



Pointwise bi-slant Riemannian maps ($\mathcal{PBSRM}$) from almost Hermitian manifolds

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Abstract. In 2022, the notion of pointwise slant Riemannian maps were introduced by Y. Gündüzalp and M. A. Akyol (Journal of Geometry and Physics, 179, 104589, 2022) as a natural generalization of slant Riemannian maps, slant Riemannian submersions, slant submanifolds. As a generalization of pointwise slant Riemannian maps and many subclasses notions, we introduce pointwise bi-slant Riemannian maps (briefly, $\mathcal{PBSRM}$) from almost Hermitian manifolds to Riemannian manifolds, giving a non-trivial (proper) example and investigate some properties of the map, we deal with their properties: the J-pluriharmonicity, the J-invariant, and the totally geodesicness of the map. Finally, we study some curvature relations in complex space form, involving Chen inequalities and Casorati curvatures for $\mathcal{PBSRM}$, respectively.

1. The first section

In differential geometry, it is useful to define appropriate maps in order to compare differentiable manifolds. In this respect, there are some important maps between manifolds such as isometric immersions, Riemannian submersions and Riemannian maps which are natural generalizations of isometric immersions and Riemannian submersions.

The notion of isometric immersions included many subclasses of submanifolds including important submanifolds of Kaehler manifolds. More precisely, holomorphic and totally real submanifolds were submanifolds examples of Kaehler manifolds. As a generalization of holomorphic and totally real submanifolds, slant submanifolds were introduced by B. Y. Chen in [18]. We recall that a submanifold M is called slant submanifold if for all non-zero vector X tangent to M the angle $\theta(X)$ between JX and T_pM is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM$.

In the 1889's, Casorati introduced Casorati curvature which is a very natural concept for regular surfaces in the three-dimensional Euclidean space in [17]. In a Riemannian manifold, this curvature is defined as the normalized square of the length of the second fundamental form, and it is well known that this is

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an extrinsic invariant. Afterwards, many geometers studied some optimal inequalities involving Casorati curvatures in various ambient spaces, for example see ([11], [12], [32], [33], [34], [58], [59], [64], [65]).

In the 1960's, B. O'Neill [41] and A. Gray [24] independently introduced Riemannian submersions. More precisely, a differentiable map $\pi : (M_1, g_1) \longrightarrow (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a Riemannian submersion if π_* is onto and it satisfies

$$g_{M_2}(\pi_*X_1, \pi_*X_2) = g_{M_1}(X_1, X_2) \quad (1)$$

for X_1, X_2 vector fields tangent to M_1 , where π_* denotes the derivative map. The theory is also very active research field not only in mathematics but also in mathematical physics. More precisely, some of them are the Yang-Mills theory ([14], [62]), the Kaluza-Klein theory ([15], [38]), supergravity and superstring theories ([39], [40]), etc.

In the 1990's, F. Etayo introduced the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds in [22] and B. Y. Chen and O. Garay studied this kind of submanifolds and investigated the geometrical characterizations in [21].

In the 1990's, B. Y. Chen established some inequalities between the main extrinsic (the squared mean curvature) and main intrinsic invariants (the scalar curvature and the Ricci curvature) of a submanifold in a real space form [19]. The author also established a relation between the Ricci curvature and the squared mean curvature for a submanifold [20]. For the inequalities, see: ([3, 4, 10, 36, 37, 56, 60, 61]).

In the 1992's, A. E. Fischer [31] defined the notion of Riemannian maps as a generalization of isometric immersions and Riemannian submersions. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For the geometry of Riemannian maps between various Riemannian manifolds and their applications in spacetime geometry, see: ([1, 2, 5–9, 23, 26–29, 42–44, 49–54]).

In the 2010's, B. Şahin introduced anti-invariant Riemannian submersions, semi-invariant Riemannian submersions and slant submersions from almost Hermitian manifolds to Riemannian manifolds as an analogue of anti-invariant submanifolds, semi-invariant submanifolds and slant submanifolds, respectively in [53]. Afterwards, as a natural generalization of slant submersions, the notion of hemi-slant submersions has defined by Taştan et.al in [57].

In the 2014's, J. W. Lee and B. Şahin defined the notion of pointwise slant submersions, as a generalization of slant submersions which can be seen analogue of pointwise slant submanifolds and obtained several basic results in this setting in [35]. More precisely, let σ be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2) . If, at each given point $p \in M_1$, the Wirtinger angle $\theta(X)$ between J_1X and the space $(\ker \sigma_*)_p$ is independent of the choice of the nonzero vector $X \in (\ker \sigma_*)$, then we say that σ is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on M_1 , which is called the slant function of the pointwise slant submersion. One can find many papers related to this notion see: ([45], [46], [47], [48]).

In [51], B. Şahin introduced slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic Riemannian maps and anti-invariant Riemannian maps, anti-invariant submanifolds, anti-invariant Riemannian submersions, slant submanifolds, slant submersions, then he studied the geometry of such maps. As a generalization of these notions, he also defined the notion of hemi-slant Riemannian maps in [55].

In 2022, the present authors [26] introduced the notion of pointwise slant Riemannian maps as a generalization of many notions including slant submanifolds, slant Riemannian submersions, slant Riemannian maps, pointwise slant submanifolds, pointwise slant submersions. The aim of the present paper is to introduce and study a new class of Riemannian maps are called pointwise bi-slant Riemannian maps (briefly, $\mathcal{PBSRM}$) as a generalization of many concepts mentioned in the abstract.

The paper is structured as follows. In Section 2 we recall some notions, which will be used in the following sections. In Section 3 we define the notion of $\mathcal{PBSRM}$ from almost Hermitian manifolds to Riemannian manifolds, giving a non-trivial (proper) example and investigate some properties of the map, we deal with their properties: the J -pluriharmonicity of $\mathcal{PBSRM}$, the J -invariant of $\mathcal{PBSRM}$ and the totally geodesic maps of $\mathcal{PBSRM}$. In Section 5 we study some curvature relations in complex space form, involving Chen inequalities and Casorati curvatures for $\mathcal{PBSRM}$, respectively.

2. Preliminaries

In this section, recall some basic materials from [13, 31, 55, 63].

A $2n$ -dimensional Riemannian manifold (M_1, g_1, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on M such that $J^2 = -I$ and

$$g_1(X, Y) = g_1(JX, JY), \quad \forall X, Y \in \Gamma(TM_1), \quad (2)$$

where I denotes the identity transformation of $T_p M_1$. Consider an almost Hermitian manifold (M_1, g_1, J) and denote by ∇ the Levi-Civita connection on M_1 with respect to g_1 . Then M_1 is called a Kaehler manifold [63] if J is parallel with respect to ∇ , i.e.

$$(\nabla_X J)Y = 0, \quad (3)$$

$\forall X, Y \in \Gamma(TM_1)$.

As a generalization of isometric immersions and Riemannian submersions, the notion of Riemannian maps was defined by Fischer in [31] as follows;

Let σ be a C^∞ -map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . The second fundamental form of σ is given by

$$(\nabla \sigma_*)(X, Y) = \nabla_X^\sigma \sigma_* Y - \sigma_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM_1), \quad (4)$$

where ∇^σ is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_1 and g_2 [13].

We call the map σ a totally geodesic map if $(\nabla \sigma_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM_1)$ [13].

Denote the range of σ_* by $\text{range } \sigma_*$ as a subset of the pullback bundle $\sigma^{-1}TM_2$. With its orthogonal complement $(\text{range } \sigma_*)^\perp$ we obtain the following decomposition

$$\sigma^{-1}TM_2 = \text{range } \sigma_* \oplus (\text{range } \sigma_*)^\perp.$$

Moreover, we have

$$TM_1 = \ker \sigma_* \oplus (\ker \sigma_*)^\perp.$$

Finally, B. Şahin proved the following lemma in [49].

Theorem 2.1. [49] Let σ be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then

$$(\nabla \sigma_*)(X, Y) \in \Gamma((\text{range } \sigma_*)^\perp) \quad \text{for } X, Y \in \Gamma((\ker \sigma_*)^\perp). \quad (5)$$

Let σ be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then, we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{T}_{\xi_1} \xi_2 = h \nabla_{v \xi_1} v \xi_2 + v \nabla_{v \xi_1} h \xi_2 \quad (6)$$

and

$$\mathcal{A}_{\xi_1} \xi_2 = v \nabla_{h \xi_1} h \xi_2 + h \nabla_{h \xi_1} v \xi_2 \quad (7)$$

for every $\xi_1, \xi_2 \in \Gamma(TM_1)$, where ∇ is the Levi-Civita connection of g_1 . In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $\xi_1 \in \Gamma(TM_1)$, $\mathcal{T}_{\xi_1}$ and $\mathcal{A}_{\xi_1}$ are skew-symmetric operators on $(\Gamma(TM_1), g_1)$ reversing the horizontal and the vertical distributions. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_{\eta_1} \eta_2 = \mathcal{T}_{\eta_2} \eta_1, \quad \mathcal{A}_{\xi_1} \xi_2 = -\mathcal{A}_{\xi_2} \xi_1, \quad \forall \eta_1, \eta_2 \in \ker \sigma_*, \forall \xi_1, \xi_2 \in (\ker \sigma_*)^\perp. \quad (8)$$

Using (6) and (7), we obtain

$$\nabla_{\eta_1} \eta_2 = \mathcal{T}_{\eta_1} \eta_2 + \hat{\nabla}_{\eta_1} \eta_2; \quad (9)$$

$$\nabla_{\eta_1} \xi_1 = \mathcal{T}_{\eta_1} \xi_1 + h \nabla_{\eta_1} \xi_1; \quad (10)$$

$$\nabla_{\xi_1} \eta_1 = \mathcal{A}_{\xi_1} \eta_1 + v \nabla_{\xi_1} \eta_1; \quad (11)$$

$$\nabla_{\xi_1} \xi_2 = \mathcal{A}_{\xi_1} \xi_2 + h \nabla_{\xi_1} \xi_2, \quad (12)$$

for any $\xi_1, \xi_2 \in \Gamma((\ker \sigma_*)^\perp)$, $\eta_1, \eta_2 \in \Gamma(\ker \sigma_*)$, here $\hat{\nabla}_{\eta_1} \eta_2 = v \nabla_{\eta_1} \eta_2$.

3. $\mathcal{PBSRM}$ from Kaehler Manifolds

In this section, we are going to introduce pointwise bi-slant Riemannian maps (briefly, $\mathcal{PBSRM}$) from almost Hermitian manifolds to Riemannian manifolds, provide some examples and investigate the geometry of foliations and their geometric properties. We first obtain necessary and sufficient conditions for the image of φ_* to be a local product Riemannian manifold and give necessary and sufficient conditions for φ to be totally geodesic. Finally, we give some theorems on the harmonicity of the $\mathcal{PBSRM}$ maps.

Definition 3.1. Let (M_1, g_1, J) be an almost Hermitian manifold and (M_2, g_2) be a Riemannian manifold. Then we say that a Riemannian map $\varphi : M_1 \rightarrow M_2$ is a pointwise bi-slant Riemannian map ($\mathcal{PBSRM}$) if there exists a pair of orthogonal distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ on $\ker \varphi_*$ such that

1. The space $\ker \varphi_*$ admits the orthogonal direct decomposition $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$.
2. $J(\mathcal{D}^{\theta_1}) \perp \mathcal{D}^{\theta_2}$ and $J(\mathcal{D}^{\theta_2}) \perp \mathcal{D}^{\theta_1}$,
3. The distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are pointwise slant with slant functions θ_1 and θ_2 , respectively.

In this case, the angle θ_i , $i = 1, 2$ can be regarded as a functions on M_1 , which is called the bi-slant function of the $\mathcal{PBSRM}$.

We now give a non-trivial example for $\mathcal{PBSRM}$.

Example 3.2. Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J_1, J_2\}$ a pair of almost complex structures on $\mathbb{R}^8$ and $J_1 J_2 = -J_2 J_1$, here

$$J_1(l_1, \dots, l_8) = (-l_3, -l_4, l_1, l_2, -l_7, -l_8, l_5, l_6)$$

and

$$J_2(l_1, \dots, l_8) = (-l_2, l_1, l_4, -l_3, -l_6, l_5, l_8, -l_7).$$

For any function $f : \mathbb{R}^8 \rightarrow \mathbb{R}$, we describe almost complex structure J_f on $\mathbb{R}^8$ by $J_f = (\cos f)J_1 + (\sin f)J_2$. Then $\mathbb{R}_f^8 = (\mathbb{R}^8, J_f, g_{\mathbb{R}^8})$ is an almost Hermitian manifold.

$\Psi : \mathbb{R}_f^8 \rightarrow \mathbb{R}^8$ by

$$\Psi(x_1, \dots, x_8) = (0, 0, x_3, x_4, 0, 0, x_6, x_8).$$

Ψ is a proper $\mathcal{PBSRM}$ with the bi-slant functions $\theta_1 = \arccos \sin f$ and $\theta_2 = f$ such that

$$\mathcal{D}^{\theta_1} = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle, \text{ and } \mathcal{D}^{\theta_2} = \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} \right\rangle.$$

Also, we obtain

$$(\ker \varphi_*)^\perp = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \right\rangle,$$

here $x_1, \dots, x_8$ are the local coordinates on $\mathbb{R}^8$.

Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then for any $V \in \Gamma(\ker \varphi_*)$, we put

$$JV = \phi V + \omega V, \quad (13)$$

where $\phi V \in \Gamma(\ker \varphi_*)$ and $\omega V \in \Gamma(\ker \varphi_*)^\perp$. Also for any $X \in (\ker \varphi_*)^\perp$, we have

$$JX = \mathcal{B}X + CX, \quad (14)$$

where $\mathcal{B}X \in \Gamma(\ker \varphi_*)$ and $CX \in \Gamma(\ker \varphi_*)^\perp$.

The proof of the following result is exactly the same as that for slant immersions (see [18] or [16] for Sasakian case), so we omit its proof.

Theorem 3.3. *Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then φ is a $\mathcal{PBSRM}$ if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$\phi^2 U = \lambda U \quad (15)$$

for $U \in \Gamma(\mathcal{D}^{\theta_i}), i = 1, 2$. If φ is a $\mathcal{PBSRM}$, then $\lambda = -\cos^2 \theta_i, i = 1, 2$.

By using the above theorem, it is easy to see that

$$\begin{aligned} g_2(\phi \varphi_*(U), \phi \varphi_*(V)) &= \cos^2 \theta_i g_1(U, V), \\ g_2(\omega \varphi_*(U), \omega \varphi_*(V)) &= \sin^2 \theta_i g_1(U, V), \end{aligned}$$

for any $U, V \in \Gamma(\mathcal{D}^{\theta_i}), i = 1, 2$.

Now, we are going to investigate the J -pluriharmonicity of the $\mathcal{PBSRM}$ with respect to the distributions on the total space. First, we have the following definition.

Definition 3.4. *Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function $\theta_i, i = 1, 2$. A $\mathcal{PBSRM}$ is called J -pluriharmonic, $(\ker \varphi_*)^\perp$ - J -pluriharmonic, $\ker \varphi_*$ - J -pluriharmonic, $\mathcal{D}^{\theta_1}$ - J -pluriharmonic, $\mathcal{D}^{\theta_2}$ - J -pluriharmonic and $((\ker \varphi_*)^\perp - \ker \varphi_*)$ - J -pluriharmonic if*

$$(\nabla \varphi_*)(X, Y) + (\nabla \varphi_*)(JX, JY) = 0 \quad (16)$$

for any $X, Y \in \Gamma(TM_1)$, for any $X, Y \in \Gamma((\ker \varphi_*)^\perp)$, for any $X, Y \in \Gamma(\ker \varphi_*)$, for any $X, Y \in \Gamma(\mathcal{D}^{\theta_i}), i = 1, 2$ for any $X \in \Gamma((\ker \varphi_*)^\perp), Y \in \Gamma(\ker \varphi_*)$.

We first have the following theorem.

Theorem 3.5. *Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ_1 . Suppose that the map φ is a $\mathcal{D}^{\theta_1}$ - J -pluriharmonic. Then the map φ is a $\omega \mathcal{D}^{\theta_1}$ -geodesic map if and only if $\mathcal{T}_U V + \mathcal{T}_{\phi U} \phi V + \mathcal{H} \nabla_{\phi V} \omega W + \mathcal{A}_{\omega V} \phi W = 0$.*

Proof. Given $U, V \in \Gamma(\mathcal{D}^{\theta_1})$, since $\mathcal{D}^{\theta_1}$ - J -pluriharmonic, by virtue of (4) we obtain

$$\begin{aligned} 0 &= (\nabla \varphi_*)(V, W) + (\nabla \varphi_*)(JV, JW) \\ &= -\varphi_*(\mathcal{T}_V W) + (\nabla \varphi_*)(\omega V, \omega W) - \varphi_*(\mathcal{T}_{\phi V} \phi W + \mathcal{H} \nabla_{\phi V} \omega W + \mathcal{A}_{\omega V} \phi W) \\ (\nabla \varphi_*)(\omega V, \omega W) &= -\varphi_*(\mathcal{T}_V W + \mathcal{T}_{\phi V} \phi W + \mathcal{H} \nabla_{\phi V} \omega W + \mathcal{A}_{\omega V} \phi W) \end{aligned}$$

which completes the proof. $\square$

In a similar way, we can obtain the above theorem for $\mathcal{D}^{\theta_2}$. Now, for $((\ker \varphi_*)^\perp - \ker \varphi_*)$ - J -pluriharmonicity, we have the following theorem.

Theorem 3.6. Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_i, i = 1, 2$. Suppose that the map φ is a $(\ker\varphi_*)^\perp$ - $\ker\varphi_*$ - J -pluriharmonic. Then the following assertions are equivalent:

- (i) The horizontal distribution $(\ker\varphi_*)^\perp$ defines a totally geodesic foliations on M_1 .
- (ii) $(\nabla\varphi_*)(CX, \omega U) = -\varphi_*(\mathcal{T}_{\mathcal{B}X}\phi U + \mathcal{H}\nabla_{\mathcal{B}X}\omega U + \mathcal{A}_{CX}\phi U)$

for any $X \in \Gamma(\ker\varphi_*)^\perp$ and $U \in \Gamma(\ker\varphi_*)$.

Proof. For $X \in \Gamma(\ker\varphi_*)^\perp$ and $U \in \Gamma(\ker\varphi_*)$, since the map φ is a $((\ker\varphi_*)^\perp - \ker\varphi_*)$ - J -pluriharmonic, by using (4), we get

$$\begin{aligned} 0 &= (\nabla\varphi_*)(X, U) + (\nabla\varphi_*)(JX, JU) \\ &= -\varphi_*(\nabla_X U) + (\nabla\varphi_*)(\mathcal{B}X, \phi U) + (\nabla\varphi_*)(\mathcal{B}X, \omega U) \\ &\quad + (\nabla\varphi_*)(CX, \phi U) + (\nabla\varphi_*)(CX, \omega U) \\ &= -\varphi_*(\nabla_X U) - \varphi_*(\mathcal{T}_{\mathcal{B}X}\phi U) - \varphi_*(\mathcal{H}\nabla_{\mathcal{B}X}\omega U) \\ &\quad - \varphi_*(\mathcal{A}_{CX}\phi U) + (\nabla\varphi_*)(CX, \omega U) \\ (\nabla\varphi_*)(CX, \omega U) &= -\varphi_*(\nabla_X U) - \varphi_*(\mathcal{T}_{\mathcal{B}X}\phi U + \mathcal{H}\nabla_{\mathcal{B}X}\omega U + \mathcal{A}_{CX}\phi U) \end{aligned}$$

which completes the proof. $\square$

Finally, we will find necessary and sufficient conditions for the $\mathcal{PBSRM}$ to be the J -invariant of the distributions on the total space. First, we have the following definition.

Definition 3.7. Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function $\theta_i, i = 1, 2$. A $\mathcal{PBSRM}$ is called J -invariant, $(\ker\varphi_*)^\perp$ - J -invariant, $\ker\varphi_*$ - J -invariant, $\mathcal{D}^{\theta_1}$ - J -invariant, $\mathcal{D}^{\theta_2}$ - J -invariant and $((\ker\varphi_*)^\perp - \ker\varphi_*)$ - J -invariant if

$$(\nabla\sigma_*)(Z, W) = (\nabla\sigma_*)(JZ, JW) \quad (17)$$

for any $Z, W \in \Gamma(TM_1)$, for any $Z, W \in \Gamma((\ker\varphi_*)^\perp)$, for any $Z, W \in \Gamma(\ker\varphi_*)$, for any $Z, W \in \Gamma(\mathcal{D}^{\theta_1})$, for any $Z, W \in \Gamma(\mathcal{D}^{\theta_2})$, for any $Z \in \Gamma((\ker\varphi_*)^\perp)$, $W \in \Gamma(\ker\varphi_*)$.

We first have the following theorem.

Theorem 3.8. Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ_1 . Suppose map φ is a $\mathcal{D}^{\theta_1}$ - J -invariant. The following assertions are equivalent.

- (i) The fibres are totally geodesic submanifolds in M_1 .
- (ii) $\nabla\varphi_*(\omega U, \omega V) = \varphi_*(\mathcal{T}_{\phi U}\phi V + \mathcal{H}\nabla_{\phi U}\omega V - \mathcal{A}_{\omega U}\phi V)$

for any $U, V \in \Gamma(\mathcal{D}^{\theta_1})$.

Proof. Given $U, V \in \Gamma(\mathcal{D}^{\theta_1})$, since $\mathcal{D}^{\theta_1}$ - J -invariant, by virtue of (4), we obtain

$$\begin{aligned} (\nabla\varphi_*)(U, V) &= (\nabla\varphi_*)(JU, JV) \\ -\varphi_*(\nabla_U V) &= (\nabla\varphi_*)(\phi U, \phi V) + (\nabla\varphi_*)(\phi U, \omega V) + (\nabla\varphi_*)(\omega U, \phi V) + (\nabla\varphi_*)(\omega U, \omega V) \\ -\varphi_*(\nabla_U V) &= -\varphi_*(\nabla_{\phi U}\phi V) - \varphi_*(\nabla_{\phi U}\omega V) - \varphi_*(\nabla_{\omega U}\phi V) - \varphi_*(\nabla_{\omega U}\omega V) \\ -\varphi_*(\nabla_U V) &= -\varphi_*(\mathcal{T}_{\phi U}\phi V + \mathcal{H}\nabla_{\phi U}\omega V - \mathcal{A}_{\omega U}\phi V) - \varphi_*(\nabla_{\omega U}\omega V). \end{aligned}$$

which completes the proof. $\square$

In a similar way, we can obtain the above theorem for $\mathcal{D}^{\theta_2}$. Finally, for $((\ker\varphi_*)^\perp - \ker\varphi_*)$ - J -invariant, we have the following theorem.

Theorem 3.9. Let φ be a $\mathcal{PBSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function $\theta_i, i = 1, 2$. If the map φ is a $((\ker \varphi_*)^\perp - \ker \varphi_*)$ - J -invariant if and only if $C(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{CX}U) + \omega(\hat{\nabla}_{\mathcal{B}X}U + \mathcal{V}\nabla_{CX}U) + \mathcal{A}_XU = 0$ for any $X \in \Gamma(\ker \varphi_*)^\perp$ and $U \in \Gamma(\ker \varphi_*)$.

Proof. Given $X \in \Gamma(\ker \varphi_*)^\perp$ and $U \in \Gamma(\ker \varphi_*)$. We assume that the map is invariant. In this case, by virtue of (4) we have

$$\begin{aligned} (\nabla \varphi_*)(X, U) &= (\nabla \varphi_*)(JX, JU) \\ -\varphi_*(\nabla_X U) &= (\nabla \varphi_*)(\mathcal{B}X, JU) + (\nabla \varphi_*)(CX, JU) \\ -\varphi_*(\nabla_X U) &= -\varphi_*(\nabla_{\mathcal{B}X}JU) - \varphi_*(\nabla_{CX}JU) \\ -\varphi_*(\nabla_U V) &= -\varphi_*(J(\mathcal{T}_{\mathcal{B}X}U + \nabla_{\mathcal{B}X}U) + J(\mathcal{A}_{CX}U + \mathcal{V}\nabla_{CX}U)) \\ 0 &= \varphi_*(C(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{CX}U) + \omega(\hat{\nabla}_{\mathcal{B}X}U + \mathcal{V}\nabla_{CX}U + \mathcal{A}_XU)) \end{aligned}$$

which completes the proof. $\square$

Recall that a map φ is called totally geodesic if $(\nabla \varphi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM_1)$. Geometrically the notion implies that for each geodesic β in M_1 the image $\varphi(\beta)$ is a geodesic in M_2 .

Theorem 3.10. Let φ be a $\mathcal{PBSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then σ is totally geodesic if and only if

$$\begin{aligned} \sin 2\theta U(\theta)W + \mathcal{H}\nabla_U \omega \phi W + C\mathcal{H}\nabla_U \omega W + \omega \mathcal{T}_U \omega W &= 0 \\ \sin 2\theta X(\theta)W + \mathcal{H}\nabla_X \omega \phi W + C\mathcal{H}\nabla_X \omega W + \omega \mathcal{A}_X \omega W &= 0 \end{aligned}$$

and

$$\nabla_X^\varphi \varphi_*(Y) = -\varphi_*(\mathcal{A}_X \phi \mathcal{B}Y + \mathcal{H}\nabla_X \omega \mathcal{B}Y) + C\mathcal{H}\nabla_X CY + \omega \mathcal{A}_X CY$$

for $U, V \in \Gamma(\ker \varphi_*)$, $Z \in \Gamma(\mathcal{D}^{\theta_1})$ and $X, Y \in \Gamma(\ker \varphi_*)^\perp$.

Proof. For $U \in \Gamma(\ker \varphi_*)$ and $W \in \Gamma(\mathcal{D}^{\theta_1})$, (4), (3) and (13) imply

$$(\nabla \varphi_*)(U, W) = \varphi_*(\nabla_U \phi^2 W + \nabla_U \omega \phi W + \omega \mathcal{T}_U \omega W + C\mathcal{H}\nabla_U \omega W).$$

Then by using (15), we derive

$$\sin^2 \theta (\nabla \varphi_*)(U, W) = \varphi_*(\sin 2\theta U(\theta)W + \mathcal{H}\nabla_U \omega \phi W + C\mathcal{H}\nabla_U \omega W + \omega \mathcal{T}_U \omega W). \quad (18)$$

In a similar way, for $X \in \Gamma((\ker \varphi_*)^\perp)$ and $W \in \Gamma(\mathcal{D}^{\theta_1})$, we obtain

$$\sin^2 \theta (\nabla \varphi_*)(X, W) = \varphi_*(\sin 2\theta X(\theta)W + \mathcal{H}\nabla_X \omega \phi W + C\mathcal{H}\nabla_X \omega W + \omega \mathcal{A}_X \omega W). \quad (19)$$

For $X, Y \in \Gamma((\ker \varphi_*)^\perp)$, from (4), (3) and (11), we have

$$\begin{aligned} (\nabla \varphi_*)(X, Y) &= \nabla_X^\varphi \varphi_*(Y) + \varphi_*(\nabla_X J \mathcal{B}Y) + \varphi_*(J \nabla_X CY) \\ &= \nabla_X^\varphi \varphi_*(Y) + \varphi_*(\mathcal{A}_X \phi \mathcal{B}Y + \mathcal{H}\nabla_X \omega \mathcal{B}Y + C\mathcal{H}\nabla_X CY + \omega \mathcal{A}_X CY). \end{aligned} \quad (20)$$

Thus proof is complete due to (18)-(20). $\square$

4. Chen-Ricci inequality of $\mathcal{PBSRM}$

In the present section, we aim to obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for $\mathcal{PBSRM}$ from a Kaehler manifold to a Riemannian manifold. We also consider the equality cases of these inequalities.

$$\begin{aligned}\hat{R}(U, V, F, W) &= \frac{c}{4} \{g_1(V, F)g_1(U, W) - g_1(U, F)g_1(V, W) \\ &\quad + g_1(U, JF)g_1(JV, W) - g_1(V, JF)g_1(JU, W) \\ &\quad + 2g_1(U, JV)g_1(JF, W)\} - g_1(\mathcal{T}_U W, \mathcal{T}_V F) + g_1(\mathcal{T}_V W, \mathcal{T}_U F),\end{aligned}\quad (21)$$

for all vector fields $U, V, F, W \in \Gamma(\ker \varphi_*)$ and

$$\begin{aligned}R^*(X, Y, Z, H) &= \frac{c}{4} \{g_1(Y, Z)g_1(X, H) - g_1(X, Z)g_1(Y, H) \\ &\quad + g_1(JY, Z)g_1(JX, H) - g_1(JX, Z)g_1(JY, H) \\ &\quad + 2g_1(X, JY)g_1(JZ, H)\} + g_1(\mathcal{A}_X Y, \mathcal{A}_Z H) - g_1(\mathcal{A}_Y Z, \mathcal{A}_X H) \\ &\quad + g_1(\mathcal{A}_X Z, \mathcal{A}_Y H)\end{aligned}\quad (22)$$

for all vector fields $X, Y, Z, H \in \Gamma(\ker \varphi_*)^\perp$.

Let $(M_1^m(c), g_1, J_1)$ be a complex space form, (M_2, g_2) a Riemannian manifold and $\varphi : M_1(c) \rightarrow M_2$ be a $\mathcal{PBSRM}$ with $(\text{range} \varphi_*)^\perp = \{0\}$ and $\dim(\ker \varphi_*) = p = 2k_1 + 2k_2$. For every $q \in M_1$, we consider $\{E_1, E_2 = \sec \theta_1 E_1, \dots, E_{2k_1-1}, E_{2k_1} = \sec \theta_1 E_{2k_1-1}, E_{2k_1+1}, E_{2k_1+2} = \sec \theta_2 E_{2k_1+1}, \dots, E_{2k_1+2k_2-1}, E_p = \sec \theta_2 E_{2k_1+2k_2-1}\}$ and $\{E_{p+1}, E_{p+2}, \dots, E_{b_1}\}$ two orthonormal bases of $(\ker \varphi_*)$ and $(\ker \varphi_*)^\perp$, respectively. One can get easily,

$$g_1^2(JE_k, E_{k+1}) = \begin{cases} \cos^2 \theta_1, & \text{for } k \in \{1, 2, \dots, 2k_1 - 1\}; \\ \cos^2 \theta_2, & \text{for } k \in \{2k_1 + 1, \dots, 2k_1 + 2k_2 - 1\}. \end{cases}$$

Then

$$\sum_{k,s=1}^p g_1^2(JE_k, E_{k+1}) = 2(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2). \quad (23)$$

Let's denote $\mathcal{T}_{ks}^\alpha$ by

$$\mathcal{T}_{ks}^\alpha = g_1(\mathcal{T}_{E_k} E_s, E_\alpha) \quad (24)$$

where $1 \leq k, s \leq p$ and $p+1 \leq \alpha \leq b_1$.

Now, for $\ker \varphi_*$ using (21), since φ is a proper $\mathcal{PBSRM}$ with $(\text{range} \varphi_*)^\perp = \{0\}$ then, for each unit vector $F_1 \in \ker \varphi_*$ we arrive at

$$\begin{aligned}\text{Ric}^{\ker \varphi_*}(F_1) &= \frac{c}{4} [(p-1) + 3(\cos^2 \theta_1 + \cos^2 \theta_2)] \\ &\quad - pg_1(\mathcal{T}_{F_1} F_1, \mathcal{H}) + \sum_{k=1}^p g_1(\mathcal{T}_{E_k} F_1, \mathcal{T}_{F_1} E_k).\end{aligned}\quad (25)$$

From here, we get:

Theorem 4.1. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range} \varphi_*)^\perp = \{0\}$. Then, we have

$$\text{Ric}^{\ker \varphi_*}(F_1) \geq \frac{c}{4} [(p-1) + 3(\cos^2 \theta_1 + \cos^2 \theta_2)] - pg_1(\mathcal{T}_{F_1} F_1, \mathcal{H}).$$

For a unit vertical vector $F_1 \in \ker \varphi_*$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Applying the Theorem 4.1, we can get:

Corollary 4.2. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = 0$ and $(\text{range}\varphi_*)^\perp = \{0\}$. Then, we have

$$\text{Ric}^{\ker\varphi_*}(F_1) \geq \frac{c}{4}(p+5) - pg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}).$$

For a unit vertical vector $F_1 \in \ker\varphi_*$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Corollary 4.3. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $(\text{range}\varphi_*)^\perp = \{0\}$. Then, we get

$$\text{Ric}^{\ker\varphi_*}(F_1) \geq \frac{c}{4}(p-1) - pg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}).$$

For a unit vertical vector $F_1 \in \ker\varphi_*$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Corollary 4.4. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \frac{\pi}{2}$, $0 < \theta_2 < \frac{\pi}{2}$ and $(\text{range}\varphi_*)^\perp = \{0\}$. Then, we obtain

$$\text{Ric}^{\ker\varphi_*}(F_1) \geq \frac{c}{4}[(p-1) + 3\cos^2\theta_2] - pg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}).$$

For a unit vertical vector $F_1 \in \ker\varphi_*$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

By polarization, using (25), we obtain:

Theorem 4.5. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range}\varphi_*)^\perp = \{0\}$. Then, the Ricci tensor $S^{\ker\varphi_*}$ on $\ker\varphi_*$ satisfies

$$S^{\ker\varphi_*}(F_1, F_2) \geq \frac{c}{4}[(p-1) + 3(\cos^2\theta_1 + \cos^2\theta_2)]g_1(F_1, F_2) - pg_1(\mathcal{T}_{F_1}F_2, \mathcal{H}).$$

For $F_1, F_2 \in \ker\varphi_*$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Similarly, by using (21), we obtain

$$\begin{aligned} 2\rho^{\ker\varphi_*} &= \frac{c}{4}[p(p-1) + 6(k_1\cos^2\phi_1 + k_2\cos^2\phi_2)] \\ &\quad - p^2\|\mathcal{H}\|^2 + \sum_{k,s=1}^p g_1(\mathcal{T}_{E_k}E_s, \mathcal{T}_{E_k}E_s), \end{aligned} \quad (26)$$

here $\rho^{\ker\varphi_*} = \sum_{1 \leq k < s \leq p} R^{\ker\varphi_*}(E_k, E_s, E_s, E_k)$.

Therefore, we can state the following result.

Theorem 4.6. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range}\varphi_*)^\perp = \{0\}$. Then, we have

$$2\rho^{\ker\varphi_*} \geq \frac{c}{4}[p(p-1) + 6(k_1\cos^2\phi_1 + k_2\cos^2\phi_2)] - p^2\|\mathcal{H}\|^2.$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

By using (24) and (26), we arrive at

$$\begin{aligned} 2\rho^{ker\varphi_*} &= \frac{c}{4}[p(p-1) + 6(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2)] \\ &\quad - p^2 \|\mathcal{H}\|^2 + \sum_{\alpha=p+1}^m \sum_{k,s=1}^p (\mathcal{T}_{ks}^\alpha)^2. \end{aligned} \quad (27)$$

From ([25]), we know that

$$\begin{aligned} \sum_{\alpha=p+1}^{b_1} \sum_{k,s=1}^p (\mathcal{T}_{ks}^\alpha)^2 &= \frac{1}{2} p^2 \|\mathcal{H}\|^2 + \frac{1}{2} \sum_{\alpha=p+1}^m [\mathcal{T}_{11}^\alpha - \mathcal{T}_{22}^\alpha - \dots - \mathcal{T}_{pp}^\alpha]^2 \\ &\quad + 2 \sum_{\alpha=p+1}^m \sum_{s=2}^p (\mathcal{T}_{1s}^\alpha)^2 - 2 \sum_{\alpha=p+1}^m \sum_{2 \leq k < s \leq p} [\mathcal{T}_{kk}^\alpha \mathcal{T}_{ss}^\alpha - (\mathcal{T}_{ks}^\alpha)^2]. \end{aligned} \quad (28)$$

If we put (28) in (27), we obtain

$$\begin{aligned} 2\rho^{ker\varphi_*} &= \frac{c}{4}[p(p-1) + 6(k_1 \cos^2 \phi_1 + k_2 \cos^2 \phi_2)] \\ &\quad - \frac{1}{2} p^2 \|\mathcal{H}\|^2 + \frac{1}{2} \sum_{\alpha=p+1}^m [\mathcal{T}_{11}^\alpha - \mathcal{T}_{22}^\alpha - \dots - \mathcal{T}_{pp}^\alpha]^2 \\ &\quad + 2 \sum_{\alpha=p+1}^m \sum_{s=2}^p (\mathcal{T}_{1s}^\alpha)^2 - 2 \sum_{\alpha=p+1}^m \sum_{2 \leq k < s \leq p} [\mathcal{T}_{kk}^\alpha \mathcal{T}_{ss}^\alpha - (\mathcal{T}_{ks}^\alpha)^2]. \end{aligned}$$

From here, we get

$$\begin{aligned} 2\rho^{ker\varphi_*} &\geq \frac{c}{4}[p(p-1) + 6(k_1 \cos^2 \phi_1 + k_2 \cos^2 \phi_2)] \\ &\quad - \frac{1}{2} p^2 \|\mathcal{H}\|^2 - 2 \sum_{\alpha=p+1}^m \sum_{2 \leq k < s \leq p} [\mathcal{T}_{kk}^\alpha \mathcal{T}_{ss}^\alpha - (\mathcal{T}_{ks}^\alpha)^2]. \end{aligned} \quad (29)$$

On the other hand, using (7), taking $F_1 = F_4 = E_k, F_2 = F_3 = E_s$ and from (24), we have

$$\begin{aligned} 2 \sum_{2 \leq k < s \leq p} R^{M_1}(E_k, E_s, E_s, E_k) &= 2 \sum_{2 \leq k < s \leq p} R^{ker\varphi_*}(E_k, E_s, E_s, E_k) \\ &\quad + 2 \sum_{\alpha=p+1}^{b_1} \sum_{2 \leq k < s \leq p} [\mathcal{T}_{kk}^\alpha \mathcal{T}_{ss}^\alpha - (\mathcal{T}_{ks}^\alpha)^2]. \end{aligned}$$

From the last equality, (29) can be written as

$$\begin{aligned} 2\rho^{ker\varphi_*} &\geq \frac{c}{4}[p(p-1) + 6(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2)] - \frac{1}{2} p^2 \|\mathcal{H}\|^2 \\ &\quad + 2 \sum_{2 \leq k < s \leq p} R^{ker\varphi_*}(E_k, E_s, E_s, E_k) - 2 \sum_{2 \leq k < s \leq p} R^{M_1}(E_k, E_s, E_s, E_k). \end{aligned} \quad (30)$$

Also, using the equality

$$2\rho^{ker\varphi_*} = 2 \sum_{2 \leq k < s \leq p} R^{ker\varphi_*}(E_k, E_s, E_s, E_k) + 2 \sum_{s=1}^p R^{ker\varphi_*}(E_1, E_s, E_s, E_1).$$

If we put the last equality in (30), then we have

$$2\text{Ric}^{ker\varphi_*}(E_1) \geq \frac{c}{4}[p(p-1) + 6(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2)] \\ - \frac{1}{2}p^2\|\mathcal{H}\|^2 - 2 \sum_{2 \leq k < s \leq p} R^{M_1}(E_k, E_s, E_s, E_k).$$

Since M_1 is a complex space form, curvature tensor R^{M_1} of M_1 provides equation (21), therefore we acquire

$$\text{Ric}^{ker\varphi_*}(E_1) \geq \frac{c}{4}(p-1) + \frac{3c}{4}(\cos^2 \theta_1 + \cos^2 \theta_2) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

Thus, we can give the following result:

Theorem 4.7. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range}\varphi_*)^\perp = \{0\}$. Then we have

$$\text{Ric}^{ker\varphi_*}(E_1) \geq \frac{c}{4}(p-1) + \frac{3c}{4}(\cos^2 \theta_1 + \cos^2 \theta_2) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only

$$\mathcal{T}_{11}^\alpha = \mathcal{T}_{22}^\alpha + \dots + \mathcal{T}_{pp}^\alpha$$

$$\mathcal{T}_{1s}^\alpha = 0, s = 2, \dots, p.$$

Corollary 4.8. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $(\text{range}\varphi_*)^\perp = \{0\}$. Then we have

$$\text{Ric}^{ker\varphi_*}(E_1) \geq \frac{c}{4}(p-1) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only

$$\mathcal{T}_{11}^\alpha = \mathcal{T}_{22}^\alpha + \dots + \mathcal{T}_{pp}^\alpha$$

$$\mathcal{T}_{1s}^\alpha = 0, s = 2, \dots, p.$$

Corollary 4.9. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = 0$ and $(\text{range}\varphi_*)^\perp = \{0\}$. Then we have

$$\text{Ric}^{ker\varphi_*}(E_1) \geq \frac{c}{4}(p+5) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only

$$\mathcal{T}_{11}^\alpha = \mathcal{T}_{22}^\alpha + \dots + \mathcal{T}_{pp}^\alpha$$

$$\mathcal{T}_{1s}^\alpha = 0, s = 2, \dots, p.$$

Now, we give Chen-Ricci inequality on $(ker\varphi_*)^\perp$ for $\mathcal{PBSRM}$ with $(\text{range}\varphi_*)^\perp = \{0\}$.

Let's denote $\mathcal{A}_{ks}^\alpha$ by

$$\mathcal{A}_{ks}^\alpha = g_1(\mathcal{A}_{E_k} E_s, E_\alpha) \tag{31}$$

Using (1.28) of ([30]) and (31) we have

$$2\rho^{(ker\varphi_*)^\perp} = \frac{c}{4}\{m_1(m_1 + 2) + 3tr(nB)\} - 3 \sum_{\alpha=1}^p \sum_{k,s=p+1}^m (\mathcal{A}_{ks}^\alpha)^2, \quad (32)$$

here $m - p = m_1$. From (32), we get

$$2\rho^{(ker\varphi_*)^\perp} = \frac{c}{4}\{m_1(m_1 + 2) + 3tr(nB)\} - 6 \sum_{\alpha=1}^p \sum_{s=p+2}^m (\mathcal{A}_{1s}^\alpha)^2 - 6 \sum_{\alpha=1}^p \sum_{p+2 \leq k < s \leq m} (\mathcal{A}_{ks}^\alpha)^2. \quad (33)$$

Moreover, from (1.28) of ([30]) and (31), taking $Z_1 = Z_4 = E_k, Z_2 = Z_3 = E_s$ we obtain

$$\begin{aligned} 2 \sum_{p+2 \leq k < s \leq m} R^{M_1}(E_k, E_s, E_s, E_k) &= 2 \sum_{p+2 \leq k < s \leq m} R^{(ker\varphi_*)^\perp}(E_k, E_s, E_s, E_k) \\ &+ 6 \sum_{\alpha=1}^p \sum_{p+2 \leq k < s \leq m} (\mathcal{A}_{ks}^\alpha)^2. \end{aligned} \quad (34)$$

If we consider (34) in (33), then we have

$$\begin{aligned} 2\rho^{(ker\varphi_*)^\perp} &= \frac{c}{4}\{m_1(m_1 + 2) + 3tr(nB)\} - 6 \sum_{\alpha=1}^p \sum_{s=p+2}^m (\mathcal{A}_{1s}^\alpha)^2 \\ &- 2 \sum_{p+2 \leq k < s \leq m} R^{M_1}(E_k, E_s, E_s, E_k) \\ &+ 2 \sum_{p+2 \leq k < s \leq m} R^{(ker\varphi_*)^\perp}(E_k, E_s, E_s, E_k). \end{aligned}$$

Since M_1 is a complex space form, we get

$$2Ric^{(ker\varphi_*)^\perp}(E_{p+1}) = \frac{c}{4}(2m_1 - 2 + 6\|CE_{p+1}\|^2) - 6 \sum_{\alpha=1}^p \sum_{s=p+2}^m (\mathcal{A}_{1s}^\alpha)^2.$$

Then, we can write

$$Ric^{(ker\varphi_*)^\perp}(E_{p+1}) \leq \frac{c}{4}(m_1 - 1 + 3\|CE_{p+1}\|^2).$$

Thus, we can give the following result:

Theorem 4.10. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(range\varphi_*)^\perp = \{0\}$. Then we have

$$Ric^{(ker\varphi_*)^\perp}(E_{p+1}) \leq \frac{c}{4}(m_1 - 1 + 3\|CE_{p+1}\|^2).$$

the equality status of the inequality satisfies if and only

$$\mathcal{A}_{1s} = 0, s = p + 2, \dots, m.$$

5. Casorati curvatures of $\mathcal{PBSRM}$

The following lemma plays a key role in the proof of our theorem:

Lemma 5.1. Let $W = \{(y_1, y_2, \dots, y_m) \in \mathbb{R}^m : y_1 + y_2 + \dots + y_m = z\}$ be a hyperplane of $\mathbb{R}^m$, and $g : \mathbb{R}^m \rightarrow \mathbb{R}$ a quadratic form given by

$$g(y_1, y_2, \dots, y_m) = c \sum_{k=1}^{m-1} (y_k)^2 + d (y_m)^2 - 2 \sum_{1 \leq k < s \leq m} y_k y_s, \quad c > 0, d > 0.$$

Then the constrained extremum problem $\min_{(y_1, y_2, \dots, y_m) \in W} g$ has the following solution:

$$y_1 = y_2 = \dots = y_{m-1} = \frac{z}{c+1}, \quad y_m = \frac{z}{d+1} = \frac{z(m-1)}{(c+1)d} = (c-m+2) \frac{z}{c+1},$$

provided that $d = \frac{m-1}{c-m+2}$ [58].

Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range } \varphi_*)^\perp = \{0\}$. Suppose $\{E_1, \dots, E_p\}$ is an orthonormal basis of the vertical space $\ker \varphi_{*q}$, for $q \in M_1$, and $\{E_{p+1}, \dots, E_m\}$ be an orthonormal basis of the horizontal space $(\ker \varphi_{*q})^\perp$.

We define the scalar curvature $\tau^{\ker \varphi_*}$ on the vertical space $\ker \varphi_{*q}$ by

$$\tau^{\ker \varphi_*} = \sum_{k,s=1}^p g_1(R^{\ker \varphi_*}(E_k, E_s)E_s, E_k)$$

and the normalized scalar curvature $\kappa^{\ker \varphi_*}$ of $\ker \varphi_{*q}$ as

$$\kappa^{\ker \varphi_*} = \frac{2\tau^{\ker \varphi_*}}{p(p-1)}.$$

Then, we can write

$$\begin{aligned} \mathcal{T}_{ks}^\beta &= g_1(\mathcal{T}(E_k, E_s), E_\beta), \quad k, s = 1, \dots, p, \quad \beta = p+1, \dots, b_2, \\ \|\mathcal{T}\|^2 &= \sum_{k,s=1}^p g_1(\mathcal{T}(E_k, E_s), \mathcal{T}(E_k, E_s)), \\ \text{trace } \mathcal{T} &= \sum_{k=1}^p \mathcal{T}(E_k, E_k), \quad \|\text{trace } \mathcal{T}\|^2 = g_1(\text{trace } \mathcal{T}, \text{trace } \mathcal{T}) \end{aligned}$$

and the squared norm of $\mathcal{T}$ over the manifold M_1 , denoted by $C^{\ker \varphi_*}$, is called the vertical Casorati curvatures of the vertical space $(\ker \varphi_*)_q$. Thus, we get

$$C^{\ker \varphi_*} = \frac{1}{p} \|\mathcal{T}\|^2 = \frac{1}{p} \sum_{\beta=p+1}^{b_1} \sum_{k,s=1}^p (\mathcal{T}_{ks}^\beta)^2.$$

Now, assume that $L^{\ker \varphi_*}$ is a t -dimensional subspace $(\ker \varphi_*)_q$, $2 \leq t$ and let $\{E_1, E_2, \dots, E_t\}$ be an orthonormal basis of $L^{\ker \varphi_*}$. Then the Casorati curvature $C^{\ker \varphi_*}(L^{\ker \varphi_*})$ of $L^{\ker \varphi_*}$ defined as

$$C^{\ker \varphi_*}(L^{\ker \varphi_*}) = \frac{1}{t} \|\mathcal{T}\|^2 = \frac{1}{t} \sum_{\beta=p+1}^{b_1} \sum_{k,s=1}^t (\mathcal{T}_{ks}^\beta)^2.$$

The normalized $\sigma^{\ker \varphi_*}$ -Casorati curvatures $\sigma_C^{\ker \varphi_*}(p-1)$ and $\bar{\sigma}_C^{\ker \varphi_*}(p-1)$ of $(\ker \varphi_*)_q$ are given by

$$\begin{aligned} [\sigma_C^{\ker \varphi_*}(p-1)]_q &= \frac{1}{2} C_q^{\ker \varphi_*} + \frac{p+1}{2p} \inf \{C^{\ker \varphi_*}(L^{\ker \varphi_*}) : L^{\ker \varphi_*} \text{ a hyperplane of } (\ker \varphi_*)_q\}, \text{ and} \\ [\bar{\sigma}_C^{\ker \varphi_*}(p-1)]_q &= 2C_q^{\ker \varphi_*} - \frac{2p-1}{2p} \inf \{C^{\ker \varphi_*}(L^{\ker \varphi_*}) : L^{\ker \varphi_*} \text{ a hyperplane of } (\ker \varphi_*)_q\}. \end{aligned}$$

Theorem 5.2. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions θ_1, θ_2 and $(\text{range}\varphi_*)^\perp = \{0\}$ with $3 \leq p$. Then the normalized σ -Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy

$$(i) \quad \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2), \quad (35)$$

$$(ii) \quad \kappa^{ker\varphi_*} \leq \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2). \quad (36)$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{E_1, \dots, E_p\}$ on $(ker\varphi_*)_q$ and $\{E_{p+1}, \dots, E_m\}$ on $((ker\varphi_*)_q)^\perp$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^\beta = \mathcal{T}_{22}^\beta = \dots = \mathcal{T}_{p-1p-1}^\beta = \frac{1}{2}\mathcal{T}_{pp}^\beta, \quad \beta \in \{p+1, p+2, \dots, m\},$$

$$\mathcal{T}_{ks}^\beta = 0, \quad k, s \in \{1, \dots, p\} (k \neq s), \quad \beta \in \{p+1, p+2, \dots, m\}.$$

Proof. Using (1.28) of ([30]) and (7), we have

$$\begin{aligned} 2\tau^{ker\varphi_*} &= \frac{c}{4}(p^2 - p) + \frac{3c}{2}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2) \\ &- pC^{ker\varphi_*} + \|\text{trace}\mathcal{T}\|^2. \end{aligned} \quad (37)$$

Now we define a function $Q^{ker\varphi_*}$ associated with the following quadratic polynomial with respect to the components of $\mathcal{T}$:

$$\begin{aligned} Q^{ker\varphi_*} &= \frac{1}{2}[(p^2 - p)C^{ker\varphi_*} + (p^2 - 1)C^{ker\varphi_*}(L^{ker\varphi_*})] \\ &- 2\tau^{ker\varphi_*} + \frac{c}{4}(p^2 - p) + \frac{3c}{2}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2). \end{aligned}$$

Without loss of generality, by supposing that the hyperplane $L^{ker\varphi_*}$ is spanned by $\{E_1, \dots, E_{p-1}\}$, one can produce

$$\begin{aligned} Q^{ker\varphi_*} &= \sum_{\beta=p+1}^{b_1} \sum_{k=1}^{p-1} [p(\mathcal{T}_{kk}^\beta)^2 + (p+1)(\mathcal{T}_{kp}^\beta)^2] \\ &+ \sum_{\beta=p+1}^{b_1} [2(p+1)\sum_{1=k < s}^{p-1} (\mathcal{T}_{ks}^\beta)^2 \\ &- 2\sum_{1=k < s}^p \mathcal{T}_{kk}^\beta \mathcal{T}_{ss}^\beta + \frac{p-1}{2}(\mathcal{T}_{pp}^\beta)^2]. \end{aligned} \quad (38)$$

Using (38), we obtain the critical points

$$\mathcal{T}^c = (\mathcal{T}_{11}^{p+1}, \mathcal{T}_{12}^{p+1}, \dots, \mathcal{T}_{pp}^{p+1}, \dots, \mathcal{T}_{11}^{b_1}, \dots, \mathcal{T}_{pp}^{b_1})$$

of $Q^{ker\varphi_*}$ are solutions of the next system of equations:

$$\left\{ \begin{aligned} \frac{\partial Q^{ker\varphi_*}}{\partial \mathcal{T}_{kk}^\beta} &= 2(r+1)\mathcal{T}_{kk}^\beta - 2\sum_{t=1}^p \mathcal{T}_{tt}^\beta = 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial \mathcal{T}_{pp}^\beta} &= (r-1)\mathcal{T}_{pp}^\beta - 2\sum_{t=1}^{p-1} \mathcal{T}_{tt}^\beta = 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial \mathcal{T}_{ks}^\beta} &= 4(r+1)\mathcal{T}_{ks}^\beta = 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial \mathcal{T}_{kp}^\beta} &= 2(r+1)\mathcal{T}_{kp}^\beta = 0 \end{aligned} \right. \quad (39)$$

here $k, s \in \{1, 2, \dots, p-1\}$, $k \neq s$ and $\beta \in \{p+1, \dots, m\}$. Frankly (39) is a system consisting only in linear homogeneous equations and it is easy to checky that every solution $\mathcal{T}^c$ has $\mathcal{T}_{ks}^\beta = 0$ for $k \neq s$, and the

determinant corresponding to the first two series of linear homogeneous equations in (39) has vanishes. Furthermore, the Hessian matrix of $Q^{ker\varphi_*}$ is defined as

$$\mathcal{H}(Q^{ker\varphi_*}) = \begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix},$$

here

$$\mathcal{H}_1 = \begin{pmatrix} 2p & -2 & \dots & -2 & -2 \\ -2 & 2p & \dots & -2 & -2 \\ \dots & \dots & \dots & \dots & \dots \\ -2 & -2 & \dots & 2p & -2 \\ -2 & -2 & \dots & -2 & p-1 \end{pmatrix},$$

0 denotes the zero matrix of suitable dimensions and the matrices $\mathcal{H}_2, \mathcal{H}_3$ are ones having the following diagonal forms

$$\begin{aligned} \mathcal{H}_2 &= \text{diag}(4(p+1), 4(p+1), \dots, 4(p+1)), \\ \mathcal{H}_3 &= \text{diag}(2(p+1), 2(p+1), \dots, 2(p+1)). \end{aligned}$$

Then a standard computation shows that the eigenvalues of $\mathcal{H}(Q^{ker\varphi_*})$ are

$$\begin{aligned} \xi_{11} &= 0, \quad \xi_{22} = p+3, \quad \xi_{33} = \dots = \xi_{pp} = 2(p+1), \quad \xi_{ks} = 4(p+1), \\ \xi_{km} &= 2(p+1), \quad \forall k, s \in \{1, 2, \dots, p-1\}, \quad k \neq s. \end{aligned}$$

Also it follows that $Q^{ker\varphi_*}$ is parabolic and achieves a global minimum value $Q^{ker\varphi_*}(c)$ for $\mathcal{T}^c$ – the solution of (39). However we obtain $Q^{ker\varphi_*}(c) = 0$ and we get $Q^{ker\varphi_*} \geq 0$. Thus,

$$\begin{aligned} 2\tau^{ker\varphi_*} &\leq \frac{1}{2}[(p^2 - p)C^{ker\varphi_*} + (p^2 - 1)C^{ker\varphi_*}(L^{ker\varphi_*})] \\ &+ \frac{c}{4}(p^2 - p) + \frac{3c}{2}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2) \end{aligned} \quad (40)$$

and using (40) we obtain

$$\kappa^{ker\varphi_*} \leq \left[\frac{1}{2}C^{ker\varphi_*} + \frac{p+1}{2p}C^{ker\varphi_*}(L^{ker\varphi_*}) \right] + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2) \quad (41)$$

for all hyperplane $L^{ker\varphi_*}$ of M_1 . Now, taking the infimum in (41) over every hyperplane $L^{ker\varphi_*}$, we get (i)

$$\kappa^{ker\varphi_*} \leq \sigma_c^{ker\varphi_*}(p-1) + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2). \quad (42)$$

Besides, simply we can check that the equality sign holds in the (42) if and only if

$$\mathcal{T}_{ks}^\beta = 0, \quad \forall k, s \in \{1, 2, \dots, p\}, \quad k \neq s, \quad \beta \in \{p+1, \dots, m\},$$

and

$$\mathcal{T}_{pp}^\beta = 2\mathcal{T}_{11}^\beta = \dots = 2\mathcal{T}_{p-1, p-1}^\beta, \quad \forall k, s \in \{p+1, p+2, \dots, m\}.$$

In a similar way we have (ii).

□

Using the Theorem 4.2, we obtain the following results:

Corollary 5.3. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $(\text{range}\varphi_*)^\perp = \{0\}$ with $3 \leq p$. Then the normalized σ -Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy

$$(i) \quad \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{c}{4}, \quad (43)$$

$$(ii) \quad \kappa^{ker\varphi_*} \leq \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{c}{4}. \quad (44)$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{E_1, \dots, E_p\}$ on $(ker\varphi_*)_q$ and $\{E_{p+1}, \dots, E_m\}$ on $((ker\varphi_*)_q)^\perp$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^\beta = \mathcal{T}_{22}^\beta = \dots = \mathcal{T}_{p-1p-1}^\beta = \frac{1}{2}\mathcal{T}_{pp}^\beta, \quad \beta \in \{p+1, p+2, \dots, m\},$$

$$\mathcal{T}_{ks}^\beta = 0, \quad k, s \in \{1, \dots, p\} (k \neq s), \quad \beta \in \{p+1, p+2, \dots, m\}.$$

Corollary 5.4. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $\theta_1 = \theta_2 = 0$ and $(\text{range}\varphi_*)^\perp = \{0\}$ with $3 \leq p$. Then the normalized σ -Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy

$$(i) \quad \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{(p+2)c}{4(p-1)}, \quad (45)$$

$$(ii) \quad \kappa^{ker\varphi_*} \leq \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{(p+2)c}{4(p-1)}. \quad (46)$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{E_1, \dots, E_p\}$ on $(ker\varphi_*)_q$ and $\{E_{p+1}, \dots, E_m\}$ on $((ker\varphi_*)_q)^\perp$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^\beta = \mathcal{T}_{22}^\beta = \dots = \mathcal{T}_{p-1p-1}^\beta = \frac{1}{2}\mathcal{T}_{pp}^\beta, \quad \beta \in \{p+1, p+2, \dots, m\},$$

$$\mathcal{T}_{ks}^\beta = 0, \quad k, s \in \{1, \dots, p\} (k \neq s), \quad \beta \in \{p+1, p+2, \dots, m\}.$$

Corollary 5.5. Let φ be a $\mathcal{PBSRM}$ from a complex space form $(M_1^m(c), g_1, J_1)$ to a Riemannian manifold (M_2, g_2) with the slant functions $0 < \theta_1 < \frac{\pi}{2}$, $\theta_2 = 0$ and $(\text{range}\varphi_*)^\perp = \{0\}$ with $3 \leq p$. Then the normalized σ -Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy

$$(i) \quad \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \phi_1 + k_2), \quad (47)$$

$$(ii) \quad \kappa^{ker\varphi_*} \leq \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{c}{4} + \frac{3c}{2p(p-1)}(k_1 \cos^2 \phi_1 + k_2). \quad (48)$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{E_1, \dots, E_p\}$ on $(ker\varphi_*)_q$ and $\{E_{p+1}, \dots, E_m\}$ on $((ker\varphi_*)_q)^\perp$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^\beta = \mathcal{T}_{22}^\beta = \dots = \mathcal{T}_{p-1p-1}^\beta = \frac{1}{2}\mathcal{T}_{pp}^\beta, \quad \beta \in \{p+1, p+2, \dots, m\},$$

$$\mathcal{T}_{ks}^\beta = 0, \quad k, s \in \{1, \dots, p\} (k \neq s), \quad \beta \in \{p+1, p+2, \dots, m\}.$$

References

- [1] R. Abraham, J.E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Applied Mathematical Sciences, Vol. 75. Springer, New York, 1988.
- [2] M. A. Akyol, *On Pointwise Riemannian Maps in Complex Geometry*, International Symposium on Differential Geometry and Its Applications, Maulana Azad National Urdu University, Gachibowli, Hyderabad - 500032, Telangana, India (2022).
- [3] M. A. Akyol, R. Demir, N. Ö. Poyraz, G. E. Vilcu, *Optimal inequalities for hemi-slant Riemannian submersions*, Mathematics, **10** (21), 2022, 3993.
- [4] M. A. Akyol and N. Ö. Poyraz, *Sharp inequalities involving Chen-Ricci inequality for slant Riemannian submersions*, Bulletin of Korean Mathematical Society, **60** (5), 2023, 1155–1179.
- [5] M. A. Akyol and Y. Gündüzalp, *Pointwise slant Riemannian maps (PSRM) to almost Hermitian manifolds*, Mediterranean Journal of Mathematics, **20**, 2023, 116.
- [6] M. A. Akyol and Y. Gündüzalp, *Pointwise hemi-slant Riemannian maps (PSRM) to almost Hermitian manifolds*, Hacettepe Journal of Mathematics, **53** (5), 2024, 1218–1237.
- [7] M. A. Akyol and B. Şahin, *Conformal anti-invariant Riemannian maps to Kaehler manifolds*, U.P.B. Sci. Bull., Series A, **80** (4), 2018, 187–198.
- [8] M. A. Akyol and B. Şahin, *Conformal semi-invariant Riemannian maps to Kaehler manifolds*, Revista de la Union Matematica Argentina, **60** (2), 2019, 459–468.
- [9] M. A. Akyol and B. Şahin, *Conformal slant Riemannian maps to Kaehler manifolds*, Tokyo J. Math. **42** (1), 2019, 225–237.
- [10] M. E. Aydin, A. Mihai, I. Mihai, *Some Inequalities on submanifolds in statistical manifolds of constant curvature*, Filomat **29** (3), (2015), 465–477.
- [11] M. Aquib, J. W. Lee, G. E. Vilcu and D. W. Yoon, *Classification of Casorati ideal Lagrangian submanifolds in complex space forms*, Difer. Geom. Appl. **63**, 2019, 30–49.
- [12] M. Aquib and M. H. Shahid, *Generalized normalized δ -Casorati curvature for statistical submanifolds in quaternion Kaehler-like statistical space forms*, J. Geom. **109** (1), 2018, 13.
- [13] P. Baird and J.C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Clarendon Press, Oxford, 2003.
- [14] J. P. Bourguignon and H. B. Lawson, *Stability and isolation phenomena for Yangmills fields*, Commun. Math. Phys. **79**, 1981, 189–230.
- [15] J. P. Bourguignon and H. B. Lawson, *A mathematician's Visit to Kaluza-Klein Theory*, pp. 143–163. Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1989).
- [16] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. **42** (1), 2000, 125–138.
- [17] F. Casorati, *Nuova definizione della curvatura delle superficie e suo confronto con quella di Gauss. (New definition of the curvature of the surface and its comparison with that of Gauss)*. Rend. Inst. Matem. Accad. Lomb. Ser. II **22** (8), 1889, 335–346.
- [18] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, (1990).
- [19] B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel) **60**, 1993, 568–578.
- [20] B.-Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J. **41** (1), (1999), 33–41.
- [21] B. Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk J. Math. **36**, 2012, 630–640.
- [22] F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold*, Publ. Math. Debrecen **53**, 1998, 217–223.
- [23] E. Garcia-Rio and D. N. K peli, *Semi-Riemannian Maps and Their Applications*, Kluwer Academic, Dordrecht, (1999).
- [24] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16**, 1967, 715–737.
- [25] M. G lbahar, Ş. E. Meri  and E. Kili , *Sharp inequalities involving the Ricci curvature for Riemannian submersions*, Kragujevac Journal of Mathematics, **41** (2), 2017, 279–293.
- [26] Y. G nd zalp and M. A. Akyol, *Pointwise slant Riemannian maps from Kaehler manifolds*, Journal of Geometry and Physics, **179**, 2022, 104589.
- [27] Y. G nd zalp and M. A. Akyol, *Pointwise semi-slant Riemannian (PSSR) maps from almost Hermitian manifolds*, Filomat, **37** (13), 2023, 4271–4286.
- [28] Y. G nd zalp and M. A. Akyol, *Remarks on conformal anti-invariant Riemannian maps to cosymplectic manifolds*, Hacet. J. Math. Stat. **50** (4), 2021, 1131–1139.
- [29] Y. G nd zalp, M. A. Akyol and B. Şahin, *Pointwise hemi-slant Riemannian (PSSR) maps into almost Hermitian manifolds and Casorati inequalities*, International Journal of Maps in Mathematics, **7** (1), 2024, 76–96.
- [30] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific, 2004.
- [31] A. E. Fischer, *Riemannian maps between Riemannian manifolds*, Contemp. Math. **132**, 1992, 331–366.
- [32] C. W. Lee, J. W. Lee and G. E. Vilcu, *Optimal inequalities for the normalized δ -Casorati curvatures of submanifolds in Kenmotsu space forms*, Adv. Geom. **17** (3), 2017, 355–362.
- [33] C. W. Lee, J. W. Lee, B. Şahin and G. E. Vilcu, *Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures*, Annali di Matematica Pura ed Applicata (1923 -) **200**, 2021, 1277–1295.
- [34] J. Lee, J. H. Park, B. Şahin and D. Y. Song, *Einstein conditions for the base of anti-invariant Riemannian submersions and Clairaut submersions*, Taiwan. J. Math. **19** (4), 2015, 1145–1160.
- [35] J. W. Lee and B. Şahin, *Pointwise slant submersions*, Bulletin of the Korean Mathematical Society, **51** (4), 2014, 115–1126.
- [36] A. Mihai and I. Mihai, *Curvature invariants for statistical submanifolds of Hessian manifolds of constant Hessian curvature*, Mathematics, **6**, 2018, 44.
- [37] A. Mihai and C.  zg r, *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection*, Taiwanese J. Math. **14** (4), 2010, 1465–1477.

- [38] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Class. Quantum Gravity **4**, 1987, 1317–1325.
- [39] S. Ianus and M. Visinescu, *Space-time compactification and Riemannian submersions* In: Rassias, G. (ed.) *The Mathematical Heritage of C. F. Gauss*, pp. 358–371. World Scientific, River Edge (1991).
- [40] M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. **41**, 2000, 6918–6929.
- [41] B. O'Neill, *The fundamental equations of a submersion*, Mich Math J. **13**, 1966, 458–469.
- [42] K. S. Park and B. Şahin, *Semi-slant Riemannian maps into almost Hermitian manifolds*, Czechoslovak Mathematical Journal, **64** (4), 2014, 1045–1061.
- [43] K. S. Park, *Almost h -semi-slant Riemannian maps*, Taiwanese J. Math. **17** (3), 2013, 937–956.
- [44] R. Prasad and S. Pandey, *Slant Riemannian maps from an almost contact manifold*, Filomat, **31** (13), 2017, 3999–4007.
- [45] S. A. Sepet and H. G. Bozok, *Pointwise semi-slant submersion*, Differential Geometry - Dynamical Systems, **22**, 2020, 1–10.
- [46] S. A. Sepet and M. Ergüt, *Pointwise bi-slant submersions from cosymplectic manifolds*, Commun.Fac.Sci.Univ.Ank.Ser.A1 Math.Stat. **69** (2), 2020, 1310–1319.
- [47] S. A. Sepet and M. Ergüt, *Pointwise slant submersions from cosymplectic manifolds*, Turkish Journal of Mathematics, **40** (3), 2016, 582–593.
- [48] S. A. Sepet and M. Ergüt, *Pointwise slant submersions from almost product Riemannian manifolds*, Journal of Interdisciplinary Mathematics, **23** (3), 2019, 639–655.
- [49] B. Şahin, *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, Acta Appl. Math. **109** (3), 2010, 829–847.
- [50] B. Şahin, *Invariant and anti-invariant Riemannian maps to Kahler manifolds*, Int. J. Geom. Meth. Mod. Phys. **7** (3), 2010, 1–19.
- [51] B. Şahin, *Slant Riemannian maps from almost Hermitian manifolds*, Quaestiones Mathematicae, **36** (3), 2013, 449–461.
- [52] B. Şahin, *Slant Riemannian maps to Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. **10**, 2013, 1250080.
- [53] B. Şahin, *Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and their Applications*, Elsevier, Academic Press, (2017).
- [54] B. Şahin and Ş. Yanan, *Conformal Riemannian maps from almost Hermitian manifolds*, Turk J Math **42**, 2018, 2436–2451.
- [55] B. Şahin, *Hemi-slant Riemannian maps*, Mediterranean Journal of Mathematics. **14**, 2017, 10.
- [56] B. Şahin, *Chen's first inequality for Riemannian maps*, Ann. Polon. Math. **117** (3), 2016, 249–258.
- [57] H. M. Taştan, B. Şahin and Ş. Yanan, *Hemi-Slant Submersions*, Mediterranean Journal of Mathematics. **13**, 2016, 2171–2184.
- [58] M. M. Tripathi, *Inequalities for algebraic Casorati curvatures and their applications*, Note Mat. **37** (1), 2017, 161–186.
- [59] G. E. Vilcu, *An optimal inequality for Lagrangian submanifolds in complex space forms involving Casorati curvatures*, J. Math. Anal. Appl. **465** (2), 2018, 1209–1222.
- [60] G. E. Vilcu, B.-Y. Chen *inequalities for slant submanifolds in quaternionic space forms*, Turkish J. Math. **34**, 2010, 115–128.
- [61] G. E. Vilcu, *On Chen invariants and inequalities in quaternionic geometry*, Journal of Inequalities and Applications, **66**, (2013).
- [62] B. Watson, G. G' -Riemannian submersions and nonlinear gauge field equations of general relativity. In: Rassias, T. (ed.) Global Analysis—Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., **57**, 1983, 324–349.
- [63] K. Yano and M. Kon, *Structures on manifolds*, World scientific, 1985.
- [64] L. Zhang, X. Pan and P. Zhang, *Inequalities for Casorati curvature of Lagrangian submanifolds in complex space forms*, Adv. Math. (China) **45** (5), 2016, 767–777.
- [65] P. Zhang and L. Zhang, *Inequalities for Casorati curvatures of submanifolds in real space forms*, Adv. Geom. **16**(3), 2016, 329–335.