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Backward error analysis of generalized eigenvalue problems preserving block structures of matrices

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Abstract. This paper considers the backward error analysis of an approximate eigenpair of blockwise structured matrix pencils that becomes an exact eigenpair of an appropriately minimal perturbed block matrix pencil. The obtained perturbed pencil preserves the structures of different blocks for the Frobenius norm. In application, we discuss the different pencils arising in continuous-time linear quadratic optimal control problems, discrete-time linear quadratic optimal control, and port-Hamiltonian descriptor systems in optimal control. We also present several numerical examples to illustrate our framework.

1. Introduction

Backward error analysis is crucial for understanding the computed solution's quality and the numerical algorithm's stability. By backward error analysis, we know how far the computed solution stands from the original solution in an algorithm. A literature series is available where unstructured and structured eigenpair backward errors of matrix pencils and polynomials have been discussed (see, [1, 3, 4, 11]). In these papers, the blocks' symmetry has been ignored if it is present in the matrices. In [21], the author has discussed blockwise perturbation for a symmetric matrix. In many applications (see [14, 17, 19]) where we get block-structured matrices, those blocks have some physical significance. We often need algorithms that take care of the blocks to preserve their physical importance. To study the quality of the computed solutions and the sensitivity of the algorithms that preserve the block structure of the matrices, we must study the blockwise structured, backward error. In this paper, we consider the perturbation of matrix pencils. In particular, we derive structured backward errors for eigenpairs of structured matrix pencil arising in optimal control theory (see, [15, 17, 18]). Consider the linear quadratic optimal control problems for the continuous case. Here the associated matrix pencil is of the form $L_c(z) = M_c + zN_c$, where

$$M_c := \begin{bmatrix} 0 & A & B \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \text{ and } N_c := \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

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with $A, E, Q \in \mathbb{C}^{n\times n}$, $S, B \in \mathbb{C}^{n\times m}$, and $R \in \mathbb{C}^{m\times m}$. Here, Q and R are positive definite and $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ is positive semidefinite (see e.g., [13, 17]). In continuous time case, the problem is to minimize the functional

$$\frac{1}{2}\int_{t_0}^{\infty} [y(t)^* Q y(t) + u(t)^* R u(t) + u(t)^* S^* y(t) + y(t)^* S u(t)]dt$$

subject to the constraints $E\dot{x}(t) = Ax(t) + Bu(t), \dot{x}(t_0) = x^0, y(t) = x(t)$. If we consider the linear quadratic optimal control problem for the discrete case, the associated matrix pencil is of the form $L_d(z) = M_d + zN_d$, where

$$M_d := \begin{bmatrix} 0 & A & B \\ -E^* & Q & S \\ 0 & S^* & R \end{bmatrix} \text{ and } N_d := \begin{bmatrix} 0 & E & 0 \\ -A^* & 0 & 0 \\ -B^* & 0 & 0 \end{bmatrix}, \tag{2}$$

$A, E, Q \in \mathbb{C}^{n\times n}$, $S, B \in \mathbb{C}^{n\times m}$, and $R \in \mathbb{C}^{m\times m}$, the problem is to minimize the functional

$$\frac{1}{2}\sum_{k=0}^{\infty}[y_k^* Q y_k + u_k^* R u_k + u_k^* S^* y_k + y_k^* S u_k],$$

subject to the difference equation $Ex_{k+1} = Ax_k + Bu_k$, $x_{k_0} = x^0$, $y_k = x_k$. We also consider the matrix pencil $L_p(z)$ of the form

$$L_p(z) = M_p + zN_p := \begin{bmatrix} 0 & J-R & B \\ (J-R)^* & Q & 0 \\ B^* & 0 & S \end{bmatrix} + z\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

where $J, R, E, Q \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n,m}$ and $S \in \mathbb{C}^{m,m}$ satisfy $J^* = -J, R^* = R, E^* = E, Q^* = Q$, and $S^* = S > 0$, i.e., S is positive definite. These types of pencils arise in optimal control and H_∞ control problems and in the passivity analysis of dynamical systems. If we consider the optimal control problem of minimizing the cost functional

$$\int_{t_0}^{\infty} [x^* Q x + u^* S u]dt$$

subject to the constraint $E\dot{x}(t) = Ax(t) + Bu(t), \dot{x}(t_0) = x^0, y(t) = x(t)$. Here, we assume that $E = E^*$. Then, partition it into skew-symmetric and symmetric parts $A = J - R$. The above system is the port Hamiltonian descriptor system (see, e.g., [8, 10]). In this paper, we do perturbation analysis on matrix pencils given in the equations (1), (2), and (3), respectively. The main highlights of the paper are given as follows:

- The authors in [1, 3, 5–7] have discussed eigenpair backward error, but those frameworks cannot be implemented for the blockwise structured perturbations.
- We have formulated a framework designed to uphold both the block structure and the internal structure within each block within our theory. By our framework, we also maintain the sparsity within the block structures.
- Mehl et al. in [16] derive blockwise eigenpair for $\lambda \in i\mathbb{R}$ for the pencil (3), where they have considered the matrix $Q = 0$. Our framework calculates the structured blockwise eigenpair backward error for any such $\lambda \in \mathbb{C}$ and $Q \in \mathbb{C}^{n\times n}$. In some cases, bounds of the backward error have been derived in their paper, but by our framework, we find the exact backward errors.
- In this paper, we introduce a method for perturbing our block matrices without altering the definiteness of the individual blocks due to these perturbations. For instance, in equation (1), the block $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ maintains its positive semidefinite property. In subsection 3.1, we demonstrate how we preserve the original block structure's semidefiniteness. Still, if we want to perturb all the blocks, the same framework can be applied to the rest to calculate the corresponding backward error.

- In [8], linear port-Hamiltonian descriptor systems are discussed; in that paper and the reference within the paper, we can find many block-structured matrices, and this analysis can help to find the blockwise backward error of those structures.
- Recently, in [20], the authors consider block structured pencils obtained from optimal and robust control, and it discusses eigenvalue characterization under linear perturbation. Our framework can be used to study the blockwise eigenpair backward error for those structures.

The paper is organized as follows: Section 2 reviews some definitions and preliminary results required in the later part of the paper. In Section 3, we derive formulae for the blockwise eigenpair backward error for various block structures in different subsections. Finally, in Section 4, we have illustrated numerical examples based on our theory and compared it with previous literature.

2. Notation and Preliminaries

Throughout the paper, we follow the following notation: $\mathbb{C}^{n\times m}$ is the set of all $n \times m$ complex matrices. Define $A^\dagger$ as the pseudoinverse of the matrix $A \in \mathbb{C}^{n\times m}$. For $A \in \mathbb{C}^{n\times m}$, A^T and A^* denote the transpose and the conjugate transpose of A, respectively. Let $\operatorname{sgn} a = 1$ when $a \neq 0$, otherwise $\operatorname{sgn} a = 0$. For a matrix $A_j := (a^{(j)}_{pq}) \in \mathbb{C}^{n\times m}$, $j = 1, 2$, where $p = 1, \ldots, n, q = 1, \ldots, m$, we define $\operatorname{sgn} A_j = (\operatorname{sgn} a^{(j)}_{pq}) \in \mathbb{C}^{n\times m}$. For $A = (a_{ij}) \in \mathbb{C}^{n\times m}$ and $B = (b_{ij}) \in \mathbb{C}^{n\times m}$, we define $A \circ B := (a_{ij}b_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. We denote $\mathrm{SHerm}(n)$ and $\mathrm{Herm}(n)$ to be the set of all skew-Hermitian and Hermitian matrices of size $n \times n$, respectively. We define spectral and Frobenius norm on $\mathbb{C}^{n\times n}$ by $\|A\|_2 := \max_{\|x\|=1} \|Ax\|_2$ and $\|A\|_F := \sqrt{\operatorname{trace}(A^*A)}$, respectively. For a vector $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{C}^n$, $\operatorname{diag}(v)$ denotes a diagonal matrix of size $n \times n$ with diagonal entries v_j for $j = 1, \ldots, n$. If $A = (a_{ij}) \in \mathbb{C}^{n\times m}$, then $\operatorname{vec}(A) := [a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nm}]^T \in \mathbb{C}^{nm}$. If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is symmetric matrix, then define $\operatorname{vec}(A) := [a_{11}, \ldots, a_{1n}, a_{22}, \ldots, a_{2n}, \ldots, a_{(n-1)(n-1)}, a_{(n-1)n}, a_{nn}]$, where $\operatorname{vec}(A) \in \mathbb{C}^{(n^2+n)/2}$. If A is skew-symmetric matrix, then we define $\operatorname{vec}(A) := [a_{12}, \ldots, a_{1n}, a_{23}, \ldots, a_{2n}, \ldots, a_{(n-2)n}, a_{(n-1)n}]^T$, where $\operatorname{vec}(A) \in \mathbb{C}^{(n^2-n)/2}$. We set $L(\mathbb{C}^{n\times n})$ as the space of matrix pencils.

Definition 2.1. *For the pencil $L \in L(\mathbb{C}^{n\times n})$ of the form $L(z) = A + zB$, the norm of the pencil is defined by*

$$|\!|\!|L|\!|\!| := (\|A\|_F^2 + \|B\|_F^2)^{1/2}.$$

Definition 2.2. *Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ be the exact eigenpair of the perturbed pencil $L - \Delta L \in L(\mathbb{C}^{n\times n})$. Then the unstructured and structured eigenpair backward errors of (λ, x) for the pencil L are defined by*

$$\begin{aligned} \eta_F(L, \lambda, x) &= \inf\{|\!|\!|\Delta L|\!|\!| : (L(\lambda) - \Delta L(\lambda))x = 0\} \text{ and} \\ \eta^{\mathbb{S}}_F(L, \lambda, x) &= \inf\{|\!|\!|\Delta L|\!|\!| : \Delta L \in \mathbb{S}, (L(\lambda) - \Delta L(\lambda))x = 0\}, \end{aligned}$$

respectively, where $\mathbb{S}$ denotes some specific structure and $\Delta L(z) = \Delta A + z\Delta B$.

Note: The statement $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ is the exact eigenpair of the perturbed pencil $L - \Delta L \in L(\mathbb{C}^{n\times n})$ can be rewritten as $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$ to be the approximate eigenpair of the unperturbed matix pencil $L \in L(\mathbb{C}^{n\times n})$.

For further development of our theory we define the following matrices

$$P_s = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \cdots & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & 1 \end{bmatrix}, Q_s = \begin{bmatrix} 1 & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & 1 & \cdots & \sqrt{2} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{2} & \sqrt{2} & \cdots & 1 \end{bmatrix}, P_{ss} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \cdots & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & 0 \end{bmatrix}, Q_{ss} = \begin{bmatrix} 0 & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & 0 & \cdots & \sqrt{2} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{2} & \sqrt{2} & \cdots & 0 \end{bmatrix}.$$

and the vectors generated from these matrices are

$$\left[1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{2}}, 1\right]^T, \left[1, \sqrt{2}, \ldots, \sqrt{2}, 1\right]^T, \left[\frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{2}}\right]^T \text{and} \left[\sqrt{2}, \ldots, \sqrt{2}\right]^T,$$

respectively. To maintain the sparsity within the blocks we introduce sgn function as follows:

Remark 2.3. *Consider the perturbed system of equation*

$$\begin{aligned}\delta a_{11}x^{(1)} + \delta a_{12}x^{(2)} + \delta a_{13}x^{(3)} &= b^{(1)}\\ \delta a_{21}x^{(1)} + \delta a_{22}x^{(2)} + \delta a_{23}x^{(3)} &= b^{(2)}\\ \delta a_{31}x^{(1)} + \delta a_{32}x^{(2)} + \delta a_{33}x^{(3)} &= b^{(3)}.\end{aligned}$$

Now, if we want to maintain the sparsity in the perturbed system of equations, we can write the equivalent system of equations as follows:

$$\begin{bmatrix}\delta a_{11} & \delta a_{12} & \delta a_{13}\\ \delta a_{21} & \delta a_{22} & \delta a_{23}\\ \delta a_{31} & \delta a_{32} & \delta a_{33}\end{bmatrix} \circ \begin{bmatrix}\operatorname{sgn} a_{11} & \operatorname{sgn} a_{12} & \operatorname{sgn} a_{13}\\ \operatorname{sgn} a_{21} & \operatorname{sgn} a_{22} & \operatorname{sgn} a_{23}\\ \operatorname{sgn} a_{31} & \operatorname{sgn} a_{32} & \operatorname{sgn} a_{33}\end{bmatrix}\begin{bmatrix}x^{(1)}\\ x^{(2)}\\ x^{(3)}\end{bmatrix} = \begin{bmatrix}b^{(1)}\\ b^{(2)}\\ b^{(3)}\end{bmatrix}.$$

Now, we discuss all possible cases in which we write the system of equations in their equivalent form in the following lemmas. Firstly, we write this for general unstructured block matrices.

Lemma 2.4. *Let $A_1, \Delta A_1 \in \mathbb{R}^{n\times n}$, and $A_2, \Delta A_2 \in \mathbb{R}^{n\times m}$ be generated by $[a^{(1)}_{11}, a^{(1)}_{12}, \dots, a^{(1)}_{nn}]^T \in \mathbb{R}^{n^2}$, $[\delta a^{(1)}_{11}, \delta a^{(1)}_{12}, \dots, \delta a^{(1)}_{nn}]^T \in \mathbb{R}^{n^2}$, $[a^{(2)}_{11}, a^{(2)}_{12}, \dots, a^{(2)}_{nm}]^T \in \mathbb{R}^{nm}$, and $[\delta a^{(2)}_{11}, \delta a^{(2)}_{12}, \dots, \delta a^{(2)}_{nm}]^T \in \mathbb{R}^{nm}$, respectively. Let $x = \begin{bmatrix}x^{(1)}\\ x^{(2)}\end{bmatrix} \in \mathbb{R}^{n+m}$, where $x^{(1)} = [x^{(1)}_{11}, \dots, x^{(1)}_{n1}]^T \in \mathbb{R}^n$, $x^{(2)} = [x^{(2)}_{11}, \dots, x^{(2)}_{m1}]^T \in \mathbb{R}^m$, and $b \in \mathbb{R}^n$, then*

$$\begin{bmatrix} \Delta A_1 \circ \operatorname{sgn} A_1 \mid \Delta A_2 \circ \operatorname{sgn} A_2 \end{bmatrix}_{n\times(n+m)} \begin{bmatrix}x^{(1)}\\ x^{(2)}\end{bmatrix}_{(n+m)\times 1} = b$$

is equivalent to

$$\begin{bmatrix} X_{x^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_1)) \mid X_{x^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_2)) \end{bmatrix}_{n\times(n^2+nm)} \begin{bmatrix}\operatorname{vec}(\Delta A_1 \circ \operatorname{sgn} A_1)\\ \operatorname{vec}(\Delta A_2 \circ \operatorname{sgn} A_2)\end{bmatrix} = b,$$

where

$$X_{x^{(1)}} := \left[\begin{array}{cccc|cccc|c|cccc} x^{(1)}_{11} & x^{(1)}_{21} & \cdots & x^{(1)}_{n1} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0\\ 0 & 0 & \cdots & 0 & x^{(1)}_{11} & x^{(1)}_{21} & \cdots & x^{(1)}_{n1} & \cdots & 0 & \cdots & \cdots & 0\\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x^{(1)}_{11} & \cdots & \cdots & x^{(1)}_{n1} \end{array}\right]_{n\times n^2}$$

and

$$X_{x^{(2)}} := \left[\begin{array}{cccc|cccc|c|cccc} x^{(2)}_{11} & x^{(2)}_{21} & \cdots & x^{(2)}_{m1} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0\\ 0 & 0 & \cdots & 0 & x^{(2)}_{11} & x^{(2)}_{21} & \cdots & x^{(2)}_{m1} & \cdots & 0 & \cdots & \cdots & 0\\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x^{(2)}_{11} & \cdots & \cdots & x^{(2)}_{m1} \end{array}\right]_{n\times nm}.$$

Proof. Consider

$$\begin{bmatrix} \Delta A_1 \circ \operatorname{sgn} A_1 \mid \Delta A_2 \circ \operatorname{sgn} A_2 \end{bmatrix}_{n\times(n+m)} \begin{bmatrix}x^{(1)}\\ x^{(2)}\end{bmatrix}_{(n+m)\times 1} = b,$$

where

$$\Delta A_1 \circ \operatorname{sgn} A_1 := \begin{bmatrix} \delta a^{(1)}_{11}\operatorname{sgn} a^{(1)}_{11} & \delta a^{(1)}_{12}\operatorname{sgn} a^{(1)}_{12} & \cdots & \delta a^{(1)}_{1n}\operatorname{sgn} a^{(1)}_{1n}\\ \delta a^{(1)}_{21}\operatorname{sgn} a^{(1)}_{21} & \delta a^{(1)}_{22}\operatorname{sgn} a^{(1)}_{1,22} & \cdots & \delta a^{(1)}_{2n}\operatorname{sgn} a^{(1)}_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ \delta a^{(1)}_{n1}\operatorname{sgn} a^{(1)}_{n1} & \delta a^{(1)}_{n2}\operatorname{sgn} a^{(1)}_{n2} & \cdots & \delta a^{(1)}_{nn}\operatorname{sgn} a^{(1)}_{nn} \end{bmatrix}$$

and

$$\Delta A_2 \circ \mathrm{sgn} A_2 := \begin{bmatrix} \delta a_{11}^{(1)} \mathrm{sgn}\, a_{11}^{(1)} & \delta a_{12}^{(1)} \mathrm{sgn}\, a_{12}^{(1)} & \cdots & \delta a_{1m}^{(1)} \mathrm{sgn}\, a_{1m}^{(1)} \\ \delta a_{21}^{(1)} \mathrm{sgn}\, a_{21}^{(1)} & \delta a_{22}^{(1)} \mathrm{sgn}\, a_{1,22}^{(1)} & \cdots & \delta a_{2m}^{(1)} \mathrm{sgn}\, a_{2m}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \delta a_{n1}^{(1)} \mathrm{sgn}\, a_{n1}^{(1)} & \delta a_{n2}^{(1)} \mathrm{sgn}\, a_{n2}^{(1)} & \cdots & \delta a_{nm}^{(1)} \mathrm{sgn}\, a_{nm}^{(1)} \end{bmatrix}.$$

Now, from the given system of equation we get

$$\left[\begin{array}{cccc|cccc} \delta a_{11}^{(1)} \mathrm{sgn}\, a_{11}^{(1)} & \delta a_{12}^{(1)} \mathrm{sgn}\, a_{12}^{(1)} & \cdots & \delta a_{1n}^{(1)} \mathrm{sgn}\, a_{1n}^{(1)} & \delta a_{11}^{(2)} \mathrm{sgn}\, a_{11}^{(2)} & \delta a_{12}^{(2)} \mathrm{sgn}\, a_{12}^{(2)} & \cdots & \delta a_{1m}^{(2)} \mathrm{sgn}\, a_{1m}^{(2)} \\ \delta a_{21}^{(1)} \mathrm{sgn}\, a_{21}^{(1)} & \delta a_{22}^{(1)} \mathrm{sgn}\, a_{22}^{(1)} & \cdots & \delta a_{2n}^{(1)} \mathrm{sgn}\, a_{1n}^{(1)} & \delta a_{21}^{(2)} \mathrm{sgn}\, a_{21}^{(2)} & \delta a_{22}^{(2)} \mathrm{sgn}\, a_{22}^{(2)} & \cdots & \delta a_{2m}^{(2)} \mathrm{sgn}\, a_{2m}^{(2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \delta a_{n1}^{(1)} \mathrm{sgn}\, a_{n1}^{(1)} & \delta a_{n2}^{(1)} \mathrm{sgn}\, a_{n2}^{(1)} & \cdots & \delta a_{nn}^{(1)} \mathrm{sgn}\, a_{nn}^{(1)} & \delta a_{n1}^{(2)} \mathrm{sgn}\, a_{n1}^{(2)} & \delta a_{n2}^{(2)} \mathrm{sgn}\, a_{n2}^{(2)} & \cdots & \delta a_{nm}^{(2)} \mathrm{sgn}\, a_{nm}^{(2)} \end{array}\right] \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = b.$$

Now, by rearranging the system of equations so that the elements of the matrices ΔA_1 and ΔA_2 are treated as unknowns, we obtain the system of equations as

$$\left[\begin{array}{c|c} X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & X_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2)) \end{array}\right] \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn} A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2) \end{bmatrix} = b,$$

where

$$X_{x^{(1)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{n1}^{(1)} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{n1}^{(1)} & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x_{11}^{(1)} & \cdots & \cdots & x_{n1}^{(1)} \end{array}\right]_{n \times n^2},$$

$$X_{x^{(2)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(2)} & x_{21}^{(2)} & \cdots & x_{m1}^{(2)} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{11}^{(2)} & x_{21}^{(2)} & \cdots & x_{m1}^{(2)} & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x_{11}^{(2)} & \cdots & \cdots & x_{m1}^{(2)} \end{array}\right]_{n \times nm}.$$

and

$$\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) = \begin{bmatrix} \mathrm{sgn} a_{11}^{(1)} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{sgn} a_{12}^{(1)} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathrm{sgn} a_{nn}^{(1)} \end{bmatrix}_{n^2 \times n^2},$$

$$\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2)) = \begin{bmatrix} \mathrm{sgn} a_{11}^{(2)} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{sgn} a_{12}^{(2)} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathrm{sgn} a_{nm}^{(2)} \end{bmatrix}_{nm \times nm}. \blacksquare$$

Note: In the above equivalent system of linear equations $X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1))$ and $X_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2))$ are of full row rank matrices, follows from the construction that each matrix has linearly independent rows, and if we combine these two matrices we get a matrix $\begin{bmatrix} X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & X_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2)) \end{bmatrix}$ is of full row rank.

In the following lemma, we rewrite the system of equations into its equivalent form consisting of the transpose of the matrices.

Lemma 2.5. *Let* $A_1, \Delta A_1^T \in \mathbb{R}^{n\times n}, A_2^T, \Delta A_2^T \in \mathbb{R}^{n\times m}$ *be generated by* $[a_{11}^{(1)}, a_{12}^{(1)}, \ldots, a_{nn}^{(1)}]^T \in \mathbb{R}^{n^2}, [\delta a_{11}^{(1)}, \delta a_{12}^{(1)}, \ldots, \delta a_{nn}^{(1)}]^T$, $[a_{11}^{(2)}, a_{12}^{(2)}, \ldots, a_{mn}^{(2)}]^T \in \mathbb{R}^{nm}$, *and* $[\delta a_{11}^{(2)}, \delta a_{12}^{(2)}, \ldots, \delta a_{mn}^{(2)}]^T \in \mathbb{R}^{nm}$, *respectively. Let* $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in \mathbb{R}^{n+m}$, *where* $x^{(1)} = [x_{11}^{(1)}, \ldots, x_{n1}^{(1)}]^T \in \mathbb{R}^n$, $x^{(2)} = [x_{11}^{(2)}, \ldots, x_{m1}^{(2)}]^T \in \mathbb{R}^m$, *and* $b \in \mathbb{R}^n$. *Then*

$$\begin{bmatrix} \Delta A_1^T \circ \mathrm{sgn} A_1^T & \Delta A_2^T \circ \mathrm{sgn} A_2^T \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = b,$$

is equivalent to

$$\begin{bmatrix} \widetilde{X}_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & \widetilde{X}_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2)) \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn} A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2) \end{bmatrix} = b,$$

where

$$\widetilde{X}_{x^{(1)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(1)} & 0 & \cdots & 0 & x_{21}^{(1)} & 0 & \cdots & 0 & \cdots & x_{n1}^{(1)} & 0 & \cdots & 0 \\ 0 & x_{11}^{(1)} & \cdots & 0 & 0 & x_{21}^{(1)} & \cdots & 0 & \cdots & 0 & x_{n1}^{(1)} & \cdots & 0 \\ & & \ddots & & & & \ddots & & & & & \ddots & \\ 0 & 0 & \cdots & x_{11}^{(1)} & 0 & 0 & \cdots & x_{21}^{(1)} & \cdots & 0 & 0 & \cdots & x_{n1}^{(1)} \end{array}\right]_{n\times n^2}$$

and

$$\widetilde{X}_{x^{(2)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(2)} & 0 & \cdots & 0 & x_{21}^{(2)} & 0 & \cdots & 0 & \cdots & x_{m1}^{(2)} & 0 & \cdots & 0 \\ 0 & x_{11}^{(2)} & \cdots & 0 & 0 & x_{21}^{(2)} & \cdots & 0 & \cdots & 0 & x_{m1}^{(2)} & \cdots & 0 \\ & & \ddots & & & & \ddots & & & & & \ddots & \\ 0 & 0 & \cdots & x_{11}^{(2)} & 0 & 0 & \cdots & x_{21}^{(2)} & \cdots & 0 & 0 & \cdots & x_{m1}^{(2)} \end{array}\right]_{n\times nm}.$$

Proof. The proof follows from the proof method of Lemma 2.4.■

Next, we rewrite the system of equations into its equivalent form consisting of unstructured and symmetric structures.

Lemma 2.6. *Let* $A_1, \Delta A_1 \in \mathbb{R}^{n\times m}$ *and* $A_2^T = A_2, \Delta A_2^T = \Delta A_2 \in \mathbb{R}^{n\times n}$ *be generated by* $[a_{11}^{(1)}, a_{12}^{(1)}, \ldots, a_{nm}^{(1)}]^T$, $[\delta a_{11}^{(1)}, \delta a_{12}^{(1)}, \ldots, \delta a_{nm}^{(1)}]^T \in \mathbb{R}^{nm}$, *and* $[a_{11}^2, a_{12}^2 \ldots, a_{nn}^{(2)}]^T$, $[\delta a_{11}^{(2)}, \delta a_{12}^{(2)}, \ldots, \delta a_{nn}^{(2)}]^T \in \mathbb{R}^{n(n+1)/2}$ *respectively. Let* $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in \mathbb{R}^{m+n}$, , *where* $x^{(1)} = [x_{11}^{(1)}, \ldots, x_{m1}^{(1)}]^T \in \mathbb{R}^m$ *and* $x^{(2)} = [x_{11}^{(2)}, \ldots, x_{n1}^{(2)}]^T \in \mathbb{R}^n$, *and* $b \in \mathbb{R}^n$. *Then* $\begin{bmatrix} \Delta A_1 \circ \mathrm{sgn} A_1 & \Delta A_2 \circ \mathrm{sgn} A_2 \circ P_s \circ Q_s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$ *is equivalent to*

$$\begin{bmatrix} X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & X_{x^{(2)}}^s \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2 \circ P_s)) \end{bmatrix}_{n\times (nm+\frac{n(n+1)}{2})} \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn} A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2 \circ Q_s) \end{bmatrix} = b,$$

where

$$X_{x^{(1)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{m1}^{(1)} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{m1}^{(1)} & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x_{11}^{(1)} & \cdots & \cdots & x_{m1}^{(1)} \end{array}\right]_{n\times nm}$$

and

$$X_{x^{(2)}}^s := \begin{bmatrix} x_{11}^{(2)} & x_{21}^{(2)} & \cdots & x_{n1}^{(2)} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & x_{11}^{(2)} & \cdots & 0 & x_{21}^{(2)} & x_{31}^{(2)} & \cdots & \cdots & 0 \\ \vdots & & & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & x_{11}^{(2)} & 0 & \cdots & \cdots & x_{(n-1)1}^{(2)} & x_{n1}^{(2)} \end{bmatrix}_{n\times \frac{n(n+1)}{2}}.$$

Proof. The proof is similar to the proof method of Lemma 2.4.■
In the next lemma, we rewrite the system of equations into its equivalent form consisting of unstructured and skew-symmetric structure.

Lemma 2.7. *Let $A_1, \Delta A_1 \in \mathbb{R}^{n\times m}$ and $A_2^T = -A_2, \Delta A_2^T = -\Delta A_2 \in \mathbb{R}^{n\times n}$ be generated by $[a_{11}^{(1)}, a_{12}^{(1)}, \ldots, \delta a_{nm}^{(1)}]^T$, $[\delta a_{11}^{(1)}, \delta a_{12}^{(1)} \ldots, \delta a_{nm}^{(1)}]^T \in \mathbb{R}^{nm}$, and $[a_{12}^{(2)}, \ldots a_{(n-1)n}^{(2)}]^T, [\delta a_{12}^{(2)}, \ldots, \delta a_{(n-1)n}^{(2)}]^T \in \mathbb{R}^{\frac{n(n-1)}{2}}$, respectively. Let $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \in \mathbb{R}^{m+n}$ where $x^{(1)} = [x_{11}^{(1)}, \ldots, x_{m1}^{(1)}]^T \in \mathbb{R}^m$ and $x_2 = [x_{11}^{(2)}, \ldots, x_{n1}^{(2)}]^T \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$. Then we have*

$$\begin{bmatrix} \Delta A_1 \circ \mathrm{sgn} A_1 & \Delta A_2 \circ \mathrm{sgn} A_2 \circ P_{ss} \circ Q_{ss} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = b \text{ is equivalent to}$$

$$\begin{bmatrix} X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & X^{ss}_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2 \circ P_{ss})) \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn} A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2 \circ Q_{ss}) \end{bmatrix} = b,$$

where

$$X_{x^{(1)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{m1}^{(1)} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{11}^{(1)} & x_{21}^{(1)} & \cdots & x_{m1}^{(1)} & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x_{11}^{(1)} & \cdots & \cdots & x_{m1}^{(1)} \end{array}\right]_{n\times nm}$$

and

$$X^{ss}_{x^{(2)}} := \begin{bmatrix} x_{21}^{(2)} & x_{31}^{(2)} & \cdots & x_{n1}^{(2)} & 0 & \cdots & \cdots & \cdots & 0 \\ -x_{11}^{(2)} & 0 & \cdots & 0 & x_{31}^{(2)} & \cdots & \cdots & \cdots & 0 \\ \vdots & & & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & -x_{11}^{(2)} & 0 & \cdots & \cdots & \cdots & x_{(n-1)1}^{(2)} \end{bmatrix}_{n\times \frac{n(n-1)}{2}}.$$

For more clarity, we write the proof of the previous lemma when $n = 3$ and $m = 2$.

Remark 2.8. *Consider*

$$\begin{bmatrix} \Delta A_1 \circ \mathrm{sgn} A_1 & \Delta A_2 \circ \mathrm{sgn} A_2 \circ P_{ss} \circ Q_{ss} \end{bmatrix}_{3\times 5} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}_{5\times 1} = b.$$

The first block of the system of equations can be tackled using Remark 2.3. Consider the second block of the system of equation i.e., $(\Delta A_2 \circ \mathrm{sgn} A_2 \circ P_{ss} \circ Q_{ss}) x^{(2)}$,

$$\begin{bmatrix} 0 & \delta a_{12}^{(2)} & \delta a_{13}^{(2)} \\ -\delta a_{12}^{(2)} & 0 & \delta a_{23}^{(2)} \\ -\delta a_{13}^{(2)} & -\delta a_{23}^{(2)} & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & \mathrm{sgn} a_{12}^{(2)} & \mathrm{sgn} a_{13}^{(2)} \\ \mathrm{sgn} a_{12}^{(2)} & 0 & \mathrm{sgn} a_{23}^{(2)} \\ \mathrm{sgn} a_{13}^{(2)} & \mathrm{sgn} a_{23}^{(2)} & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{31}^{(2)} \end{bmatrix}.$$

After multiplication, we get

$$\begin{array}{l} (\sqrt{2}\delta a_{12}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{12}^{(2)}) x_{21}^{(2)} + (\sqrt{2}\delta a_{13}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{13}^{(2)}) x_{31}^{(2)} \\ -(\sqrt{2}\delta a_{12}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{12}^{(2)}) x_{11}^{(2)} + (\sqrt{2}\delta a_{23}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{23}^{(2)}) x_{31}^{(2)} \\ -(\sqrt{2}\delta a_{13}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{13}^{(2)}) x_{11}^{(2)} - (\sqrt{2}\delta a_{23}^{(2)})(\frac{1}{\sqrt{2}} \mathrm{sgn} a_{23}^{(2)}) x_{21}^{(2)}. \end{array}$$

Now, by rearranging, we get

$$\begin{bmatrix} x_{21}^{(2)} & x_{31}^{(2)} & 0 \\ -x_{11}^{(2)} & 0 & x_{31}^{(2)} \\ 0 & -x_{11}^{(2)} & -x_{21}^{(2)} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{sgn} a_{12}^{(2)}}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\mathrm{sgn} a_{13}^{(2)}}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{\mathrm{sgn} a_{23}^{(2)}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2}\delta a_{12}^{(2)} \mathrm{sgn} a_{12}^{(2)} \\ \sqrt{2}\delta a_{13}^{(2)} \mathrm{sgn} a_{13}^{(2)} \\ \sqrt{2}\delta a_{23}^{(2)} \mathrm{sgn} a_{23}^{(2)} \end{bmatrix},$$

which implies

$$X_{x^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2) \circ P_{ss}) \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2 \circ Q_{ss}).$$

Hence, by combining both blocks, we get

$$\begin{bmatrix} X_{x^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) & X_{x^{(2)}}^{ss} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2 \circ P_{ss})) \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn} A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn} A_2 \circ Q_{ss}) \end{bmatrix} = b,$$

where

$$X_{x^{(1)}} := \begin{bmatrix} x_{11}^{(1)} & x_{21}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{11}^{(1)} & x_{21}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{11}^{(1)} & x_{21}^{(1)} \end{bmatrix}_{3\times 6},$$

$$X_{x^{(2)}} := \begin{bmatrix} x_{21}^{(2)} & x_{31}^{(2)} & 0 \\ -x_{11}^{(2)} & 0 & x_{31}^{(2)} \\ 0 & -x_{11}^{(2)} & -x_{21}^{(2)} \end{bmatrix}_{3\times 3}$$

and

$$\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1)) = \begin{bmatrix} \mathrm{sgn} a_{11}^{(1)} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{sgn} a_{12}^{(1)} & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathrm{sgn} a_{32}^{(1)} \end{bmatrix}_{6\times 6},$$

$$\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2 \circ P_{ss})) = \begin{bmatrix} \frac{\mathrm{sgn} a_{12}^{(2)}}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\mathrm{sgn} a_{13}^{(2)}}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{\mathrm{sgn} a_{23}^{(2)}}{\sqrt{2}} \end{bmatrix}_{3\times 3}.$$

Note: For arranging unstructured blocks, we need Lemma 2.4, and to arrange unstructured with transpose blocks, we need Lemma 2.5. Lemma 2.6 to arrange unstructured with symmetric blocks. Similarly, for the other cases, we need the respective Lemmas. To calculate the backward error, we need to apply the combinations of these Lemmas derived in this section. These lemmas will help us to preserve blockwise perturbation and maintain the sparsity within the blocks.

The next lemma provides the condition for the minimum norm solution of systems of equations.

Lemma 2.9. *[9] The system of linear equation $Ax = b$ is consistent if and only if $AA^{\dagger}b = b$ where $A^{\dagger}$ is the generalized inverse of $A \in \mathbb{R}^{n\times m}$, and $b \in \mathbb{R}^n$. A general solution is given by*

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y$$

where $y \in \mathbb{R}^m$ is an arbitrary vector. Moreover, $Ax = b$ has a unique solution if and only if $AA^{\dagger} = I_n, AA^{\dagger}b = b$ and solution is given by

$$x = A^{\dagger}b.$$

3. Structured eigenpair backward errors

In this section, we consider structured matrix pencils of various forms that arise in optimal control theory. Firstly, we discuss the basic definitions of eigenpair backward error. For matrix pencils of the form (3), if we are given $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ as an approximate eigenpair then we define backward error by

$$\eta(L_p, \lambda, x) = \inf\{\|\Delta L_p\| \Delta M_p, \Delta N_p \in \mathbb{C}^{2n+m,2n+m}, (L_p(\lambda) - \Delta L_p(\lambda))x = 0\}.$$

Here, we have ignored the structure of the pencil. Now, if M_p is Hermitian and N_p is skew-Hermitian, i.e., the matrix pencil is $*$-even, then we define structured eigenpair backward error for even pencil by

$$\eta^{*-\text{even}}(L_p, \lambda, x) = \inf\{\|\Delta L_p\| \Delta M_p \in \text{Herm}(2n+m), \Delta N_p \in \text{SHerm}(2n+m), (L_p(\lambda) - \Delta L_p(\lambda))x = 0\},$$

where $L_p(\lambda) = M_p + \lambda N_p, \Delta L_p(\lambda) = \Delta M_p + \lambda \Delta N_p$. ΔM_p and ΔN_p can be found from (3).

In [12], the authors discuss various structured polynomial matrices and their linearizations. A matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, $A_0, \ldots, A_k \in \mathbb{C}^{n\times n}$ is said to be $*$-even if $P(\lambda) = P^*(-\lambda)$. For our given matrix pencil $L_p(z) = M_p + zN_p, M_p \in \text{Herm}(2n+m)$ and $N_p \in SHerm(2n+m)$, which shows that our matrix pencil is a $*$-even matrix pencil. Our $*$-even pencil has some special structured blocks associated with it. The unstructured and structured backward error results are in [1, 3, 6, 7]. The authors consider structured matrix pencils and polynomials in these articles and obtain the corresponding eigenpair backward error. However, the framework used in those papers does not preserve the block structures within the matrices they considered. In this paper, we focus on matrix pencils with specific block structures. We have preserved the structure of the pencils and maintained the block structures while perturbing the coefficient matrices.

3.1. Perturbation on pencils arising in continious time linear quadratic optimal control problems

Let $L_c(z)$ be of the form (1) and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$. Let $\mathcal{B}$ be the collection of the pencils of the form $\Delta L_c(z) = \Delta M_c + z\Delta N_c$, where ΔM_c and ΔN_c are given by

$$\Delta M_c = \begin{bmatrix} 0 & \Delta A & \Delta B \\ \Delta A^* & 0 & 0 \\ \Delta B^* & 0 & 0 \end{bmatrix} \text{ and } \Delta N_c = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

If we are given $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$, where $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}, x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$, then we define eigenpair backward error for perturbation along the blocks by

$$\begin{aligned} \eta^{\mathcal{B}}(M_c, N_c, \lambda, x) = \inf\{\|(\Delta A, \Delta E, \Delta B)\| \mid \Delta A, \Delta E \in \mathbb{C}^{n\times n}, \Delta B \in \mathbb{C}^{n\times m}, \Delta L_c \in \mathcal{B}, \\ (L_c(\lambda) - \Delta L_c(\lambda))x = 0\}, \end{aligned} \tag{5}$$

where $\|(\Delta A, \Delta E, \Delta B)\| = \sqrt{\|\Delta A\|_F^2 + \|\Delta E\|_F^2 + \|\Delta B\|_F^2}$, and we set $x^{(1)} = x_1^{(1)} + ix_2^{(1)}, x^{(2)} = x_1^{(2)} + ix_2^{(2)}, x^{(3)} = x_1^{(3)} + ix_2^{(3)}, \lambda x^{(1)} = u^{(1)} = u_1^{(1)} + iu_2^{(1)}, \lambda x^{(2)} = u^{(2)} = u_1^{(2)} + iu_2^{(2)}, \lambda x^{(3)} = u^{(3)} = u_1^{(3)} + iu_2^{(3)}$, where $u_1^{(j)}$ is real part of $u^{(j)}$ and $u_2^{(j)}$ is imaginary part of $u^{(j)}$. for $j = 1, 2$. In this case, we perturb the block matrices in such a way that the definiteness of the block matrix $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ won't be hampered.

Given that our matrices are expressed in block form, the following lemma rewrites the system of equations $(L_c(\lambda) - \Delta L_c(\lambda))x = 0$ into an equivalent system of equations. This reformulation will be instrumental in determining the corresponding eigenpair backward error.

Lemma 3.1. *Let $L_c(z)$ be of the form (1), and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$. If $\Delta A, \Delta E \in \mathbb{C}^{n\times n}$ and $\Delta B \in \mathbb{C}^{n\times m}$, there exists $\Delta L_c(z) = \Delta M_c + z\Delta N_c \in \mathcal{B}$, such that $(L_c(\lambda) - \Delta L_c(\lambda))x = 0$ if and only if*

$$\begin{aligned} (\Delta A + \lambda \Delta E)x^{(2)} + \Delta B x^{(3)} &= r^{(1)}, & (6)\\ (\Delta A^* - \lambda \Delta E^*)x^{(1)} &= r^{(2)}, & (7)\\ \Delta B^* x^{(1)} &= r^{(3)}, & (8) \end{aligned}$$

where $r^{(1)} := (A + \lambda E)x^{(2)} + Bx^{(3)}, r^{(2)} := (A^* - \lambda E^*)x^{(1)} + Qx^{(2)} + Sx^{(3)}$, and $r^{(3)} := B^*x^{(1)} + S^*x^{(2)} + Rx^{(3)}$.

Proof. The proof is very straightforward. We have

$$\begin{aligned} (L_c(\lambda) - \Delta L_c(\lambda))x &= 0,\\ \iff ((M_c - \Delta M_c) + \lambda(N_c - \Delta N_c))x &= 0.\\ \iff \begin{bmatrix} 0 & \Delta A + \lambda\Delta E & \Delta B \\ \Delta A^* - \lambda \Delta E^* & 0 & 0 \\ \Delta B^* & 0 & 0 \end{bmatrix}\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix} &= \begin{bmatrix} r^{(1)} \\ r^{(2)} \\ r^{(3)} \end{bmatrix}. \end{aligned}$$

Now, Multiplying blockwise we get

$$\begin{aligned} (\Delta A + \lambda \Delta E)x^{(2)} + \Delta B x^{(3)} &= r^{(1)},\\ (\Delta A^* - \lambda \Delta E^*)x^{(1)} &= r^{(2)},\\ \Delta B^* x^{(1)} &= r^{(3)}. \blacksquare \end{aligned}$$

Using the above system of equations, we can calculate the corresponding blockwise backward error.

Theorem 3.2. *Let $L_c(z)$ be a sparse matrix pencil of the form (1). Let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$ be an approximate eigenpair of L_c. Then there exists minimum norm ΔL_c of the form (4) with sparsity such that $(L_c(\lambda) - \Delta L_c(\lambda))x = 0$, if $M\Delta = r$ is solvable and the backward error is given by $\eta^{\mathcal{B}}(M_c, N_c, \lambda, x) = \|M^{\dagger} r\|_F$, where M is defined in Appendix A and $r = \begin{bmatrix} r_1^{(1)T} & r_2^{(1)T} & r_1^{(2)T} & r_2^{(2)T} & r_1^{(3)T} & r_2^{(3)T} \end{bmatrix}^T$ where $r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}, r^{(3)} = r_1^{(3)} + ir_2^{(3)}$.*

Proof. Let $\Delta A = \Delta A_1 + i\Delta A_2, \Delta E = \Delta E_1 + i\Delta E_2$ and $\Delta B = \Delta B_1 + i\Delta B_2$. Then from, equation (6), we get,

$$(\Delta A_1 + i\Delta A_2)(x_1^{(2)} + ix_2^{(2)}) + (\Delta E_1 + i\Delta E_2)(u_1^{(2)} + iu_2^{(2)}) + (\Delta B_1 + i\Delta B_2)(x_1^{(3)} + ix_2^{(3)}) = r_1^{(1)} + ir_2^{(1)}, \tag{9}$$

where $(A - \lambda E)x^{(2)} + Bx^{(3)} = r^{(1)} = r_1^{(1)} + ir_2^{(1)}$. Now, after multiplication and comparing real and imaginary parts from equation (9) we get the following equations,

$$\begin{aligned} \Delta A_1 x_1^{(2)} - \Delta A_2 x_2^{(2)} + \Delta E_1 u_1^{(2)} - \Delta E_2 u_2^{(2)} + \Delta B_1 x_1^{(3)} - \Delta B_2 x_2^{(3)} &= r_1^{(1)}, & (10)\\ \Delta A_1 x_2^{(2)} + \Delta A_2 x_1^{(2)} + \Delta E_1 u_2^{(2)} + \Delta E_2 u_1^{(2)} + \Delta B_1 x_2^{(3)} + \Delta B_2 x_1^{(3)} &= r_2^{(1)}. & (11) \end{aligned}$$

Hence, equations (10) and (11) can be written as

$$\begin{bmatrix} \Delta A_1 & \Delta A_2 & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix}\begin{bmatrix} x_1^{(2)} \\ -x_2^{(2)} \\ u_1^{(2)} \\ -u_2^{(2)} \\ x_1^{(3)} \\ -x_2^{(3)} \end{bmatrix} = r_1^{(1)}, \tag{12}$$

$$\begin{bmatrix} \Delta A_1 & \Delta A_2 & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_2^{(2)} \\ x_1^{(2)} \\ u_2^{(2)} \\ u_1^{(2)} \\ x_2^{(3)} \\ x_1^{(3)} \end{bmatrix} = r_2^{(1)}. \tag{13}$$

Now, since we want to preserve the sparsity within the perturbed system then by using Remark 2.3 and Lemma 2.4 clockwise, we rewrite equations (12) and (13) as

$$\begin{bmatrix} \left(X_{x_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1))\right)^T \\ -\left(X_{x_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2))\right)^T \\ \left(X_{u_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1))\right)^T \\ -\left(X_{u_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2))\right)^T \\ \left(X_{x_1^{(3)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} B_1))\right)^T \\ -\left(X_{x_2^{(3)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} B_2))\right)^T \end{bmatrix}^T \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn}\, A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn}\, A_2) \\ \mathrm{vec}(\Delta E_1 \circ \mathrm{sgn}\, E_1) \\ \mathrm{vec}(\Delta E_2 \circ \mathrm{sgn}\, E_2) \\ \mathrm{vec}(\Delta B_1 \circ \mathrm{sgn}\, B_1) \\ \mathrm{vec}(\Delta B_2 \circ \mathrm{sgn}\, B_2) \end{bmatrix} = r_1^{(1)}, \tag{14}$$

$$\begin{bmatrix} \left(X_{x_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_1))\right)^T \\ \left(X_{x_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_2))\right)^T \\ \left(X_{u_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1))\right)^T \\ \left(X_{u_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2))\right)^T \\ \left(X_{x_2^{(3)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} B_1))\right)^T \\ \left(X_{x_1^{(3)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} B_2))\right)^T \end{bmatrix}^T \begin{bmatrix} \mathrm{vec}(\Delta A_1 \circ \mathrm{sgn}\, A_1) \\ \mathrm{vec}(\Delta A_2 \circ \mathrm{sgn}\, A_2) \\ \mathrm{vec}(\Delta E_1 \circ \mathrm{sgn}\, E_1) \\ \mathrm{vec}(\Delta E_2 \circ \mathrm{sgn}\, E_2) \\ \mathrm{vec}(\Delta B_1 \circ \mathrm{sgn}\, B_1) \\ \mathrm{vec}(\Delta B_2 \circ \mathrm{sgn}\, B_2) \end{bmatrix} = r_2^{(1)}. \tag{15}$$

From equation (7) we get,

$$(\Delta A_1^T - i\Delta A_2^T)(x_1^{(1)} + ix_2^{(1)}) - (\Delta E_1^T - i\Delta E_2^T)(u_1^{(1)} + iu_2^{(1)}) = r_1^{(2)} + ir_2^{(2)}, \tag{16}$$

where $r^{(2)} = r_1^{(2)} + ir_2^{(2)} = (A^* - \lambda E^*)x^{(1)} + Qx^{(2)} + Sx^{(3)}$. After multiplication and comparing both sides of equation (16) we get,

$$\begin{bmatrix} \Delta A_1^T & \Delta A_2^T & \Delta E_1^T & \Delta E_2^T & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ -u_1^{(1)} \\ -u_2^{(1)} \\ 0 \\ 0 \end{bmatrix} = r_1^{(2)}, \tag{17}$$

$$\begin{bmatrix} \Delta A_1^T & \Delta A_2^T & \Delta E_1^T & \Delta E_2^T & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_2^{(1)} \\ -x_1^{(1)} \\ -u_2^{(1)} \\ u_1^{(1)} \\ 0 \\ 0 \end{bmatrix} = r_2^{(2)}. \tag{18}$$

Now, using Remark 2.3 and Lemma 2.5, we write equations (17) and (18) as

$$\begin{bmatrix}\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0\end{bmatrix} \begin{bmatrix}\operatorname{vec}(\Delta A_1\circ\operatorname{sgn}A_1)\\ \operatorname{vec}(\Delta A_2\circ\operatorname{sgn}A_2)\\ \operatorname{vec}(\Delta E_1\circ\operatorname{sgn}E_1)\\ \operatorname{vec}(\Delta E_2\circ\operatorname{sgn}E_2)\\ \operatorname{vec}(\Delta B_1\circ\operatorname{sgn}B_1)\\ \operatorname{vec}(\Delta B_2\circ\operatorname{sgn}B_2)\end{bmatrix} = r_1^{(2)}, \tag{19}$$

$$\begin{bmatrix}\widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & \widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0\end{bmatrix} \begin{bmatrix}\operatorname{vec}(\Delta A_1\circ\operatorname{sgn}A_1)\\ \operatorname{vec}(\Delta A_2\circ\operatorname{sgn}A_2)\\ \operatorname{vec}(\Delta E_1\circ\operatorname{sgn}E_1)\\ \operatorname{vec}(\Delta E_2\circ\operatorname{sgn}E_2)\\ \operatorname{vec}(\Delta B_1\circ\operatorname{sgn}B_1)\\ \operatorname{vec}(\Delta B_2\circ\operatorname{sgn}B_2)\end{bmatrix} = r_2^{(2)}. \tag{20}$$

From equation (8) we get,

$$(\Delta B_1^T - i\Delta B_2^T)(x_1^{(1)} + ix_2^{(1)}) = r_1^{(3)} + ir_2^{(3)}, \tag{21}$$

where $B^*x^{(1)} + S^*x^{(2)} + Rx^{(3)} = r^{(3)} = r_1^{(3)} + ir_2^{(3)}$. After multiplication and comparing real and imaginary parts of equation (21). Now, using Remark 2.3 and Lemma 2.5 we can write equation (21) as

$$\begin{bmatrix}0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{sgn}B_2)\end{bmatrix} \begin{bmatrix}\operatorname{vec}(\Delta A_1\circ\operatorname{sgn}A_1)\\ \operatorname{vec}(\Delta A_2\circ\operatorname{sgn}A_2)\\ \operatorname{vec}(\Delta E_1\circ\operatorname{sgn}E_1)\\ \operatorname{vec}(\Delta E_2\circ\operatorname{sgn}E_2)\\ \operatorname{vec}(\Delta B_1\circ\operatorname{sgn}B_1)\\ \operatorname{vec}(\Delta B_2\circ\operatorname{sgn}B_2)\end{bmatrix} = r_1^{(3)}, \tag{22}$$

$$\begin{bmatrix}0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2))\end{bmatrix} \begin{bmatrix}\operatorname{vec}(\Delta A_1\circ\operatorname{sgn}A_1)\\ \operatorname{vec}(\Delta A_2\circ\operatorname{sgn}A_2)\\ \operatorname{vec}(\Delta E_1\circ\operatorname{sgn}E_1)\\ \operatorname{vec}(\Delta E_2\circ\operatorname{sgn}E_2)\\ \operatorname{vec}(\Delta B_1\circ\operatorname{sgn}B_1)\\ \operatorname{vec}(\Delta B_2\circ sgn\, B_2)\end{bmatrix} = r_2^{(3)}, \tag{23}$$

where we adjusted the size of the 0 matrices according to the size of the other matrices. Now for calculating the backward error according to the block perturbations, we combine the equations (14), (15), (19), (20), (22), and (23) and get the following system of equation

$$M\Delta = r. \tag{24}$$

Where M is given in Appendix A and $\Delta = \begin{bmatrix}\operatorname{vec}(\Delta A_1\circ\operatorname{sgn}A_1)\\ \operatorname{vec}(\Delta A_2\circ\operatorname{sgn}A_2)\\ \operatorname{vec}(\Delta E_1\circ\operatorname{sgn}E_1)\\ \operatorname{vec}(\Delta E_2\circ\operatorname{sgn}E_2)\\ \operatorname{vec}(\Delta B_1\circ\operatorname{sgn}B_1)\\ \operatorname{vec}(\Delta B_2\circ\operatorname{sgn}B_2)\end{bmatrix}$, and $r = \begin{bmatrix}r_1^{(1)}\\ r_2^{(1)}\\ r_1^{(2)}\\ r_2^{(2)}\\ r_1^{(3)}\\ r_2^{(3)}\end{bmatrix}$. Now, when M is a full row rank matrix or the system $M\Delta = r$ is consistent, then from equation (24), we get the minimum norm solution as $\Delta = M^{\dagger}r$ by Lemma 2.9. Backward error is given by

$$\inf |\!|\!|(\Delta A, \Delta E, \Delta B)|\!|\!| = \inf\|\Delta\|_F = \|M^{\dagger}r\|_F. \blacksquare$$

Note: We can construct minimum norm $\Delta A, \Delta E, \Delta B$ by extracting the elements from Δ.

Remark 3.3. *Since we have $x^{(2)} \in \mathbb{C}^n$ then we have $x_1^{(2)} \in \mathbb{C}^n$. Now, let $x_1^{(2)} = \left[x_{1,11}^{(2)}, \dots, x_{1,n1}^{(2)}\right]^T$ then we have*

$$X_{x_1^{(2)}} := \left[\begin{array}{cccc|cccc|c|cccc} x_{1,11}^{(2)} & x_{1,21}^{(2)} & \cdots & x_{1,n1}^{(2)} & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{1,11}^{(2)} & x_{1,21}^{(2)} & \cdots & x_{1,n1}^{(2)} & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & x_{1,11}^{(2)} & \cdots & \cdots & x_{1,n1}^{(2)} \end{array}\right]_{n\times n^2}.$$

Similarly, we get the other components of the block matrices.

Remark 3.4. *When M is a full row rank matrix, the rank of M automatically equals the rank of the augmented system for the equation $M\Delta = r$. If M is not a full row rank matrix but the rank of M is still equal to the rank of the augmented matrix, meaning the system is consistent, we also obtain a minimum norm solution.*

3.2. Perturbation on pencils arising in discrete-time linear quadratic optimal control problems

Let $L_d(z)$ be of the form (2) and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$. Let $\mathcal{B}$ be the collection of the pencils of the form $\Delta L_d(z) = \Delta M_d + z\Delta N_d$, where ΔM_d and ΔN_d are given by

$$\Delta M_d = \begin{bmatrix} 0 & \Delta A & \Delta B \\ -(\Delta E)^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \Delta N_d = \begin{bmatrix} 0 & \Delta E & 0 \\ -(\Delta A)^* & 0 & 0 \\ -(\Delta B)^* & 0 & 0 \end{bmatrix}. \tag{25}$$

If we are given $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$, where $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix} \in \mathbb{R}^{2n+m}$, where $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$, then we define eigenpair backward error for perturbation along the blocks by

$$\begin{aligned} \eta^{\mathcal{B}}(M_d, N_d, \lambda, x) = \inf\{ |\!|\!|(\Delta A, \Delta E, \Delta B)|\!|\!| \; | \Delta A, \Delta E \in \mathbb{C}^{n\times n}, \Delta B \in \mathbb{C}^{n\times m}, L_d \in \mathcal{B}, \\ (L_d(\lambda) - \Delta L_d(\lambda))x = 0\}, \end{aligned} \tag{26}$$

where $|\!|\!|(\Delta A, \Delta E, \Delta B)|\!|\!| = \sqrt{\|\Delta A\|_F^2 + \|\Delta E\|_F^2 + \|\Delta B\|_F^2}$ and $\|.\|_F$ denotes the Frobenious norm of a matrix. Now by the help of the next lemma we formulate the problem into a set of equations.

Lemma 3.5. *Let $L_d(z)$ a sparse matrix pencil of the form (2). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$. If $\Delta A, \Delta E \in \mathbb{C}^{n\times n}$ and $\Delta B \in \mathbb{C}^{n\times m}$, then there exists $\Delta L_d(z) = \Delta M_d + z\Delta N_d \in \mathcal{B}$ such that $(L_d(\lambda) - \Delta L_d(\lambda))x = 0$ if and only if*

$$(\Delta A + \lambda\Delta E)x^{(2)} + \Delta B x^{(3)} = r^{(1)}, \tag{27}$$

$$(-(\Delta E)^* - \lambda \Delta A^*)x^{(1)} = r^{(2)}, \tag{28}$$

$$-\lambda(\Delta B)^* x^{(1)} = r^{(3)}, \tag{29}$$

where $r^{(1)} := (A + \lambda E)x^{(2)} + Bx^{(3)}, r^{(2)} := (-E^* - \lambda A^*)x^{(1)} + Qx^{(2)} + Sx^{(3)}, r^{(3)} := -\lambda B^* x^{(1)} + S^* x^{(2)} + Rx^{(3)}$.

We can calculate the corresponding blockwise error using the above system of equations.

Theorem 3.6. *Let $L_d(z)$ be a sparse matrix pencil of the form (2). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_d. Then there exists minimum norm ΔL_d of the form (25) with sparsity such that $(L_d(\lambda) - \Delta L_d(\lambda))x = 0$, if $M\Delta = r$ is solvable and the backward error is given by $\eta^{\mathcal{B}}(H_d, N_d, \lambda, x) = \|M^\dagger r\|_F$, where M is defined in Appendix B $r = \begin{bmatrix} r_1^{(1)T} & r_2^{(1)T} & r_1^{(2)T} & r_2^{(2)T} & r_1^{(3)T} & r_2^{(3)T} \end{bmatrix}^T$ $r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}, r^{(3)} = r_1^{(3)} + ir_2^{(3)}$.*

Proof. The proof is similar to the proof method of Theorem 3.2.■

3.3. Port-Hamiltonian descriptor system in Control systems

In this section, we perturb different blocks of the port-Hamiltonian system and calculate its blockwise eigenpair backward error.

3.4. Perturbation on J, E

Let $L_p(z)$ be of the form (3) and $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$. We define eigenpair backward errors $\eta^{\mathcal{B}}(J, E, \lambda, x)$ and $\eta^{\mathcal{S}}(J, E, \lambda, x)$ by

$$\begin{aligned} \eta^{\mathcal{B}}(J, E, \lambda, x) &= \inf\{|\!|\!|(\Delta J, \Delta E)|\!|\!| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta M_p + z\Delta N_p \in \mathcal{B}\}, \\ \eta^{\mathcal{S}}(J, E, \lambda, x) &= \inf\{|\!|\!|(\Delta J, \Delta E)|\!|\!| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta M_p + z\Delta N_p \in \mathcal{S}\}, \end{aligned}$$

where $\eta^{\mathcal{B}}(J, E, \lambda, x)$ is blockwise and $\eta^{\mathcal{S}}(J, E, \lambda, x)$ is blockwise symmetry preserving backward error respectively. We set $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}$ where $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$. The set of all pencils $\Delta L_p(z) = \Delta M_p + z\Delta N_p$, with

$$\Delta M_p = \begin{bmatrix} 0 & \Delta J & 0 \\ \Delta J^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \Delta N_p = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{30}$$

is denoted by $\mathcal{B}$. When we have $\Delta J \in \mathrm{SHerm}(n)$, and $\Delta E \in \mathrm{Herm}(n)$ that set denoted by $\mathcal{S}$. Here we perturb the block matrices similar to the perturbation defined in [16]. Now, with the help of the next lemma, we formulate the problem into a set of equations.

Lemma 3.7. *Let $L_p(z)$ be of the form (3) and let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$. If $\Delta J, \Delta E \in \mathbb{C}^{n\times n}$, furthermore $\Delta L_p(z) = \Delta M_p + z\Delta N_p \in \mathcal{B}$, then $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$ if and only if*

$$\Delta J x^{(2)} + \lambda \Delta E x^{(2)} = r^{(1)}, \tag{31}$$

$$\Delta J^* x^{(1)} - \lambda \Delta E^* x^{(1)} = r^{(2)}, \tag{32}$$

$$B^* x^{(1)} + S x^{(3)} = 0, \tag{33}$$

where $r^{(1)} := (J - R + \lambda E)x^{(2)} + Bx^{(3)}$ and $r^{(2)} := (-J - R - \lambda E)x^{(1)} + Qx^{(2)}$.

Now, using the above system of equations, we can calculate the corresponding blockwise backward error.

Theorem 3.8. *Let $L_p(z)$ be a sparse matrix pencil of the form (3). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_p. Then there exists minimum norm ΔL_p of the form (30) with sparsity such that $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$, if $M\Delta = r$ is solvable and the backward error is given by $\eta^{\mathcal{B}}(J, E, \lambda, x) = \|M^\dagger r\|_F$, where*

$$M = \begin{bmatrix} X_{x_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_1)) & -X_{x_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_2)) & X_{u_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1)) & -X_{u_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2)) \\ X_{x_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_1)) & X_{x_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_2)) & X_{u_2^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1)) & X_{u_1^{(2)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2)) \\ \widetilde{X}_{x_1^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_1)) & \widetilde{X}_{x_2^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_2)) & -\widetilde{X}_{u_1^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1)) & -\widetilde{X}_{u_2^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2)) \\ \widetilde{X}_{x_2^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_1)) & -\widetilde{X}_{x_1^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} J_2)) & -\widetilde{X}_{u_2^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_1)) & \widetilde{X}_{u_1^{(1)}} \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} E_2)) \end{bmatrix}$$

and $r = \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_1^{(2)} \\ r_2^{(2)} \end{bmatrix}, r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}.$

Proof. The proof follows from the proof method of Theorem 3.2. ∎

The next theorem calculates the structured blocwise backward error.

Theorem 3.9. *Consider $L_p(z)$ of the form (3). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_p. Then there exists minimum norm ΔL_p, of the form (30) with sparsity if $M\Delta = r$ is solvable and $\eta^{\mathcal{S}}(J, E, \lambda, x) = \|M^\dagger r\|_F$, where M is defined in Appendix C and*

$$r = \begin{bmatrix} r_1^{(1)T} & r_2^{(1)T} & r_1^{(2)T} & r_2^{(2)T} \end{bmatrix}^T,$$

$r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}.$

Proof. Let $\Delta J = \Delta J_1 + i\Delta J_2$, and $\Delta E = \Delta E_1 + i\Delta E_2$. Then from, equation (31), we get,

$$(\Delta J_1 + i\Delta J_2)(x_1^{(2)} + ix_2^{(2)}) + (\Delta E_1 + i\Delta E_2)(u_1^{(2)} + iu_2^{(2)}) = r_1^{(1)} + ir_2^{(1)}, \tag{34}$$

where $r_1 := (J - R + \lambda E)x_2 + Bx_3$.

Now, after multiplication and comparing real and imaginary parts from equation (34) we get,

$$\begin{bmatrix}\Delta J_1 & \Delta J_2 & \Delta E_1 & \Delta E_2\end{bmatrix}\begin{bmatrix} x_1^{(2)} \\ -x_2^{(2)} \\ u_1^{(2)} \\ -u_2^{(2)} \end{bmatrix} = r_1^{(1)}, \tag{35}$$

$$\begin{bmatrix}\Delta J_1 & \Delta J_2 & \Delta E_1 & \Delta E_2\end{bmatrix}\begin{bmatrix} x_2^{(2)} \\ x_1^{(2)} \\ u_2^{(2)} \\ u_1^{(2)} \end{bmatrix} = r_2^{(1)}. \tag{36}$$

Since we need to arrange symmetric and skew-symmetric block structures, we first use Remark 2.3 then apply the combination of Lemmas 2.6 and 2.7. Using those Lemmas we write equations (35) and (36) as

$$\begin{bmatrix} X^{ss}_{x_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}J_2 \circ P_{s})) & X^{s}_{u_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2 \circ P_{ss}))\end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta J_1 \circ \mathrm{sgn}\, J_1 \circ Q_{ss}) \\ \mathrm{vec}(\Delta J_2 \circ \mathrm{sgn}\, J_2 \circ Q_{s}) \\ \mathrm{vec}(\Delta E_1 \circ \mathrm{sgn}\, E_1 \circ Q_{s}) \\ \mathrm{vec}(\Delta E_2 \circ \mathrm{sgn}\, E_2 \circ Q_{ss}) \end{bmatrix} = r_1^{(1)}, \tag{37}$$

$$\begin{bmatrix} X^{ss}_{x_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}J_1 \circ P_{ss})) & X^{s}_{x_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}J_2 \circ P_{s})) & X^{s}_{u_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1 \circ P_{s})) & X^{ss}_{u_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2 \circ P_{ss}))\end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta J_1 \circ \mathrm{sgn} J_1 \circ Q_{ss}) \\ \mathrm{vec}(\Delta J_2 \circ \mathrm{sgn}\, J_2 \circ Q_{s}) \\ \mathrm{vec}(\Delta E_1 \circ \mathrm{sgn}\, E_1 \circ Q_{s}) \\ \mathrm{vec}(\Delta E_2 \circ \mathrm{sgn}\, E_2 \circ Q_{ss}) \end{bmatrix} = r_2^{(1)}. \tag{38}$$

From equation (32) we get,

$$-(\Delta J_1 + i\Delta J_2)(x_1^{(1)} + ix_2^{(1)}) - (\Delta E_1 + i\Delta E_2)(u_1^{(1)} + iu_2^{(1)}) = r_1^{(2)} + ir_2^{(2)}, \tag{39}$$

where $r^{(2)} = r_1^{(2)} + ir_2^{(2)} = (-J - R - \lambda E)x^{(1)} + Qx^{(2)}$. After multiplication comparing both sides of equation (39) we get,

$$\begin{bmatrix}\Delta J_1 & \Delta J_2 & \Delta E_1 & \Delta E_2\end{bmatrix}\begin{bmatrix} -x_1^{(1)} \\ x_2^{(1)} \\ -u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = r_1^{(2)} \tag{40}$$

and

$$\begin{bmatrix}\Delta J_1 & \Delta J_2 & \Delta E_1 & \Delta E_2\end{bmatrix}\begin{bmatrix} -x_2^{(1)} \\ -x_1^{(1)} \\ -u_2^{(1)} \\ -u_1^{(1)} \end{bmatrix} = r_2^{(2)}. \tag{41}$$

Again Remark 2.3 and combining Lemmas 2.6 and 2.7 we can write equations (40) and (41) as

$$\begin{bmatrix} -X^{ss}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & X^{s}_{x_2^{(1)}}\operatorname{diag}(\operatorname{sgn}J_2 \circ \operatorname{vec}(P_{s})) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\Delta J_1 \circ \operatorname{sgn} J_1 \circ Q_{ss}) \\ \operatorname{vec}(\Delta J_2 \circ \operatorname{sgn} J_2 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_1 \circ \operatorname{sgn} E_1 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_2 \circ \operatorname{sgn} E_2 \circ Q_{ss}) \end{bmatrix} = r_1^{(2)} \quad (42)$$

$$\begin{bmatrix} -X^{ss}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\Delta J_1 \circ \operatorname{sgn} J_1 \circ Q_{ss}) \\ \operatorname{vec}(\Delta J_2 \circ \operatorname{sgn} J_2 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_1 \circ \operatorname{sgn} E_1 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_2 \circ \operatorname{sgn} E_2 \circ Q_{ss}) \end{bmatrix} = r_2^{(2)}. \quad (43)$$

Now, for calculating the backward errors we combine the equations (37), (38), (42), (43), we get

$$\begin{bmatrix} X^{ss}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & X^{s}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ X^{ss}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & X^{s}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ -X^{ss}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & X^{s}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ -X^{ss}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\Delta J_1 \circ \operatorname{sgn} J_1 \circ Q_{ss}) \\ \operatorname{vec}(\Delta J_2 \circ \operatorname{sgn} J_2 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_1 \circ \operatorname{sgn} E_1 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_2 \circ \operatorname{sgn} E_2 \circ Q_{ss}) \end{bmatrix} = \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_1^{(2)} \\ r_2^{(2)} \end{bmatrix}. \quad (44)$$

Let $\Delta = \begin{bmatrix} \operatorname{vec}(\Delta J_1 \circ \operatorname{sgn} J_1 \circ Q_{ss}) \\ \operatorname{vec}(\Delta J_2 \circ \operatorname{sgn} J_2 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_1 \circ \operatorname{sgn} E_1 \circ Q_{s}) \\ \operatorname{vec}(\Delta E_2 \circ \operatorname{sgn} E_2 \circ Q_{ss}) \end{bmatrix}$ and $r = \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_1^{(2)} \\ r_2^{(2)} \end{bmatrix}$, then by equation (44) we get $r = M\Delta$. Now when M is a full row rank matrix or the system $M\Delta = r$ is consistent then the minimum norm solution is given by $\Delta = M^{\dagger}r$. Backward error is given by

$$\inf|\!|\!|(\Delta J, \Delta E, \Delta B)|\!|\!|_F = \inf\|\Delta\|_F = \|M^{\dagger}r\|_F. \blacksquare$$

Remark 3.10. *Similar calculations can be made if we perturb the matrices R and E.*

3.5. Perturbation on J, R, E and B

Consider $L_p(z)$ of the form (3). Here we perturb the matrices J, R, E, and B of the block pencil $L_p(z)$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair. We define eigenpair backward errors $\eta^{\mathcal{B}}(J, R, E, B, \lambda, x)$ and $\eta^{\mathcal{S}}(J, R, E, B, \lambda, x)$ by

$$\begin{aligned} \eta^{\mathcal{B}}(J, R, E, B, \lambda, x) &= \inf\{|\!|\!|(\Delta J, \Delta R, \Delta E, \Delta B)|\!|\!| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{B}\} \\ \eta^{\mathcal{S}}(J, R, E, B, \lambda, x) &= \inf\{|\!|\!|(\Delta J, \Delta R, \Delta E, \Delta B)|\!|\!| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{S}\}, \end{aligned}$$

respectively, where $\eta^{\mathcal{B}}(J, R, E, B, \lambda, x)$ is blockwise and $\eta^{\mathcal{S}}(J, R, E, B, \lambda, x)$ is blockwise symmetry preserving. We set $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}$, where $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$. Let the collection of all pencils $\Delta L_p(z) = \Delta M_p + z\Delta N_p$, where

$$\Delta M_p = \begin{bmatrix} 0 & \Delta J - \Delta R & \Delta B \\ (\Delta J - \Delta R)^* & 0 & 0 \\ \Delta B^* & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_p = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (45)$$

is denoted by $\mathcal{B}$. When we have $\Delta J \in \mathrm{SHerm}(n)$, and $\Delta R, \Delta E \in \mathrm{Herm}(n)$. then the set is denoted by $\mathcal{S}$.

Now by applying next lemma we formulate the problem into a set of equations.

Lemma 3.11. *Consider $L_p(z)$ of the form (3), and let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$. If $\Delta B \in \mathbb{C}^{n,m}$, $\Delta J, \Delta R$ and $\Delta E \in \mathbb{C}^{n\times n}$, furthermore $\Delta L_p(z) = \Delta M_p + z\Delta N_p \in \mathcal{B}$, then $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$ if and only if*

$$\begin{aligned} \Delta A x^{(2)} + \lambda \Delta E x^{(2)} + \Delta B x^{(3)} &= r^{(1)}, && (46)\\ \Delta A^* x^{(1)} - \lambda \Delta E^* x^{(1)} &= r^{(2)}, && (47)\\ \Delta B^* x^{(1)} &= r^{(3)}, && (48) \end{aligned}$$

where $\Delta A := (\Delta J - \Delta R)$, $r^{(1)} := (J - R + \lambda E)x^{(2)} + Bx^{(3)}, r^{(2)} := (-J - R - \lambda E)x^{(1)} + Qx^{(2)}, r^{(3)} := B^* x^{(1)} + Sx^{(3)}$.

Now using the above system of equations we can calculate the corresponding blockwise error.

Theorem 3.12. *Let $L_p(z)$ be a sparse matrix pencil of the form (3). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_p. Then there exists minimum norm ΔL_p of the form (45) with sparsity such that $(L_P(\lambda) - \Delta L_p(\lambda))x = 0$ if $M\Delta = r$ is solvable and the backward error is given by $\eta^{\mathcal{S}}(J, R, E, B, \lambda, x) = \|M^{\dagger} r\|_F$, where M is given in Appendix D and $r = \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_1^{(2)} \\ r_2^{(2)} \\ r_1^{(3)} \\ r_2^{(3)} \end{bmatrix}$, $r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}, r^{(3)} = r_1^{(3)} + ir_2^{(3)}$.*

Proof. From, equation (46), we get,

$$(\Delta A_1 + i\Delta A_2)(x_1^{(2)} + ix_2^{(2)}) + (\Delta E_1 + i\Delta E_2)(u_1^{(2)} + iu_2^{(2)}) + (\Delta B_1 + i\Delta B_2)(x_1^{(3)} + ix_2^{(3)}) = r_1^{(1)} + ir_2^{(1)}, \tag{49}$$

where $r^{(1)} := (J - R + \lambda E)x^{(2)} + Bx^{(3)}$. Now, after multiplication and comparing real and imaginary parts from equation (49) we get,

$$\begin{bmatrix} \Delta A_1 & \Delta A_2 & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ -x_2^{(2)} \\ u_1^{(2)} \\ -u_2^{(2)} \\ x_1^{(3)} \\ -x_2^{(3)} \end{bmatrix} = r_1^{(1)}, \tag{50}$$

$$\begin{bmatrix} \Delta A_1 & \Delta A_2 & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_2^{(2)} \\ x_1^{(2)} \\ u_2^{(2)} \\ u_1^{(2)} \\ x_2^{(3)} \\ x_1^{(3)} \end{bmatrix} = r_2^{(1)}. \tag{51}$$

Here we need to arrange unstructured, symmetric, and skew-symmetric block structures, so firstly we use Remark 2.3 to introduce sgn function then apply a combination of Lemmas 2.4, 2.6, and 2.7. Using those

Lemmas we combine equations (50) and (51) as

$$\begin{bmatrix} \left(X_{x_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_1))\right)^T \\ -\left(X_{x_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_2))\right)^T \\ \left(X^s_{u_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1\circ P_s))\right)^T \\ -\left(X^{ss}_{u_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2\circ P_{ss}))\right)^T \\ \left(X_{x_1^{(3)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_1))\right)^T \\ -\left(X_{x_2^{(3)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_2))\right)^T \end{bmatrix}^T \begin{bmatrix} \mathrm{vec}(\Delta A_1\circ \mathrm{sgn}\,A_1) \\ \mathrm{vec}(\Delta A_2\circ \mathrm{sgn}\,A_2) \\ \mathrm{vec}(\Delta E_1\circ \mathrm{sgn}\,E_1\circ Q_s) \\ \mathrm{vec}(\Delta E_2\circ \mathrm{sgn}\,E_2\circ Q_{ss}) \\ \mathrm{vec}(\Delta B_1\circ \mathrm{sgn}\,B_1) \\ \mathrm{vec}(\Delta B_2\circ \mathrm{sgn}\,B_2) \end{bmatrix} = r_1^{(1)} \tag{52}$$

and

$$\begin{bmatrix} \left(X_{x_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_1))\right)^T \\ \left(X_{x_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_2))\right)^T \\ \left(X^s_{u_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1\circ P_s))\right)^T \\ \left(X^{ss}_{u_1^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2\circ P_{ss}))\right)^T \\ \left(X_{x_2^{(3)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_1))\right)^T \\ \left(X_{x_1^{(3)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_2))\right)^T \end{bmatrix}^T \begin{bmatrix} \mathrm{vec}(\Delta A_1\circ \mathrm{sgn}\,A_1) \\ \mathrm{vec}(\Delta A_2\circ \mathrm{sgn}\,A_2) \\ \mathrm{vec}(\Delta E_1\circ \mathrm{sgn}\,E_1\circ Q_s) \\ \mathrm{vec}(\Delta E_2\circ \mathrm{sgn}\,E_2\circ Q_{ss}) \\ \mathrm{vec}(\Delta B_1\circ \mathrm{sgn}\,B_1) \\ \mathrm{vec}(\Delta B_2\circ \mathrm{sgn}\,B_2) \end{bmatrix} = r_2^{(1)}. \tag{53}$$

From equation (47) we get,

$$(\Delta A_1^T - i\Delta A_2^T)(x_1^{(1)} + ix_2^{(1)}) - (\Delta E_1 + i\Delta E_2)(u_1^{(1)} + iu_2^{(1)}) = r_1^{(2)} + ir_2^{(2)}, \tag{54}$$

where $r^{(2)} = r_1^{(2)} + ir_2^{(2)} = (-J - R - \lambda E)x^{(1)} + Qx^{(2)}$. After multiplication and comparing both sides of equation (54) we get,

$$\begin{bmatrix} \Delta A_1^T & \Delta A_2^T & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ -u_1^{(1)} \\ u_2^{(1)} \\ 0 \\ 0 \end{bmatrix} = r_1^{(2)} \text{ and} \tag{55}$$

$$\begin{bmatrix} \Delta A_1^T & \Delta A_2^T & \Delta E_1 & \Delta E_2 & \Delta B_1 & \Delta B_2 \end{bmatrix} \begin{bmatrix} x_2^{(1)} \\ -x_1^{(1)} \\ -u_2^{(1)} \\ -u_1^{(1)} \\ 0 \\ 0 \end{bmatrix} = r_2^{(2)}. \tag{56}$$

Since we need to arrange blocks with transpose matrices, symmetric and skew-symmetric block matrices, we introduce sgn function using the logic given in Remark 2.3 then need to apply the combination of Lemmas 2.5, 2.6, 2.7. Using these Lemmas we combine equations (55) and (56) as

$$\begin{bmatrix} \widetilde{X}_{x_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_1)) & \widetilde{X}_{x_2^{(1)}\mathrm{diag}(\mathrm{sgn}A_2)} & -X^s_{u_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1\circ P_s)) & X^{ss}_{u_2^{(2)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2\circ P_{ss})) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A_1\circ \mathrm{sgn}A_1) \\ \mathrm{vec}(\Delta A_2\circ \mathrm{sgn}A_2) \\ \mathrm{vec}(\Delta E_1\circ \mathrm{sgn}E_1\circ Q_s) \\ \mathrm{vec}(\Delta E_2\circ \mathrm{sgn}E_2\circ Q_{ss}) \\ \mathrm{vec}(\Delta B_1\circ \mathrm{sgn}B_1) \\ \mathrm{vec}(\Delta B_2\circ \mathrm{sgn}B_2) \end{bmatrix} = r_1^{(2)}, \tag{57}$$

and

$$\begin{bmatrix}\widetilde{X}_{x_2^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_1)) & -\widetilde{X}_{x_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}A_2)) & -X^{s}_{u_2^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_1\circ P_s)) & -X^{ss}_{u_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}E_2\circ P_{ss})) & 0 & 0\end{bmatrix}$$
$$\begin{bmatrix}\mathrm{vec}(\Delta A_1\circ\mathrm{sgn}A_1)\\ \mathrm{vec}(\Delta A_2\circ\mathrm{sgn}A_2)\\ \mathrm{vec}(\Delta E_1\circ\mathrm{sgn}E_1\circ Q_s)\\ \mathrm{vec}(\Delta E_2\circ\mathrm{sgn}E_2\circ Q_{ss})\\ \mathrm{vec}(\Delta B_1\circ\mathrm{sgn}B_1)\\ \mathrm{vec}(\Delta B_2\circ\mathrm{sgn}B_2)\end{bmatrix} = r_2^{(2)}. \tag{58}$$

From equation (48) we get,

$$(\Delta B_1^T - i\Delta B_2^T)(x_1^{(1)} + ix_2^{(1)}) = r_1^{(3)} + ir_2^{(3)}, \tag{59}$$

where $B^*x^{(1)} + Sx^{(3)} = r^{(3)} = r_3^1 + ir_2^{(3)}$. After multiplication and comparing real and imaginary parts of equation (59) and using Lemma 2.5 we write

$$\begin{bmatrix}0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_2))\end{bmatrix}\begin{bmatrix}\mathrm{vec}(\Delta A_1\circ\mathrm{sgn}A_1)\\ \mathrm{vec}(\Delta A_2\circ\mathrm{sgn}A_2)\\ \mathrm{vec}(\Delta E_1\circ\mathrm{sgn}E_1\circ Q_s)\\ \mathrm{vec}(\Delta E_2\circ\mathrm{sgn}E_2\circ Q_{ss})\\ \mathrm{vec}(\Delta B_1\circ\mathrm{sgn}B_1)\\ \mathrm{vec}(\Delta B_2\circ\mathrm{sgn}B_2)\end{bmatrix} = r_1^{(3)}, \tag{60}$$

and

$$\begin{bmatrix}0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}B_2))\end{bmatrix}\begin{bmatrix}\mathrm{vec}(\Delta A_1\circ\mathrm{sgn}A_1)\\ \mathrm{vec}(\Delta A_2\circ\mathrm{sgn}A_2)\\ \mathrm{vec}(\Delta E_1\circ\mathrm{sgn}E_1\circ Q_s)\\ \mathrm{vec}(\Delta E_2\circ\mathrm{sgn}E_2\circ Q_{ss})\\ \mathrm{vec}(\Delta B_1\circ\mathrm{sgn}B_1)\\ \mathrm{vec}(\Delta B_2\circ\mathrm{sgn}B_2)\end{bmatrix} = r_2^{(3)}. \tag{61}$$

Combining equations (52), (53), (57), (58), (60), and (61) we get

$$M\Delta = r, \tag{62}$$

where $\Delta = \begin{bmatrix}\mathrm{vec}(\Delta A_1\circ\mathrm{sgn}A_1)\\ \mathrm{vec}(\Delta A_2\circ\mathrm{sgn}A_2)\\ \mathrm{vec}(\Delta E_1\circ\mathrm{sgn}E_1\circ Q_s)\\ \mathrm{vec}(\Delta E_2\circ\mathrm{sgn}E_2\circ Q_{ss})\\ \mathrm{vec}(\Delta B_1\circ\mathrm{sgn}B_1)\\ \mathrm{vec}(\Delta B_2\circ\mathrm{sgn}B_2)\end{bmatrix}$ and $r = \begin{bmatrix}r_1^{(1)}\\ r_2^{(1)}\\ r_1^{(2)}\\ r_2^{(2)}\\ r_1^{(3)}\\ r_2^{(3)}\end{bmatrix}$. Now, if M is a full row rank matrix or $M\Delta = r$ is consistent then, the minimum norm solution is given by $\Delta = M^{\dagger}r$. Backward error is given by

$$\inf\|(\Delta A, \Delta E, \Delta B)\|_F = \inf\|(\Delta J, \Delta R, \Delta E, \Delta B)\|_F = \|\Delta\|_F = \|M^{\dagger}r\|_F.$$

where $\Delta J = (\Delta A - \Delta A^*)/2$ and $\Delta R = -(\Delta A + \Delta A^*)/2$. ∎

Note: For calculation of $\eta^{\mathcal{B}}(J, R, E, B, \lambda, x)$ we can use the method used in Theorem 3.8.

3.6. Perturbation on J, E and B

Consider $L_p(z)$ of the form (3). Here we allow perturbation on the matrices J, E and B of the pencil L. Let $(\lambda, x) \in \mathbb{C}\times\mathbb{C}^{2n+m}\setminus\{0\}$ be a given approximate eigenpair. We define backward errors $\eta^{\mathcal{B}}(J, E, B, \lambda, x)$ and $\eta^{\mathcal{S}}(J, E, B, \lambda, x)$ by

$$\begin{aligned}\eta^{\mathcal{B}}(J, E, B, \lambda, x) &= \inf\{\|(\Delta J, \Delta E, \Delta B)\| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{B}\},\\ \eta^{\mathcal{S}}(J, E, B, \lambda, x) &= \inf\{\|(\Delta J, \Delta E, \Delta B)\| \mid ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{S}\},\end{aligned}$$

where $\eta^{\mathcal{B}}(J,E,B,\lambda,x)$ is blockwise and $\eta^{\mathcal{S}}(J,E,B,\lambda,x)$ is blockwise symmetry preserving backward error respectively. We set $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}$, where $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$. The set of all pencils of the form $\Delta L_p(z) = \Delta M_p + z\Delta N_p$ with

$$\Delta M_p = \begin{bmatrix} 0 & \Delta J & \Delta B \\ \Delta J^* & 0 & 0 \\ \Delta B^* & 0 & 0 \end{bmatrix} \text{ and } \Delta N_p = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{63}$$

is denoted by $\mathcal{B}$. When we have $\Delta J \in \mathrm{SHerm}(n)$, and $\Delta E \in \mathrm{Herm}(n)$ that set denoted by $\mathcal{S}$.

Lemma 3.13. *Let $L_p(z)$ be a sparce matrix pencil of the form (3), and $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$. If $\Delta B \in \mathbb{C}^{n,m}, \Delta J$ and $\Delta E \in \mathbb{C}^{n\times n}$, then there exists $\Delta L_p(z) = \Delta M + z\Delta N \in \mathcal{B}$, such that $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$ if and only if*

$$\Delta J x^{(2)} + \lambda \Delta E x^{(2)} + \Delta B x^{(3)} = r^{(1)}, \tag{64}$$

$$\Delta J^* x^{(1)} - \lambda \Delta E^* x^{(1)} = r^{(2)}, \tag{65}$$

$$\Delta B^* x^{(1)} = r^{(3)}, \tag{66}$$

where $r^{(1)} := (J - R + \lambda E)x^{(2)} + Bx^{(3)}, r^{(2)} := (-J - R - \lambda E)x^{(1)} + Qx^{(2)}$, and $r^{(3)} := B^* x^{(1)} + Sx^{(3)}$.

Theorem 3.14. *Let $L_p(z)$ be a sparse matrix pencil of the form (3). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_p. Then there exists minimum norm ΔL_p of the form (63) with sparsity such that $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$, if $M\Delta = r$ is solvable and backward error is given by $\eta^{\mathcal{S}}(J,E,B,\lambda,x) = \|M^\dagger r\|_F$, where $r = \begin{bmatrix} r_1^{(1)T} & r_2^{(1)T} & r_1^{(2)T} & r_2^{(2)T} & r_1^{(3)T} & r_2^{(3)} \end{bmatrix}^T$ with $r^{(1)} = r_1^{(1)} + ir_2^{(1)}, r^{(2)} = r_1^{(2)} + ir_2^{(2)}, r^{(3)} = r_1^{(3)} + ir_2^{(3)}$ and M is given in Appendix E.*

Proof. We can adopt the proof method of the strcuture block matrices Δ_J and Δ_E as given in Theorem 3.9 and ΔB can be rearranged using the proof method of Theorem 3.12.■

Remark 3.15. *When $\lambda \in i\mathbb{R}$ then equations (64), (65) and (66) can be rearranged as*

$$\begin{bmatrix} \Delta J + \lambda \Delta E & \Delta B \end{bmatrix} \begin{bmatrix} x^{(2)} \\ x^{(3)} \end{bmatrix} = r^{(1)}$$

and

$$\begin{bmatrix} \Delta J + \lambda \Delta E & \Delta B \end{bmatrix}^* x^{(1)} = \begin{bmatrix} (-J - R - \lambda E)x^{(1)}1 + Qx^{(2)} \\ B^* x^{(1)} + Sx^{(3)} \end{bmatrix} := s$$

Now, let $\Delta = \begin{bmatrix} \Delta J + \lambda \Delta E \end{bmatrix}$ and $y = \begin{bmatrix} x^{(2)} \\ x^{(3)} \end{bmatrix}$, then we get $\Delta y = r^{(1)}$, and $\Delta^* x^{(1)} = s$, which implies $y^* s = r^{(1)} x^{(1)}$, after simplification we get, $(x^{(2)})^* Q x^{(2)} + (x^{(3)})^* S x^{(3)} = 0$.

3.7. Perturbation on R, E and B

Consider $L_p(z)$ of the form (3). Here we perturb on the matrices R, E and B of the pencil L. Let $(\lambda, x) \in \mathbb{C}\times \in \mathbb{C}^{2n+m} \setminus \{0\}$ be a given approxiamate eigenpair. We define backward errors $\eta^{\mathcal{B}}(R,E,B,\lambda,x)$ and $\eta^{\mathcal{S}}(R,E,B,\lambda,x)$ by

$$\eta^{\mathcal{B}}(R,E,B,\lambda,x) = \inf\{\|(\Delta R, \Delta E, \Delta B)\| \,|\, ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{B}\},$$

$$\eta^{\mathcal{S}}(R,E,B,\lambda,x) = \inf\{\|(\Delta R, \Delta E, \Delta B)\| \,|\, ((M_p - \Delta M_p) + \lambda(N_p - \Delta N_p))x = 0, \Delta L_p \in \mathcal{S}\},$$

where $\eta^{\mathcal{B}}(R, E, B, \lambda, x)$ is blockwise and $\eta^{\mathcal{S}}(R, E, B, \lambda, x)$ is blockwise symmetry preserving backward error respectively. We set $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}$, where $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$. The the set of all pencils of the form $\Delta L_p(z) = \Delta M_p + z\Delta N_p$ with

$$\Delta M_p = \begin{bmatrix} 0 & -\Delta R & \Delta B \\ -\Delta R^* & 0 & 0 \\ \Delta B^* & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_p = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{67}$$

is denoted by $\mathcal{B}$. When we have $\Delta R \in \text{Herm}(n)$, and $\Delta E \in \text{Herm}(n)$. then the set is denoted by $\mathcal{S}$.

Lemma 3.16. *Let $L_p(z)$ be of the form (3), and let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$. If $\Delta B \in \mathbb{C}^{n,m}$, ΔR and $\Delta E \in \mathbb{C}^{n\times n}$, then there exists $\Delta L_p(z) = \Delta M_p + z\Delta N_p \in \mathcal{B}$ such that $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$ if and only if*

$$-\Delta R x^{(2)} + \lambda \Delta E x^{(2)} + \Delta B x^{(3)} = r^{(1)}, \tag{68}$$

$$-\Delta R^* x^{(1)} - \lambda \Delta E^* x^{(1)} = r^{(2)}, \tag{69}$$

$$\Delta B^* x^{(1)} = r^{(3)}, \tag{70}$$

where $r^{(1)} := (J - R + \lambda E)x^{(2)} + Bx^{(3)}$, $r^{(2)} := (-J - R - \lambda E)x^{(1)} + Qx^{(2)}$ and $r^{(3)} := B^* x^{(1)} + Sx^{(3)}$.

Now using the above system of equations we can calculate the corresponding blockwise error.

Theorem 3.17. *Let $L_p(z)$ be a sparse matrix pencil of the form (3). Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}$ be an approximate eigenpair of L_p. Then there exists minimum norm ΔL_p of the form (67) with sparsity such that $(L_p(\lambda) - \Delta L_p(\lambda))x = 0$, $M\Delta = r$ is solvable and backward error is given by $\eta^{\mathcal{S}}(R, E, B, \lambda, x) = \|M^\dagger r\|_F$, where M is given in Appendix F and $r = \begin{bmatrix} r_1^{(1)T} & r_2^{(1)T} & r_1^{(2)T} & , r_2^{(2)T} & r_1^{(3)T} & r_2^{(3)} \end{bmatrix}^T$, $r^{(1)} = r_1^{(1)} + ir_2^{(1)}$, $r^{(2)} = r_1^{(2)} + ir_2^{(2)}$, $r^{(3)} = r_1^{(3)} + ir_2^{(3)}$.*

Proof. The proof is similar to the proof method of Theorem 3.14.■

Remark 3.18. *When $\lambda \in i\mathbb{R}$ then equations (68), (69) and (70) can be rearranged as*

$$\begin{bmatrix} -\Delta R + \lambda \Delta E & \Delta B \end{bmatrix} \begin{bmatrix} x^{(2)} \\ x^{(3)} \end{bmatrix} = r^{(1)}$$

and

$$\begin{bmatrix} -\Delta R + \lambda \Delta E & \Delta B \end{bmatrix}^* x^{(1)} = \begin{bmatrix} (-J - R - \lambda E)x^{(1)} + Qx^{(2)} \\ B^* x^{(1)} + Sx^{(3)} \end{bmatrix} := s$$

Now, let $\Delta = \begin{bmatrix} -\Delta R + \lambda \Delta E \end{bmatrix}$ and $y = \begin{bmatrix} x^{(2)} \\ x^{(3)} \end{bmatrix}$, then we get $\Delta y = r^{(1)}$, and $\Delta^* x^{(1)} = s$, which implies $y^* s = (r^{(1)})^* x_1$, after simplification we get, $(x^{(2)})^* Q x^{(2)} + (x^{(3)})^* S x_3 = 0$.

Note: As we have discussed in the Introduction that if we want to perturb all the blocks we can do this using the same framework. We present this result in the next lemma.

Consider $L_c(z)$ of the form (1) and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$. Let $\mathcal{B}$ be the collection of the pencils of the form $\Delta L_c = \Delta M_c + z\Delta N_c$, where ΔM_c and ΔN_c are given by

$$\Delta M_c = \begin{bmatrix} 0 & \Delta A & \Delta B \\ \Delta A^* & \Delta Q & \Delta S \\ \Delta B^* & \Delta S^* & \Delta R \end{bmatrix} \text{ and } \Delta N_c = \begin{bmatrix} 0 & \Delta E & 0 \\ -\Delta E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

If we are given $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$, where $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix}$, $x^{(1)}, x^{(2)} \in \mathbb{C}^n$ and $x^{(3)} \in \mathbb{C}^m$, then we get the following lemma.

Lemma 3.19. *Let $L_c(z)$ be of the form (1), and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^{2n+m} \setminus \{0\})$. If $\Delta A, \Delta E \in \mathbb{C}^{n\times n}$ and $\Delta B \in \mathbb{C}^{n\times m}$, then there exists $\Delta L_c(z) = \Delta M_c + z\Delta N_c \in \mathcal{B}$ such that $(L_c(\lambda) - \Delta L_c(\lambda))x = 0$ if and only if*

$$(\Delta A + \lambda\Delta E)x^{(2)} + \Delta B x^{(3)} = r^{(1)}, \tag{72}$$
$$(\Delta A^* - \lambda\Delta E^*)x^{(1)} + \Delta Q x^{(2)} + \Delta S x^{(3)} = r^{(2)}, \tag{73}$$
$$\Delta B^* x^{(1)} + \Delta S^* x^{(2)} + \Delta R x^{(3)} = r^{(3)}, \tag{74}$$

where $r^{(1)} := (A + \lambda E)x^{(2)} + Bx^{(3)}, r^{(2)} := (A^* - \lambda E^*)x^{(1)} + Qx^{(2)} + Sx^{(3)}$, and $r^{(3)} := B^*x^{(1)} + S^*x^{(2)} + Rx^{(3)}$.

Now, we can further calculate the required backward errors by organizing the system of equations as per the block structures and using the combination of the Lemmas given in Section 2.

4. Numerical Illustrations

Here, we validate our theory through some numerical examples using MATLAB software. In this section, we compare our results with those from existing literature, illustrating this with several examples.

Example 4.1. *Consider a Port-Hamiltonian system where*

$$J = \begin{bmatrix} i & 2+i & 1+i \\ -2+i & 2i & 3+4i \\ -1+i & -3+4i & 3i \end{bmatrix}, R = \begin{bmatrix} 1 & 2+5i & -2+i \\ 2-5i & 2 & 3+5i \\ -2-i & 3-5i & -3 \end{bmatrix}, B = \begin{bmatrix} 1+i & 0 \\ 0 & i \\ 6 & 2-3i \end{bmatrix},$$

$$E = \begin{bmatrix} 12 & 1-i & 2+i \\ 1+i & 2 & i \\ 2-i & -i & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

and the matrix pencil $L_p(z)$ is of the form (3). Let us consider the perturbation on J, E, and B. Let (λ, x) be an approximate eigenpair of L, where $\lambda = -4 - 4i$ and, $x = [0.01 - 0.01i, -0.1 - 0.1i, 0.02 - 0.05i, 0.05 - 0.08i, 0.45 + 0.33i, -0.23 + 0.78i, 0.23 + 0.04i, -0.07 - 0.007i]^T$. Then by applying Theorem 3.14, we get M to be a full row rank matrix and the perturbed matrices are given by

$$\Delta J = \begin{bmatrix} 0.8059i & 0.1809 - 1.8546i & -0.4418 - 3.9551i \\ -0.1809 - 1.8546i & -7.9559i & -7.7487 + 5.1481i \\ 0.4418 - 3.9551i & 7.7487 + 5.1481i & 10.8433i \end{bmatrix},$$

$$\Delta B = \begin{bmatrix} 2.9596 + 3.1889i & 0 \\ 0 & -0.7405 + 0.5318i \\ 6.1971 - 4.5670i & 0.9118 + 1.7822i \end{bmatrix},$$

$$\Delta E = \begin{bmatrix} 0.4844 & -0.4893 + 0.5777i & -0.6819 - 1.0173i \\ -0.4893 - 0.5777i & -1.1819 & -0.7675 + 0.9210i \\ -0.6819 + 1.0173i & -0.7675 - 0.9210i & 2.1013 \end{bmatrix}$$

and the backward error is 22.1208. *In this example, we have preserved both the structure of the blocks and the sparsity within the block matrix B. In [16] the authors considered a special case where $Q = 0$ in (3). In contrast, our theory is applicable for any Q, as demonstrated through an example in our paper.*

Next, we consider various pairs (λ, x) and compare our results with those from previous structure-preserving literature, as shown in Table 1. We will compare Theorem 3.14 with Proposition 3.6 in [3] and Theorem 3.7 in [1].

Example 4.2. *Consider a Port-Hamiltonian system where*

$$J = \begin{bmatrix} i & 2+i & 1+i \\ -2+i & 2i & 3+4i \\ -1+i & -3+4i & 3i \end{bmatrix}, R = \begin{bmatrix} 5 & 3+4i & 1-2i \\ 3-4i & 5 & -1-2i \\ 1+2i & -1+2i & 1 \end{bmatrix}, B = \begin{bmatrix} 3+4i & 1-2i \\ 5 & -1-2i \\ -1+2i & 1 \end{bmatrix},$$

λ	$\eta(M_p, N_p, \lambda, x)$	$\eta^{\text{even}}(M_p, N_p, \lambda, x)$	$\eta^{S}(J, E, B, \lambda, x)$
-3+4i	2.6258	3.6982	30.7490
3.2-3.1i	2.5496	3.5844	28.2120
-0.65-0.2i	6.3758	8.8993	30.2778
4+5i	3.0540	4.3110	24.1087
-4-4i	0.5372	0.6925	22.1208

Table 1: Numerical examples suggest that the blockwise structured error is relatively larger than that of unstructured and componentwise structured backward error.

$$E = \begin{bmatrix} 12 & 1-i & 2+i \\ 1+i & 2 & i \\ 2-i & -i & 3 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and let the matrix pencil $L_p(z) = M_p + zN_p$ be of the form (3). Let us consider the perturbation only on J and E. Now consider different (λ, x) and compare our results with the previous structure preserving literature. Here we have taken x_1 and x_3 such that $B^*x^{(1)} + Sx^{(3)} = 0$. We now compare Theorems 3.8 and 3.9 with Theorems 4.3 and 4.6 in [16], as shown in Table 2.

λ	$\eta^{\mathcal{B}}(J, E, \lambda, x)$	$\eta^{S}(J, E, \lambda, x)$	$\eta^{\mathcal{B}}(J, E, \lambda, x)$ [16]	$\eta^{S}(J, E, \lambda, x)$[16]
i	9.5679	11.0604	9.5679	11.0717
2i	10.3078	11.6943	10.3078	11.6986
1+i	9.8718	12.6132	–	–

Table 2: Numerical examples suggest that by our framework we can find blockwise eigenpair backward error for any $\lambda \in \mathbb{C}$ and for any matrix Q.

Remark 4.3. *In Example 4.2, when we choose $\lambda \in i\mathbb{R}$, we observe that M is not a full row rank matrix. However, the rank of M is equal to the rank of the augmented matrix. Therefore, in this case, we obtain a minimum norm solution.*

Conclusion. We have considered the backward error analysis of an approximate eigenpair of blockwise structured matrix pencils that becomes an exact eigenpair of an appropriately minimal perturbed block matrix pencil. The obtained pencil preserves the structures of different blocks for the Frobenius norm. In application, we have discussed the different pencils arising in continuous-time linear quadratic optimal control problems, discrete-time linear quadratic optimal control, and port-Hamiltonian descriptor systems in control. We have also presented several numerical examples to illustrate our framework.

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Appendix A. M matrix for Theorem 3.2

$$M := \begin{bmatrix} X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & X_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & -X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & X_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0 \\ \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & \widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \end{bmatrix}$$

Appendix B. M matrix for Theorem 3.6

$$M := \begin{bmatrix} X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & X_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & -X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & X_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ -\widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & -\widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0 \\ -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & \widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -\widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1)) & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2)) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) \\ 0 & 0 & 0 & 0 & -\widetilde{X}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \end{bmatrix},$$

Appendix C. M matrix for Theorem 3.9

$$M := \begin{bmatrix} X^{ss}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & X^{s}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ X^{ss}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & X^{s}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & X_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ -X^{ss}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & X^{s}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \\ -X^{ss}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1 \circ P_{ss})) & -X^{s}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2 \circ P_{s})) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) \end{bmatrix}$$

Appendix D. M matrix for Theorem 3.12

$$M := \begin{bmatrix} X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & X^{s}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ X_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & X_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & X^{s}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) & X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) & 0 & 0 \\ \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_2)) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1 \circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2 \circ P_{ss})) & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \end{bmatrix}$$

Appendix E. *M* matrix for Theorem 3.14

$$M := \begin{bmatrix} X^{ss}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1\circ P_{ss})) & -X^{s}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2\circ P_{s})) & X^{s}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & -X^{ss}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ X^{ss}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1\circ P_{ss})) & X^{s}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2\circ P_{s})) & X^{s}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & X^{ss}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ -X^{ss}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1\circ P_{ss})) & X^{s}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2\circ P_{s})) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & 0 & 0 \\ -X^{ss}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_1\circ P_{ss})) & -X^{s}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}J_2\circ P_{s})) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \end{bmatrix}$$

Appendix F. *M* matrix for Theorem 3.17

$$M := \begin{bmatrix} -X^{s}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_1\circ P_{s})) & X^{ss}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_2\circ P_{ss})) & X_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & -X^{ss}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -X_{x_2^{(3)}}\operatorname{diag}(\operatorname{sgn}B_2) \\ -X^{s}_{x_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_1\circ P_{s})) & -X^{ss}_{x_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_2\circ P_{ss})) & X^{s}_{u_2^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & X^{ss}_{u_1^{(2)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & X_{x_2^{(3)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & X_{x_1^{(3)}}\operatorname{diag}(\operatorname{sgn}B_2) \\ -X^{s}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_1\circ P_{s})) & X^{ss}_{X_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_2\circ P_{ss})) & -X^{s}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & X^{ss}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & 0 & 0 \\ -X^{s}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_1\circ P_{s})) & -X^{ss}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}R_2\circ P_{ss})) & -X^{s}_{u_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_1\circ P_{s})) & -X^{ss}_{u_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}E_2\circ P_{ss})) & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \\ 0 & 0 & 0 & 0 & \widetilde{X}_{x_2^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_1)) & -\widetilde{X}_{x_1^{(1)}}\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}B_2)) \end{bmatrix}$$