left(\mathbb{Q}_p^n\right)$ *(p-adic mixed Morrey spaces).*
(2) If $\vec{q} = q$. *then* $\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right) = \mathcal{B}^{q,\lambda}\left(\mathbb{Q}_p^n\right)$ *(p-adic central Morrey spaces, see [11]).*

We introduce the p-adic mixed central Campanato space for the purpose of studying the Theorems 1.1-1.4.

Definition 2.3. ***(p-adic mixed central Campanato spaces)*** *Let* $1 \le \vec{q} < \infty$, $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i} < \lambda < \frac{1}{n}$, *then p-adic mixed central campanato spaces is defined by*

$$\mathfrak{C}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right) = \{f \in \mathcal{M}\left(\mathbb{Q}_p^n\right) : \|f\|_{\mathfrak{C}^{\vec{q},\lambda}} < \infty\},$$

where

$$\|f\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \frac{\|(f - f_{B_\gamma})\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^\lambda \|\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}.$$

Remark 2.3. ***(see [16])*** *If* $\vec{q} = q$, *then* $\mathfrak{C}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right) = \mathfrak{C}^{q,\lambda}\left(\mathbb{Q}_p^n\right)$ *(p-adic central Campanato spaces).*

Lemma 2.1. ***(Hölder's inequality [13])*** *Let* Q_p^n *be an n-dimensional p-adic vector space.Suppose that* $1 \le \vec{q} \le \infty$ *with* $\frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$,*and measurable functions* $f \in L^{\vec{q}}\left(\mathbb{Q}_p^n\right)$ *and* $g \in L^{\vec{q}'}\left(\mathbb{Q}_p^n\right)$. *Then there exists a positive constant C such that*

$$\int_{Q_p^n} |f(x)g(x)|dx \le C\|f\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|g\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}.$$

Lemma 2.2. *Let* $1 < \vec{q} < \infty, -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i} < \lambda < 0$ *and* $i, k \in \mathbb{Z}$. *If* $b \in \mathfrak{C}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$, *then*

$$|b(y) - b_{B_k}| \leq |b(y) - b_{B_j}| + C\max\{|B_k|_h^\lambda, |B_j|_h^\lambda\}\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.$$

Proof. Using Hölder's inequality to $\vec{q}$ and $\vec{q'}$, we get

$$\begin{aligned}
|b_{B_i} - b_{B_{i+1}}| &\leq \frac{1}{|B_i|_h}\int_{B_{i+1}}|b(y) - b_{B_{i+1}}|dy \\
&\leq \frac{1}{|B_i|_h}\|\chi_{B_{i+1}}(b - b_{B_{i+1}})\|_{L^1(\mathbb{Q}_p^n)} \\
&\leq \frac{1}{|B_i|_h}\|\chi_{B_{i+1}}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\|(b - b_{B_{i+1}})\chi_{B_{i+1}}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq \frac{|B_{i+1}|_h^\lambda}{|B_i|_h}\|\chi_{B_{i+1}}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\|\chi_{B_{i+1}}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} \\
&\leq C|B_{i+1}|_h^\lambda\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}$$

If $k < j$, then

$$\begin{aligned}
|b(y) - b_{B_k}| &\leq |b(y) - b_{B_j}| + \sum_{i=k}^{j-1}|b_{B_i} - b_{B_{i+1}}| \\
&\leq |b(y) - b_{B_j}| + C\sum_{i=k}^{j-1}|B_{i+1}|_h^\lambda\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} \\
&\leq |b(y) - b_{B_j}| + C|B_k|_h^\lambda\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}$$

If $k > j$, then

$$\begin{aligned}
|b(y) - b_{B_k}| &\leq |b(y) - b_{B_j}| + \sum_{i=j}^{k-1}|b_{B_i} - b_{B_{i+1}}| \\
&\leq |b(y) - b_{B_j}| + C\sum_{i=j}^{k-1}|B_{i+1}|_h^\lambda\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)} \\
&\leq |b(y) - b_{B_j}| + C|B_j|_h^\lambda\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Thus the proof of lemma 2.2 is finished. □

3. Proofs of the principal results

Proof. [Proof of Theorem 1.1] The proof can be divided into two steps.

(a) ⇒ (b) Given a fixed ball $B_\gamma \subset \mathbb{Q}_p^n$, the task is now to show that there exists a positive constant C such that

$$\left\|(\mathcal{H}_b^p f)\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \leq C|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}. \tag{3.1}$$

$$\left\|(\mathcal{H}_b^{p,*} f)\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \leq C|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}. \tag{3.2}$$

The definition of $\mathcal{H}_b^p$ and Minkowski's inequality give that

$$\begin{aligned}
\left\|(\mathcal{H}_b^p f)\chi_{B_\gamma}(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} &= \left\|\chi_{B_\gamma}(\cdot)\frac{1}{|\cdot|_p^n}\int_{B(0,|x|_p)}(b(\cdot)-b(y))f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le \sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\int_{B_k}(b(\cdot)-b(y))f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le C\sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\sum_{i=-\infty}^{k}\int_{S_i}(b(\cdot)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&+C\sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\sum_{i=-\infty}^{k}\int_{S_i}(b(y)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&= I+II.
\end{aligned}$$

Firstly, we estimate I . For $\frac{1}{\vec{q}}=\frac{1}{\vec{q_1}}+\frac{1}{\vec{q_2}}$ and $1=\frac{1}{\vec{q_2}}+\frac{1}{\vec{q_2'}}$, by Hölder's inequality, we get

$$\begin{aligned}
I &\le C\sum_{k=-\infty}^{\gamma}p^{-kn}\left\|\chi_{S_k}(b-b_{B_k})\sum_{i=-\infty}^{k}\int_{S_i}f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le \sum_{k=-\infty}^{\gamma}p^{-kn}\left\|\chi_{S_k}(b-b_{B_k})\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\left\|\chi_{S_k}\int_{S_i}f(y)dy\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\\
&\le \sum_{k=-\infty}^{\gamma}p^{-kn}\left\|\chi_{B_k}(b-b_{B_k})\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\left\|\chi_{B_k}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\left\|f\chi_{B_i}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\left\|\chi_{B_i}\right\|_{L^{\vec{q_2'}}(\mathbb{Q}_p^n)}\\
&\le \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{-kn}|B_k|_h^{\lambda_1}\left\|\chi_{B_k}\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}\\
&\times\sum_{i=-\infty}^{k}|B_i|_h^{\lambda_2}\left\|\chi_{B_k}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\left\|\chi_{B_i}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\left\|\chi_{B_i}\right\|_{L^{\vec{q_2'}}(\mathbb{Q}_p^n)}\\
&\le C\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{-kn}|B_k|_h^{\lambda_1}\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{\lambda_2+1}\\
&\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)},
\end{aligned}$$

where

$$\left\|\chi_{B_k}\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}\left\|\chi_{B_k}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\approx|B_k|_h^{\frac{1}{n}\sum_{i=1}^n\frac{1}{q_{1i}}+\frac{1}{n}\sum_{i=1}^n\frac{1}{q_{2i}}}\approx\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)},$$

and we have used the fact $-\frac{1}{n}\sum_{i=1}^n\frac{1}{q_i}<\lambda<0, -\frac{1}{n}\sum_{i=1}^n\frac{1}{q_{2i}}<\lambda_2<0$ and $\lambda=\lambda_1+\lambda_2$.

Next, by Hölder's inequality, the fact $\frac{1}{\vec{q}}=\frac{1}{\vec{q_1}}+\frac{1}{\vec{q_2}}$ and $1=\frac{1}{\vec{q_2}}+\frac{1}{\vec{q_2'}}$ allow us to estimate the term II as

$$II\le C\sum_{k=-\infty}^{\gamma}p^{-kn}\left\|\chi_{S_k}\sum_{i=-\infty}^{k}\int_{S_i}(b(y)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}$$

$$
\begin{aligned}
&\le C\sum_{k=-\infty}^{\gamma} p^{-kn}\left\|\chi_{S_k}\sum_{i=-\infty}^{k}\|(b-b_{B_k})f\chi_{B_i}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|\chi_{B_i}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le C\sum_{k=-\infty}^{\gamma} p^{-kn}\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\|(b-b_{B_k})\chi_{B_i}\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}\left\|f\chi_{B_i}\right\|_{L^{\vec{q}_2}(\mathbb{Q}_p^n)}\|\chi_{B_i}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\\
&\le C\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma} p^{-kn}\,|B_k|_h^{\lambda_1}\left\|\chi_{B_k}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\quad\times\sum_{i=-\infty}^{k}|B_i|_h^{\lambda_2}\left\|\chi_{B_i}\right\|_{L^{\vec{q}_2}(\mathbb{Q}_p^n)}\left\|\chi_{B_i}\right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\\
&\le C\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma} p^{kn(\lambda_1+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{1i}}-1)}\\
&\quad\times\sum_{i=-\infty}^{k} p^{in(\lambda_2+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i'}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{2i}})}\\
&\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)},
\end{aligned}
$$

and we have used the fact that $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{2i}} < \lambda_2 < 0,\ \lambda = \lambda_1+\lambda_2$, and $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i} < \lambda < 0$.

By virtue of the estimates of I and II, we can obtain (3.1).

To prove (3.2), note that

$$
\begin{aligned}
\left\|(\mathcal{H}_b^{p,*}f)\chi_{B_\gamma}(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} &= \left\|\chi_{B_\gamma}(\cdot)\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{(b(\cdot)-b(y))\,f(y)}{|y|_p^n}dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le \left\|\chi_{B_\gamma}(\cdot)\int_{p^{\gamma n}\ge|y|_p\ge|\cdot|_p}\frac{(b(\cdot)-b(y))\,f(y)}{|y|_p^n}dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\quad+\left\|\chi_{B_\gamma}(\cdot)\int_{|y|_p>p^{\gamma n}}\frac{(b(\cdot)-b(y))\,f(y)}{|y|_p^n}dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&= I'+II'.
\end{aligned}
$$

The term I' can be handled in a similar method as that of (3.1), the only difference being in the analysis of the term II'. Analysis similar to that of $\mathcal{H}_b^p$ shows

$$
\begin{aligned}
I' &\le \left\|\chi_{B_\gamma}(\cdot)\frac{1}{|\cdot|_p^n}\int_{p^{\gamma n}\ge|y|_p}(b(\cdot)-b(y))\,f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le \sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\sum_{i=-\infty}^{k}\int_{S_i}(b(\cdot)-b(y))\,f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le \sum_{k=-\infty}^{\gamma}p^{-kn}\left\|\chi_{B_k}(\cdot)\sum_{i=-\infty}^{k}\int_{B_i}(b(\cdot)-b(y))\,f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\
&\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q}_1,\lambda_1}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

For the term II', applying Minkowski's inequality, we obtain

$$\begin{aligned} II' &\le \left\| \chi_{B_\gamma}(\cdot) \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{\left(b(\cdot) - b_{B_\gamma}\right) f(y)}{|y|_p^n} dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\ &\quad + \left\| \chi_{B_\gamma}(\cdot) \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{\left(b_{B_\gamma} - b(y)\right) f(y)}{|y|_p^n} dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} = II'_1 + II'_2. \end{aligned}$$

For $\frac{1}{\vec{q}} = \frac{1}{\vec{q_1}} + \frac{1}{\vec{q_2}}$ and $1 = \frac{1}{\vec{q_2}} + \frac{1}{\vec{q_2'}}$, by Hölder's inequality, the fact $-\frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_{2i}} < \lambda_2 < 0$ and $\lambda = \lambda_1 + \lambda_2$, deduce that

$$\begin{aligned} II'_1 &= \left\| \chi_{B_\gamma}(\cdot)(b(\cdot) - b_{B_\gamma}) \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{|f(y)|}{|y|_p^n} dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\ &\le \left\| \chi_{B_\gamma}(\cdot)\left(b(\cdot) - b_{B_\gamma}\right) \right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)} \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} \left\| \chi_{S_k}(\cdot) \frac{|f(\cdot)|}{|\cdot|_p^n} \right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)} \left\| \chi_{S_k} \right\|_{L^{\vec{q_2'}}(\mathbb{Q}_p^n)} \\ &\le C|B_\gamma|_h^{\lambda_1} \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} p^{-kn} \left\| f\chi_{B_k} \right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)} \left\| \chi_{B_k} \right\|_{L^{\vec{q_2'}}(\mathbb{Q}_p^n)} \\ &\le C|B_\gamma|_h^{\lambda_1} \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\| f \right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} p^{-kn} |B_k|_h^{\lambda_2+1} \\ &\le C|B_\gamma|_h^{\lambda_1} \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\| f \right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)}. \end{aligned}$$

To estimate II'_2, we need the following decomposition

$$\begin{aligned} II'_2 &\le \left\| \chi_{B_\gamma}(\cdot) \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{\left(b(y) - b_{B_k}\right) f(y)}{|y|_p^n} dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\ &\quad + \left\| \chi_{B_\gamma}(\cdot) \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{(b_{B_\gamma} - b_{B_k}) f(y)}{|y|_p^n} dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} = II'_{21} + II'_{22}. \end{aligned}$$

We first consider II'_{21}. For $\frac{1}{\vec{q}} = \frac{1}{\vec{q_1}} + \frac{1}{\vec{q_2}}$ and $1 = \frac{1}{\vec{q_2}} + \frac{1}{\vec{q_2'}}$, using Hölder's inequality, we deduce

$$\begin{aligned} II'_{21} &\le C \left\| \chi_{B_\gamma}(\cdot) \sum_{k=\gamma}^{\infty} p^{-kn} \int_{B_k} \left(b(y) - b_{B_k}\right) f(y) dy \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\ &\le C \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} p^{-kn} \left\| (b - b_{B_k}) f\chi_{B_k} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \left\| \chi_{B_k} \right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)} \\ &\le C \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} p^{-kn} \left\| (b - b_{B_k}) \chi_{B_k} \right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)} \left\| f\chi_{B_k} \right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)} \left\| \chi_{B_k} \right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)} \\ &\le C \left\| \chi_{B_\gamma} \right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\| f \right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty} p^{-kn} |B_k|_h^{\lambda+1} \end{aligned}$$

$$\leq C|B_\gamma|_h^{\lambda} \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)},$$

where we have used the conditions $\lambda = \lambda_1 + \lambda_2$ and $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i} < \lambda < 0$.

For the term II'_{22}, we claim first that for $k > \gamma$,

$$|b_{B_\gamma} - b_{B_k}| \leq Cp^{(\gamma+1)n\lambda_1}\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)}.$$

Indeed,

$$\begin{aligned}
|b_{B_\gamma} - b_{B_k}| &\leq \sum_{j=\gamma}^{k-1} |b_{B_j} - b_{B_{j+1}}| \leq \sum_{j=\gamma}^{k-1} \frac{1}{|B_j|_h}\int_{B_{j+1}} |b(y) - b_{B_{j+1}}|dy \\
&\leq \sum_{j=\gamma}^{k-1} \frac{1}{|B_j|_h}\left\|\left(b(y) - b_{B_{j+1}}\right)\chi_{B_{j+1}}\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}\left\|\chi_{B_{j+1}}\right\|_{L^{\vec{q}_1'}(\mathbb{Q}_p^n)} \\
&\leq C\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \sum_{j=\gamma}^{k-1} \frac{|B_{j+1}|_h^{\lambda_1+1}}{|B_j|_h} \\
&\leq C\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \sum_{j=\gamma}^{k-1} p^{(j+1)n\lambda_1} \leq Cp^{(\gamma+1)n\lambda_1}\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore, for $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_{2i}} < \lambda_2 < 0$ and $\lambda = \lambda_1 + \lambda_2$, using Hölder's inequality again, we get

$$\begin{aligned}
II'_{22} &\leq C\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|\chi_{B_\gamma}(\cdot)\sum_{k=\gamma}^{\infty}\int_{S_k}\frac{p^{\gamma n\lambda_1}f(y)}{|y|_p^n}dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq C\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|\chi_{B_\gamma}(\cdot)p^{\gamma n\lambda_1}\sum_{k=\gamma}^{\infty}p^{-kn}\int_{B_k}|f(y)|dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq C|B_\gamma|_h^{\lambda_1}\|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty}p^{-kn}\left\|f\chi_{B_k}\right\|_{L^{\vec{q_2}}(\mathbb{Q}_p^n)}\left\|\chi_{B_k}\right\|_{L^{\vec{q}_2'}(\mathbb{Q}_p^n)} \\
&\leq C|B_\gamma|_h^{\lambda_1} \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)} \sum_{k=\gamma}^{\infty}p^{kn\lambda_2} \\
&\leq C|B_\gamma|_h^{\lambda} \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Summarizing, we have

$$II'_2 \leq C|B_\gamma|_h^{\lambda} \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \|b\|_{\mathfrak{C}^{\vec{q_1},\lambda_1}(\mathbb{Q}_p^n)} \left\|f\right\|_{\mathcal{B}^{\vec{q_2},\lambda_2}(\mathbb{Q}_p^n)},$$

which implies (3.2). This is the desired result.

$(b) \Rightarrow (a)$. In this case, the important point of the proof is to construct a proper commutator. We are reduced to proving that for a fixed ball B_γ,

$$\frac{\left\|(b - b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q_1}}(\mathbb{Q}_p^n)}} \leq C.$$

We conclude from (1.1) and Hölder's inequality that

$$\begin{aligned}\frac{\left\|(b-b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}} &\le \frac{\sup_{B_\gamma\ni y}|b(y)-b_{B_\gamma}|\|\chi_{B_\gamma}\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}}\\ &\le \frac{C\|(b-b_{B_\gamma})\chi_{B_\gamma}\|_{L^1(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^1(\mathbb{Q}_p^n)}}\\ &\le \frac{C\|(b-b_{B_\gamma})\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}.\end{aligned}$$

To deal with the above term, we note that

$$\begin{aligned}\|(b-b_{B_\gamma})\chi_{B_\gamma}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} &\le \left\|\left(b(\cdot)-\frac{1}{|B_\gamma|_h}\int_{B_\gamma}b(z)dz\right)\chi_{B_\gamma}(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &= \left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{B_\gamma}(b(\cdot)-b(z))\,dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le \left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{B(0,|x|_p)}(b(\cdot)-b(z))\,\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\quad+\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}(b(\cdot)-b(z))\,\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} = J_1+J_2.\end{aligned}$$

The $\left(\mathcal{B}^{\vec{q}_2,\lambda_2}\left(\mathbb{Q}_p^n\right),\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)\right)$ boundedness of $\mathcal{H}_b^p$ allows us to estimate J_1 as

$$\begin{aligned}J_1 &\le \left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}|\cdot|_p^n\frac{1}{|\cdot|_p^n}\int_{B(0,|x|_p)}(b(\cdot)-b(z))\,\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C\left\|\chi_{B_\gamma}(\cdot)\mathcal{H}_b^p(\chi_{B_\gamma})(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\left\|\mathcal{H}_b^p(\chi_{B_\gamma})\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\left\|\chi_{B_\gamma}\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}.\end{aligned}$$

By the $\left(\mathcal{B}^{\vec{q}_2,\lambda_2}\left(\mathbb{Q}_p^n\right),\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)\right)$ boundedness of $\mathcal{H}_b^{p,*}$, it is easy to check that

$$\begin{aligned}J_2 &\le \left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{(b(\cdot)-b(z))\,\chi_{B_\gamma}(z)}{|z|_p^n}|z|_p^n dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C\left\|\chi_{B_\gamma}(\cdot)\mathcal{H}_b^{p,*}(\chi_{B_\gamma})(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\left\|\mathcal{H}_b^{p,*}(\chi_{B_\gamma})\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\left\|\chi_{B_\gamma}\right\|_{\mathcal{B}^{\vec{q}_2,\lambda_2}(\mathbb{Q}_p^n)}\\ &\le C|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}.\end{aligned}$$

Thus we have established the following inequality if we combine the above estimates for J_1 and J_2,

$$\frac{\left\|(b-b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}} \leq \frac{C|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda_1}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}_1}(\mathbb{Q}_p^n)}} \leq C.$$

Thus the proof of Theorem 1.1 is finished. □

Application Lemma 2.2, we can now return to the proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2] (*a*) ⇒ (*b*) Let $f \in \mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$, $b \in \mathfrak{C}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right)$. The task is now to find a positive constant C such that for a fixed ball B_γ, the following inequalities are right

$$\left\|(\mathcal{H}_b^p f)\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \leq C|B_\gamma|_h^{2\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\max\{\vec{q},\vec{q}'\},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}. \tag{3.3}$$

$$\left\|(\mathcal{H}_b^{p,*} f)\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \leq C|B_\gamma|_h^{2\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\max\{\vec{q},\vec{q}'\},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.$$

To deal with (3.3), we note that

$$\begin{aligned}
\left\|(\mathcal{H}_b^p f)\chi_{B_\gamma}(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} &= \left\|\chi_{B_\gamma}(\cdot)\frac{1}{|\cdot|_p^n}\int_{B(0,|x|_p)}(b(\cdot)-b(y))f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq \sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\int_{B_k}(b(\cdot)-b(y))f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq C\sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\sum_{i=-\infty}^{k}\int_{S_i}(b(\cdot)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&+ C\sum_{k=-\infty}^{\gamma}\left\|\chi_{S_k}(\cdot)\frac{1}{|\cdot|_p^n}\sum_{i=-\infty}^{k}\int_{S_i}(b(y)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&= K_1 + K_2.
\end{aligned}$$

To estimate K_1. For $1 = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'}$, by the Hölder's inequality, we get

$$\begin{aligned}
K_1 &\leq C\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\left\|\chi_{S_k}(b-b_{B_k})\sum_{i=-\infty}^{k}\int_{S_i}f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)} \\
&\leq \sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\left\|\chi_{S_k}(b-b_{B_k})\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\int_{B_i}|f(y)|dy \\
&\leq \sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}|B_k|_h^{\lambda}\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\left\|f\chi_{B_i}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\left\|\chi_{B_i}\right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)} \\
&\leq \|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}|B_k|_h^{\lambda-1}\left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{\lambda+1} \\
&\leq C|B_\gamma|_h^{2\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)},
\end{aligned}$$

where we have used the fact $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i} < \lambda < 0$.

As for K_2, it follows from lemma 2.2 that

$$\begin{aligned}K_2 &\le C\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\left\|\chi_{S_k}\sum_{i=-\infty}^{k}\int_{S_i}(b(y)-b_{B_k})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\left\|\chi_{B_k}\sum_{i=-\infty}^{k}\int_{B_i}(b(y)-b_{B_i})f(y)dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\quad + C\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\left\|\chi_{B_k}\sum_{i=-\infty}^{k}\int_{B_i}|B_i|_h^{\lambda}|f(y)|dy\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &=: K_{21}+K_{22}.\end{aligned}$$

Repeated application of Hölder's inequality shows that

$$\begin{aligned}K_{21} &\le C\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\left\|(b-b_{B_i})\chi_{B_i}\right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\|f\chi_{B_i}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\left\|(b-b_{B_i})\chi_{B_i}\right\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}|B_i|_h^{\lambda}\|\chi_{B_i}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\le C\|b\|_{\mathfrak{C}^{\vec{q}',\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}\|\chi_{B_i}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|\chi_{B_i}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}|B_i|_h^{2\lambda}\\ &\le C\|b\|_{\mathfrak{C}^{\vec{q}',\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{2\lambda+1}\\ &\le C|B_\gamma|_h^{2\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q}',\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)},\end{aligned}$$

and the condition $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i}<\lambda<0$ has been used.

With the help of the fact $-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2q_i}<\lambda<0$ and Hölder's inequality, the term K_{22} can be bounded by

$$\begin{aligned}K_{22} &\le C\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{\lambda}\int_{B_i}|f(y)|dy\\ &\le C\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{\lambda}\|f\chi_{B_i}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|\chi_{B_i}\|_{L^{\vec{q}'}(\mathbb{Q}_p^n)}\\ &\le C\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}\frac{1}{|B_k|_h}\|\chi_{B_k}\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}|B_i|_h^{2\lambda+1}\\ &\le C|B_\gamma|_h^{2\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\|b\|_{\mathfrak{C}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\left\|f\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}.\end{aligned}$$

Based on the above estimate for K_{21} and K_{22}, we get (3.3).

With a slight modification of the proofs for (3.2) and (3.3) can be obtained easily, we omit its proof here for the similarity.

$(b)\Rightarrow(a)$ We divide the proof into two cases according to the range of $\vec{q}$ and $\vec{q}'$.

Case 1: $\vec{q}>\vec{q}'$. In this case, we only need to show that there is a positive canstant C such that for a fixed ball B_γ, there holds

$$\frac{\left\|(b-b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^{\lambda}\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\le C.$$

To deal with this inequality, we note that

$$\begin{aligned}\frac{\left\|(b-b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}} &\leq \frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|(b(\cdot)-\frac{1}{|B_\gamma|_h}\int_{B_\gamma}b(z)dz)\chi_{B_\gamma}(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq \frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{B_\gamma}(b(\cdot)-b(z))dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq \frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{B(0,|x|_p)}(b(\cdot)-b(z))\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\quad+\frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}(b(\cdot)-b(z))\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &=: L_1+L_2.\end{aligned}$$

The $\left(\mathcal{B}^{\vec{q},\lambda},\left(\mathbb{Q}_p^n\right),\mathcal{B}^{\vec{q},2\lambda}\left(\mathbb{Q}_p^n\right)\right)$ boundedness of $\mathcal{H}_b^p$ produces the following estimate for the term L_1,

$$\begin{aligned}L_1 &\leq \frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}|\cdot|_p^n\frac{1}{|\cdot|_p^n}\int_{B(0,|x|_p)}(b(\cdot)-b(z))\chi_{B_\gamma}(z)dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq C\frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\chi_{B_\gamma}(\cdot)\mathcal{H}_b^p(\chi_{B_\gamma})(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq C|B_\gamma|_h^\lambda\left\|\mathcal{H}_b^p(\chi_{B_\gamma})\right\|_{\mathcal{B}^{\vec{q},2\lambda}(\mathbb{Q}_p^n)}\\ &\leq C|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\\ &\leq C.\end{aligned}$$

By the $\left(\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right),\mathcal{B}^{\vec{q},2\lambda}\left(\mathbb{Q}_p^n\right)\right)$ boundedness of $\mathcal{H}_b^{p,*}$, the following can be confirmed easily

$$\begin{aligned}L_2 &\leq \frac{1}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\frac{\chi_{B_\gamma}(\cdot)}{|B_\gamma|_h}\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{(b(\cdot)-b(z))\,\chi_{B_\gamma}(z)}{|z|_p^n}|z|_p^n dz\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq \frac{C}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\left\|\chi_{B_\gamma}(\cdot)\mathcal{H}_b^{p,*}(\chi_{B_\gamma})(\cdot)\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}\\ &\leq C|B_\gamma|_h^\lambda\left\|\mathcal{H}_b^{p,*}(\chi_{B_\gamma})\right\|_{\mathcal{B}^{\vec{q},2\lambda}(\mathbb{Q}_p^n)}\\ &\leq C|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{\mathcal{B}^{\vec{q},\lambda}(\mathbb{Q}_p^n)}\\ &\leq C.\end{aligned}$$

Combining L_1 and L_2, we have

$$\frac{\left\|(b-b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^\lambda\left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}\leq C.$$

Case 2: $\vec{q}<\vec{q'}$. In view of the $\left(\mathcal{B}^{\vec{q},\lambda}\left(\mathbb{Q}_p^n\right),\mathcal{B}^{\vec{q},2\lambda}\left(\mathbb{Q}_p^n\right)\right)$ boundedness of $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$, the similar ways of Case 1 can be applied to this and show that

$$\frac{\left\|(b - b_{B_\gamma})\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}}{|B_\gamma|_h^\lambda \left\|\chi_{B_\gamma}\right\|_{L^{\vec{q}}(\mathbb{Q}_p^n)}} \leq C.$$

To sum up, we finish the proof of Theorem 1.2. □

Proof. [Proofs of Theorem 1.3 and Theorem 1.4] The ways used in the proofs of Theorem 1.1 and Theorem 1.2 remain valid for that of the following Theorem 1.3 and Theorem 1.4 with only a slight modification, thus we omit their proof. □

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