Filomat 39:2 (2025), 639–648
https://doi.org/10.2298/FIL2502639W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An A_α-spectral radius for a spanning tree with constrained leaf distance in a graph

Sufang Wang[a,*], Wei Zhang[b]

[a] *School of Public Management, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*
[b] *School of Economics and Management, Wenzhou University of Technology, Wenzhou, Zhejiang 325000, China*

Abstract. Let $\alpha \in [0, 1)$ be a real number, and let G be a connected graph of order n with $n \geq \lambda(\alpha)$, where $\lambda(\alpha) = 9$ for $0 \leq \alpha \leq \frac{2}{3}$ and $\lambda(\alpha) = \frac{4}{1-\alpha}$ for $\frac{2}{3} < \alpha < 1$. A spanning tree T of G is a subgraph of G that is a tree covers all vertices of G. The leaf distance of a tree is the minimum of distances between any two leaves of a tree. Let $A_\alpha(G) = \alpha D(G)+(1-\alpha)A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees of G. The largest eigenvalues of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$, is called A_α-spectral radius of G. In this paper, it is proved that G has a spanning tree with leaf distance at least 4 if $\rho_\alpha(G) \geq \gamma(n)$, where $\gamma(n)$ is the largest root of $x^3-(\alpha n+n+\alpha-3)x^2+(\alpha n^2+\alpha^2 n-\alpha n-n-2\alpha+1)x-\alpha^2n^2+3\alpha^2n-\alpha n+n-4\alpha^2+5\alpha-3 = 0$.

1. Introduction

Graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ denote a graph, where $V(G) = \{v_1, v_2, \cdots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The order of G is denoted by $|V(G)| = n$. A graph G is called trivial if $n = 1$. Let $i(G)$ denote the number of isolated vertices in G. For a vertex v in G, we let $d_G(v)$ denote the degree of v in G. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph of G induced by $V(G) \setminus S$. The complete graph of order n is denoted by K_n.

Let G_1 and G_2 be two vertex disjoint graphs. We use $G_1 \cup G_2$ to denote the union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph formed from $G_1 \cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. For an integer $k \geq 3$, The sequential join $G_1 \vee G_2 \vee \cdots \vee G_k$ of graphs $G_1, G_2, \cdots, G_k$ is the graph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) \cup \{e = x_i x_{i+1} : x_i \in V(G_i), x_{i+1} \in V(G_{i+1}), 1 \leq i \leq k-1\}$.

Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees of G. Let $Q(G) = D(G) + A(G)$ denote the signless Laplacian matrix of G. For any $\alpha \in [0, 1)$, Nikiforov [25] defined the A_α-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

It is clear that $A_0(G) = A(G)$ and $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$. Hence, $A_\alpha(G)$ generalizes both the adjacency matrix and the signless Laplacian matrix of G. The largest eigenvalues of $A(G)$, $Q(G)$ and $A_\alpha(G)$, denoted by $\rho(G)$,

2020 *Mathematics Subject Classification*. Primary 05C50; Secondary 05C05; 90B99.
Keywords. graph; A_α-spectral radius; spanning tree; leaf distance.
Received: 29 January 2024; Revised: 28 May 2024; Accepted: 27 October 2024
Communicated by Paola Bonacini
* Corresponding author: Sufang Wang
Email addresses: wangsufangjust@163.com (Sufang Wang), zw_wzu@163.com (Wei Zhang)
ORCID iDs: https://orcid.org/0009-0007-5527-5713 (Sufang Wang), https://orcid.org/0000-0002-8777-1291 (Wei Zhang)

$q(G)$ and $\rho_\alpha(G)$, are called the adjacency spectral radius and the signless Laplacian spectral radius and A_α-spectral radius of G, respectively. Obviously, $\rho_0(G) = \rho(G)$ and $\rho_{\frac{1}{2}}(G) = \frac{1}{2}q(G)$.

For two integers a and b with $1 \le a \le b$, an $[a,b]$-factor of G is a spanning subgraph F of G with $a \le d_F(v) \le b$ for any $v \in V(G)$. An $[a,b]$-factor is a $[1,r]$-factor if $a = 1$ and $b = r$. A $[1,r]$-factor is a 1-factor (or a perfect matching) if n is even and $r = 1$. A spanning tree T of a connected graph G is a subgraph of G that is a tree covers all vertices of G. For an integer $k \ge 2$, a spanning k-tree is a spanning tree with the maximum degree at most k, a spanning k-ended tree is a spanning tree with at most k leaves. In particular, a spanning k-tree is also a connected $[1,k]$-factor and a spanning 2-ended tree is also called a Hamilton path. Let T be a spanning tree of a connected graph G. The leaf degree of a vertex $v \in V(T)$ is defined as the number of leaves adjacent to v in T. The leaf degree of T is the maximum leaf degree among all the vertices of T. The leaf distance of T is defined as the minimum of distances between any two leaves of T.

Spanning trees and $[a,b]$-factors have attracted many researchers' attention. Some sufficient conditions for graphs with $[1,2]$-factors were obtained by many researchers [1, 5, 6, 9, 11, 18, 19, 21, 22, 32, 36, 38, 43]. Kim, O, Park and Ree [15], Kano and Saito [12], Zhou, Xu and Sun [40] showed some results for the existence of $[1,b]$-factors in graphs. Zhou and Liu [33] studied the relationship between the spectral radius of a connected graph and its odd $[1,b]$-factors, and claimed a lower bound on the existence of odd $[1,b]$-factors via the spectral radius. Many scholars investigated the properties of $[a,b]$-factors in graphs, and provided some graphic parameter conditions for graphs having $[a,b]$-factors [20, 23, 24, 28, 31, 34, 35, 41]. Win [27] established a connection between toughness and the existence of spanning k-trees in a graph. Kyaw [16] showed a degree and neighborhood condition for the existence of a spanning k-tree in a connected graph. Fan, Goryainov, Huang and Lin [7] presented a lower bound on the spectral radius of a connected graph G to ensure that G contains a spanning k-tree. Zhou and Wu [39] showed a distance spectral radius condition which guarantees the existence of a spanning k-tree in a connected graph. Zhou, Zhang and Liu [42] studied the connection between the distance signless Laplacian spectral radius and the spanning k-tree in a connected graph and verified an upper bound on the distance signless Laplacian spectral radius in a connected graph G to ensure the existence of a spanning k-tree. Broersma and Tuinstra [3] gave a degree sum condition for a connected graph to have a spanning k-ended tree. Flandrin, Kaiser, Kužel, Li and Ryjáček [8], Kyaw [17] obtained some results on the existence of spanning k-ended trees in connected graphs. Ao, Liu and Yuan [2], Wu [29] provided some tight spectral radius conditions for the existence of a spanning tree with leaf degree at most k in a connected graph. Zhou, Sun and Liu [37] provided the upper bounds for the distance spectral radius and the distance signless Laplacian spectral radius in a connected graph G to ensure that G has a spanning tree with leaf degree at most k, respectively. Kaneko [13] posed a criterion for the existence of a spanning tree with leaf degree at most k in a connected graph and a conjecture for a connected graph of order n with $n \ge d+1$ having a spanning tree with leaf distance at least d, where $d \ge 3$ is an integer. The above conjecture holds for $d = 3$ [13]. Kaneko, Kano and Suzuki [14] established a connection between the number of isolated vertices and spanning trees with leaf distance at least 4 in connected graphs, which implies that the above conjecture is true for $d = 4$. Chen, Lv, Li and Xu [4] provided a lower bound on the size of a connected graph G to ensure that G contains a spanning tree with leaf distance at least 4 and a lower bound on the spectral radius (or the signless Laplacian spectral radius) of a connected graph G to guarantee the existence of a spanning tree with leaf distance of at least 4.

Motivated by [4, 14, 26] directly, we present an A_α-spectral radius condition for the existence of a spanning tree with leaf distance at least 4 in a connected graph.

Theorem 1.1. Let $\alpha \in [0,1)$ be a real number, and let G be a connected graph of order n with $n \ge \lambda(\alpha)$, where

$$\lambda(\alpha) = \begin{cases} 9, & if\ 0 \le \alpha \le \frac{2}{3}; \\ \frac{4}{1-\alpha}, & if\ \frac{2}{3} < \alpha < 1. \end{cases}$$

If $\rho_\alpha(G) \ge \gamma(n)$, then G has a spanning tree with leaf distance at least 4, where $\gamma(n)$ is the largest root of $x^3 - (\alpha n + n + \alpha - 3)x^2 + (\alpha n^2 + \alpha^2 n - \alpha n - n - 2\alpha + 1)x - \alpha^2 n^2 + 3\alpha^2 n - \alpha n + n - 4\alpha^2 + 5\alpha - 3 = 0$.

2. Some preliminaries

In this section, we present some necessary preliminary lemmas, which are used to prove Theorem 1.1. Kaneko, Kano and Suzuki [14] established a connection between the number of isolated vertices and a spanning tree with leaf distance at least 4 in a connected graph.

Lemma 2.1 (Kaneko, Kano and Suzuki [14]). Let G be a connected graph of order n with $n \geq 5$. If

$$i(G-S) < |S|$$

for any $\emptyset \neq S \subseteq V(G)$, then G contains a spanning tree with leaf distance at least 4.

Lemma 2.2 (Nikiforov [25]). For a complete graph K_n of order n, we possess

$$\rho_\alpha(K_n) = n-1.$$

Lemma 2.3 (Nikiforov [25]). Let G be a connected graph, and H be a proper subgraph of G. Then

$$\rho_\alpha(G) > \rho_\alpha(H).$$

Let M be a real symmetric matrix whose rows and columns are indexed by $V = \{1, 2, \cdots, n\}$. Assume that M, with respect to the partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_t$, can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. Let q_{ij} denote the average row sum of M_{ij}. Then matrix $M_\pi = (q_{ij})$ is called the quotient matrix of M. If the row sum of every block M_{ij} is a constant, then the partition is equitable.

Lemma 2.4 (You, Yang, So and Xi [30]). Let M be a real symmetric matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the largest eigenvalues of M and M_π are equal.

The subsequent lemma is the well-known Cauchy Interlacing Theorem.

Lemma 2.5 (Haemers [10]). Let M be a Hermitian matrix of order s, and let N be a principal submatrix of M with order t. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$ are the eigenvalues of M and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t$ are the eigenvalues of N, then $\lambda_i \geq \mu_i \geq \lambda_{s-t+i}$ for $1 \leq i \leq t$.

3. The proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $G_0 = K_1 \vee (K_{n-2} \cup H)$, where $H = K_1$. In view of the partition $V(G_0) = V(H) \cup V(K_{n-2}) \cup V(K_1)$, the quotient matrix of $A_\alpha(G_0)$ equals

$$B_0 = \begin{pmatrix} \alpha & 0 & 1-\alpha \\ 0 & n+\alpha-3 & 1-\alpha \\ 1-\alpha & (1-\alpha)(n-2) & \alpha n-\alpha \end{pmatrix}.$$

Then the characteristic polynomial of B_0 is equal to

$$\begin{aligned} f_{B_0}(x) =& x^3 - (\alpha n + n + \alpha - 3)x^2 + (\alpha n^2 + \alpha^2 n - \alpha n - n - 2\alpha + 1)x \\ & - \alpha^2 n^2 + 3\alpha^2 n - \alpha n + n - 4\alpha^2 + 5\alpha - 3. \end{aligned}$$

According to the condition of Theorem 1.1, $\gamma(n)$ is the largest root of $f_{B_0}(x) = 0$. Since the partition $V(G_0) = V(H) \cup V(K_{n-2}) \cup V(K_1)$ is equitable, by Lemma 2.4, we have $\rho_\alpha(G_0) = \gamma(n)$. One checks that $G_0 = K_1 \vee (K_{n-2} \cup K_1)$ contains a spanning tree with leaf distance at least 4. Consequently, Theorem 1.1 holds for $G = G_0$. In what follows, we assume that $G \neq G_0$.

Suppose to the contrary that G has no spanning tree with leaf distance at least 4. According to Lemma 2.1, there exists a nonempty subset $S \subseteq V(G)$ satisfying $i(G-S) \geq |S|$. Choose a connected graph G of order n such that its A_α-spectral radius is as large as possible. Together with Lemma 2.3 and the choice of G, the induced subgraph $G[S]$ and every connected component of $G - S$ are complete graphs, respectively. Furthermore, $G = G[S] \vee (G - S)$.

For convenience, let $i(G-S) = i$ and $|S| = s$. One may see that there exists at most one nontrivial connected component in $G - S$. Otherwise, we can add edges among all nontrivial connected components to obtain a bigger nontrivial connected component. Then Lemma 2.3 deduces a contradiction to the choice of G. Next, we proceed by considering the two possible cases.

Case 1. $G - S$ has just one nontrivial connected component, say G_1.

Let $|V(G_1)| = n_1 \geq 2$. Then $G = K_s \vee (K_{n_1} \cup iK_1)$, where $n_1 = n - s - i \geq 2$. If $i \geq s+1$, then we construct a new graph H_1 obtained from G by joining every vertex of G_1 with one unique vertex in iK_1 by an edge. Then $i(H_1 - S) \geq s$ and G is a proper spanning subgraph of H_1. By virtue of Lemma 2.3, we conclude $\rho_\alpha(G) < \rho_\alpha(H_1)$, which is a contradiction to the choice of G. Consequently, $i \leq s$. Together with $i \geq s$, we deduce $i = s$, and so $n_1 = n - 2s$ and $G = K_s \vee (K_{n-2s} \cup sK_1)$. In terms of the partition $V(G) = V(sK_1) \cup V(K_{n-2s}) \cup V(K_s)$, the quotient matrix of $A_\alpha(G)$ equals

$$B_1 = \begin{pmatrix} \alpha s & 0 & (1-\alpha)s \\ 0 & n-(2-\alpha)s-1 & (1-\alpha)s \\ (1-\alpha)s & (1-\alpha)(n-2s) & \alpha n + (1-\alpha)s - 1 \end{pmatrix}.$$

Then the characteristic polynomial of B_1 equals

$$\begin{aligned} f_{B_1}(x) = & x^3 - (\alpha n + n + \alpha s - s - 2)x^2 \\ & + (\alpha n^2 + \alpha^2 sn - \alpha n - n - s^2 - 2\alpha s + s + 1)x \\ & - \alpha^2 sn^2 + (2\alpha^2 - 2\alpha + 1)s^2 n + (\alpha^2 + \alpha)sn \\ & - (3\alpha^2 - 5\alpha + 2)s^3 - (\alpha^2 - \alpha + 1)s^2 - \alpha s. \end{aligned} \quad (1)$$

Since the partition $V(G) = V(sK_1) \cup V(K_{n-2s}) \cup V(K_s)$ is equitable, according to Lemma 2.4, the largest root, say γ_1, of $f_{B_1}(x) = 0$ is equal to $\rho_\alpha(G)$. Let $\gamma_1 = \rho_\alpha(G) \geq \gamma_2 \geq \gamma_3$ be the three roots of $f_{B_1}(x) = 0$ and $Q = \mathrm{diag}(s, n-2s, s)$. By a simple calculation, we possess

$$Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}} = \begin{pmatrix} \alpha s & 0 & (1-\alpha)s \\ 0 & n-(2-\alpha)s-1 & (1-\alpha)s^{\frac{1}{2}}(n-2s)^{\frac{1}{2}} \\ (1-\alpha)s & (1-\alpha)s^{\frac{1}{2}}(n-2s)^{\frac{1}{2}} & \alpha n + (1-\alpha)s - 1 \end{pmatrix}.$$

Obviously, $Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}}$ is symmetric and also contains

$$\begin{pmatrix} \alpha s & 0 \\ 0 & n-(2-\alpha)s-1 \end{pmatrix}$$

as its submatrix. Since $Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}}$ and B_1 possess the same eigenvalues, the Cauchy interlacing theorem (see Lemma 2.5) yields that

$$\gamma_2 \leq n - (2-\alpha)s - 1 < n - 2 \quad (\alpha \in [0,1) \text{ and } s \geq 1). \quad (2)$$

If $s = 1$, then $G = K_1 \vee (K_{n-2} \cup K_1) = G_0$, which is a contradiction to $G \neq G_0$. Hence, we have $s \geq 2$.

Notice that K_{n-1} is a proper subgraph of $K_1 \vee (K_{n-2} \cup K_1)$. It follows from Inequality (2), Lemmas 2.2 and 2.3 that

$$\gamma(n) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1)) > \rho_\alpha(K_{n-1}) = n-2 > \gamma_2. \tag{3}$$

Let $\gamma = \gamma(n)$. Notice that $f_{B_0}(\gamma) = 0$. By a direct computation, we obtain

$$f_{B_1}(\gamma) = f_{B_1}(\gamma) - f_{B_0}(\gamma) = (s-1)p(\gamma), \tag{4}$$

where $p(\gamma) = (1-\alpha)\gamma^2 + (\alpha^2 n - 2\alpha - s)\gamma - \alpha^2 n^2 + (2\alpha^2 - 2\alpha + 1)sn + (3\alpha^2 - \alpha + 1)n - (3\alpha^2 - 5\alpha + 2)s^2 - (4\alpha^2 - 6\alpha + 3)s - 4\alpha^2 + 5\alpha - 3$.

Subcase 1.1. $0 \leq \alpha \leq \frac{2}{3}$.

Recall that $n = 2s + n_1 \geq 2s + 2$. By virtue of Inequality (3.3) and $s \geq 2$, we infer

$$-\frac{\alpha^2 n - 2\alpha - s}{2(1-\alpha)} < n - 2 < \gamma,$$

and so

$$\begin{aligned} p(\gamma) >& p(n-2) \\ =&(1-\alpha)n^2 + (2\alpha^2 s - 2\alpha s + \alpha^2 + \alpha - 3)n - (3\alpha^2 - 5\alpha + 2)s^2 \\ &- (4\alpha^2 - 6\alpha + 1)s - 4\alpha^2 + 5\alpha + 1 \\ =:& l(n,s). \end{aligned} \tag{5}$$

Recall that $s \geq 2$ and $n \geq 2s + 2$. We deduce

$$-\frac{2\alpha^2 s - 2\alpha s + \alpha^2 + \alpha - 3}{2(1-\alpha)} < 2s + 2 \leq n,$$

and so

$$\begin{aligned} l(n,s) \geq& l(2s+2, s) \\ =&(s^2 + 2s - 2)\alpha^2 - (3s^2 + 4s - 3)\alpha + 2s^2 + s - 1 \\ \geq& \frac{4}{9}(s^2 + 2s - 2) - \frac{2}{3}(3s^2 + 4s - 3) + 2s^2 + s - 1 \\ =& \frac{1}{9}(4s^2 - 7s + 1) \\ >& 0, \end{aligned} \tag{6}$$

where the last two inequalities hold from $\frac{3s^2+4s-3}{2(s^2+2s-2)} > \frac{2}{3} \geq \alpha$ and $s \geq 2$, respectively.

Using (4), (5), (6) and $s \geq 2$, we get

$$f_{B_1}(\gamma) = (s-1)p(\gamma) > (s-1)l(n,s) > 0.$$

As $\gamma = \gamma(n) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1)) > n - 2 > \gamma_2$ (see (3)), we deduce $\rho_\alpha(G) < \gamma(n)$ for $2 \leq s \leq \frac{n}{2} - 1$, which contradicts $\rho_\alpha(G) \geq \gamma(n)$.

Subcase 1.2. $\frac{2}{3} < \alpha < 1$.

According to Equality (1), we get

$$\begin{aligned} f_{B_1}(n-2) =&(3\alpha - 2)(1-\alpha)s^3 + (2\alpha^2 n - 2\alpha n - \alpha^2 + \alpha + 1)s^2 \\ &+ (n^2 - \alpha n^2 - \alpha^2 n + 3\alpha n - 3n - \alpha + 2)s \\ &+ \alpha n^2 - n^2 - 2\alpha n + 3n - 2 \\ =:& \varphi(s,n). \end{aligned}$$

By a simple computation, we obtain

$$\begin{aligned}\frac{\partial\varphi(s,n)}{\partial s} =&3(3\alpha-2)(1-\alpha)s^2+2(2\alpha^2n-2\alpha n-\alpha^2+\alpha+1)s\\&+n^2-\alpha n^2-\alpha^2n+3\alpha n-3n-\alpha+2.\end{aligned}$$

Notice that $\frac{2}{3}<\alpha<1$ and $n\geq\lambda(\alpha)=\frac{4}{1-\alpha}$. By a simple computation, we possess

$$\begin{aligned}\frac{\partial\varphi(s,n)}{\partial s}\Big|_{s=2} =&(1-\alpha)n^2+(7\alpha^2-5\alpha-3)n-40\alpha^2+63\alpha-18\\ \geq&(1-\alpha)\left(\frac{4}{1-\alpha}\right)^2+(7\alpha^2-5\alpha-3)\left(\frac{4}{1-\alpha}\right)-40\alpha^2+63\alpha-18\\ =&\frac{1}{1-\alpha}(40\alpha^3-75\alpha^2+61\alpha-14)\\ >&0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\varphi(s,n)}{\partial s}\Big|_{s=\frac{n}{2}-1} =&\frac{1}{4}((-\alpha^2+3\alpha-2)n^2+(12\alpha^2-28\alpha+16)n-24\alpha^2+48\alpha-24)\\ \leq&\frac{1}{4}\left((-\alpha^2+3\alpha-2)\left(\frac{4}{1-\alpha}\right)^2+(12\alpha^2-28\alpha+16)\left(\frac{4}{1-\alpha}\right)-24\alpha^2+48\alpha-24\right)\\ =&\frac{1}{4(1-\alpha)}(24\alpha^3-24\alpha^2-24\alpha+8)\\ <&0.\end{aligned}$$

This yields that $f_{B_1}(n-2)=\varphi(s,n)\geq\min\left\{\varphi(2,n),\varphi\left(\frac{n}{2}-1,n\right)\right\}$, because the highest degree coefficient of $\varphi(s,n)$ (view as a cubic polynomial of s) is positive, and $2\leq s\leq\frac{n}{2}-1$. In light of $\frac{2}{3}<\alpha<1$ and $n\geq\lambda(\alpha)=\frac{4}{1-\alpha}$, we get

$$\begin{aligned}\varphi(2,n) =&(1-\alpha)n^2+(6\alpha^2-4\alpha-3)n-28\alpha^2+42\alpha-10\\ \geq&(1-\alpha)\left(\frac{4}{1-\alpha}\right)^2+(6\alpha^2-4\alpha-3)\left(\frac{4}{1-\alpha}\right)-28\alpha^2+42\alpha-10\\ =&\frac{2}{1-\alpha}(14\alpha^3-23\alpha^2+18\alpha-3)\\ >&0,\end{aligned}$$

and

$$\begin{aligned}\varphi\left(\frac{n}{2}-1,n\right) =&\frac{1}{8}((\alpha^2-3\alpha+2)n^3-(4\alpha^2-16\alpha+14)n^2\\&-(4\alpha^2+8\alpha-24)n+16\alpha^2-24\alpha-8)\\ \geq&\frac{1}{8}((\alpha^2-3\alpha+2)\left(\frac{4}{1-\alpha}\right)^3-(4\alpha^2-16\alpha+14)\left(\frac{4}{1-\alpha}\right)^2\\&-(4\alpha^2+8\alpha-24)\left(\frac{4}{1-\alpha}\right)+16\alpha^2-24\alpha-8)\\ =&\frac{1}{8(1-\alpha)^2}(16\alpha^4-40\alpha^3+8\alpha^2+56\alpha-8)\\ >&0.\end{aligned}$$

Thus, we have $f_{B_1}(n-2) \geq \min\left\{\varphi(2,n), \varphi\left(\frac{n}{2}-1, n\right)\right\} > 0$ for $2 \leq s \leq \frac{n}{2}-1$. As $\gamma(n) = \rho_\alpha(K_1 \vee (K_{n-2} \cup K_1)) > n-2 > \gamma_2$ (see (3)), we infer $\rho_\alpha(G) < n-2 < \gamma(n)$ for $2 \leq s \leq \frac{n}{2}-1$, which is a contradiction to $\rho_\alpha(G) \geq \gamma(n)$.

Case 2. $G-S$ has no nontrivial connected component.

In this case, we have $G = K_s \vee iK_1$. If $i \geq s+2$, we can construct a new graph H_2 formed from G by adding an edge in iK_1. Then $i(H_2 - S) \geq s$ and $H_2 - S$ has exactly one nontrivial connected component. By Case 1, we deduce $\rho_\alpha(G) < \gamma(n)$, a contradiction. Hence, we obtain $i \leq s+1$. Together with $i \geq s$, we conclude $s \leq i \leq s+1$.

Subcase 2.1. $i = s$.

In this subcase, $n = 2s$ and $G = K_s \vee sK_1$. Consider the partition $V(G) = V(K_s) \cup V(sK_1)$. The corresponding quotient matrix of $A_\alpha(G)$ is equal to

$$B_2 = \begin{pmatrix} \alpha s + s - 1 & (1-\alpha)s \\ (1-\alpha)s & \alpha s \end{pmatrix},$$

whose characteristic polynomial equals

$$f_{B_2}(x) = x^2 - (2\alpha s + s - 1)x + 3\alpha s^2 - s^2 - \alpha s.$$

Notice that the partition $V(G) = V(K_s) \cup V(sK_1)$ is equitable. In terms of Lemma 2.4, $\rho_\alpha(G)$ is the largest root of $f_{B_2}(x) = 0$. By a direct calculation, we obtain

$$\rho_\alpha(G) = \frac{2\alpha s + s - 1 + \sqrt{(4\alpha^2 - 8\alpha + 5)s^2 - 2s + 1}}{2}.$$

Claim 1. $\rho_\alpha(G) < 2s-2$.

Proof. Let $M_1 = (2(2s-2) - (2\alpha s + s - 1))^2$ and $N_1 = (4\alpha^2 - 8\alpha + 5)s^2 - 2s + 1$. By a simple calculation, we get

$$M_1 - N_1 = 4(1-\alpha)s^2 - 4(4-3\alpha)s + 8.$$

Write $q_1(s) = 4(1-\alpha)s^2 - 4(4-3\alpha)s + 8$. If $0 \leq \alpha \leq \frac{2}{3}$, then $n = 2s \geq \lambda(\alpha) = 9$. Thus, we infer $s \geq 5$ and

$$\frac{4(4-3\alpha)}{8(1-\alpha)} = \frac{4-3\alpha}{2(1-\alpha)} < 5 \leq s,$$

and so

$$q_1(s) \geq q_1(5) = 28 - 40\alpha > 0.$$

If $\frac{2}{3} < \alpha < 1$, then $n = 2s \geq \lambda(\alpha) = \frac{4}{1-\alpha}$. Thus, we conclude $s \geq \frac{2}{1-\alpha}$ and

$$\frac{4(4-3\alpha)}{8(1-\alpha)} = \frac{4-3\alpha}{2(1-\alpha)} < \frac{2}{1-\alpha} \leq s.$$

Hence, we obtain

$$q_1(s) \geq q_1\left(\frac{2}{1-\alpha}\right) = \frac{1}{1-\alpha}(16\alpha - 8) > 0.$$

From the above discussion, we have $q_1(s) > 0$, which yields $M_1 > N_1$, that is,

$$\begin{aligned} 2s-2 = &\frac{2\alpha s + s - 1 + \sqrt{(2(2s-2) - (2\alpha s + s - 1))^2}}{2} \\ > &\frac{2\alpha s + s - 1 + \sqrt{(4\alpha^2 - 8\alpha + 5)s^2 - 2s + 1}}{2} \\ = &\rho_\alpha(G). \end{aligned}$$

This completes the proof of Claim 1. □

It follows from Inequality (3), $n = 2s$ and Claim 1 that

$$\rho_\alpha(G) < 2s - 2 = n - 2 < \gamma(n),$$

which contradicts $\rho_\alpha(G) \geq \gamma(n)$.

Subcase 2.2. $i = s + 1$.

In this subcase, $n = 2s + 1$ and $G = K_s \vee (s + 1)K_1$. In terms of the partition $V(G) = V(K_s) \cup V((s + 1)K_1)$, the quotient matrix of $A_\alpha(G)$ is

$$B_3 = \begin{pmatrix} \alpha s + s + \alpha - 1 & (1 - \alpha)(s + 1) \\ (1 - \alpha)s & \alpha s \end{pmatrix}.$$

Then the characteristic polynomial of B_3 is

$$f_{B_3}(x) = x^2 - (2\alpha s + s + \alpha - 1)x + (3\alpha - 1)s^2 + (\alpha - 1)s.$$

Since the partition $V(G) = V(K_s) \cup V((s + 1)K_1)$ is equitable, using Lemma 2.4, $\rho_\alpha(G)$ is the largest root of $f_{B_3}(x) = 0$. By a direct computation, we have

$$\rho_\alpha(G) = \frac{2\alpha s + s + \alpha - 1 + \sqrt{(4\alpha^2 - 8\alpha + 5)s^2 + (4\alpha^2 - 6\alpha + 2)s + (\alpha - 1)^2}}{2}.$$

Claim 2. $\rho_\alpha(G) < 2s - 1$.

Proof. Let $M_2 = (2(2s - 1) - (2\alpha s + s + \alpha - 1))^2$ and $N_2 = (4\alpha^2 - 8\alpha + 5)s^2 + (4\alpha^2 - 6\alpha + 2)s + (\alpha - 1)^2$. By a direct computation, we obtain

$$M_2 - N_2 = 4(1 - \alpha)s^2 - 4(2 - \alpha)s + 4\alpha.$$

Set $q_2(s) = 4(1 - \alpha)s^2 - 4(2 - \alpha)s + 4\alpha$. If $0 \leq \alpha \leq \frac{2}{3}$, then $n = 2s + 1 \geq \lambda(\alpha) = 9$. Thus, we possess $s \geq 4$ and

$$\frac{4(2 - \alpha)}{8(1 - \alpha)} = \frac{2 - \alpha}{2(1 - \alpha)} < 4 \leq s,$$

and so

$$q_2(s) \geq q_2(4) = 32 - 44\alpha > 0.$$

If $\frac{2}{3} < \alpha < 1$, then $n = 2s + 1 \geq \lambda(\alpha) = \frac{4}{1-\alpha}$. Therefore, we get $s \geq \frac{2}{1-\alpha} - \frac{1}{2}$ and

$$\frac{4(2 - \alpha)}{8(1 - \alpha)} = \frac{2 - \alpha}{2(1 - \alpha)} < \frac{2}{1 - \alpha} - \frac{1}{2} \leq s.$$

Thus, we conclude

$$q_2(s) \geq q_2\left(\frac{2}{1 - \alpha} - \frac{1}{2}\right) = \frac{1}{1 - \alpha}(-\alpha^2 + 12\alpha - 3) > 0.$$

From the above discussion, we deduce $q_2(s) > 0$, which implies $M_2 > N_2$, namely,

$$\begin{aligned} 2s - 1 =& \frac{2\alpha s + s + \alpha - 1 + \sqrt{(2(2s - 1) - (2\alpha s + s + \alpha - 1))^2}}{2} \\ >& \frac{2\alpha s + s + \alpha - 1 + \sqrt{(4\alpha^2 - 8\alpha + 5)s^2 + (4\alpha^2 - 6\alpha + 2)s + (\alpha - 1)^2}}{2} \\ =& \rho_\alpha(G). \end{aligned}$$

Claim 2 is proved. □

According to (3), $n = 2s + 1$ and Claim 2, we have

$$\rho_\alpha(G) < 2s - 1 = n - 2 < \gamma(n),$$

which is a contradiction to $\rho_\alpha(G) \geq \gamma(n)$. This completes the proof of Theorem 1.1. □

Declaration of competing interest

The authors declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to express their deepest gratitude to the anonymous referees for offering many helpful comments and suggestions.

References

[1] K. Ando, Y. Egawa, A. Kaneko, K. Kawarabayashi, H. Matsuda, *Path factors in claw-free graphs*, Discrete Math. **243** (2002), 195–200.
[2] G. Ao, R. Liu, J. Yuan, *Spectral radius and spanning trees of graphs*, Discrete Math. **346** (2023), 113400.
[3] H. Broersma, H. Tuinstra, *Independence trees and Hamilton cycles*, J. Graph Theory **29** (1998), 227–237.
[4] H. Chen, X. Lv, J. Li, S. Xu, *Sufficient conditions for spanning trees with constrained leaf distance in a graph*, Discuss. Math. Graph Theory, https://doi.org/10.7151/dmgt.2530
[5] G. Dai, *Toughness and isolated toughness conditions for path-factor critical covered graphs*, RAIRO Oper. Res. **57** (2023), 847–856.
[6] Y. Egawa, M. Furuya, *The existence of a path-factor without small odd paths*, Electron. J. Combin. **25**(1) (2018), #P1.40.
[7] D. Fan, S. Goryainov, X. Huang, H. Lin, *The spanning k-trees, perfect matchings and spectral radius of graphs*, Linear Multilinear Algebra **70** (2022), 7264–7275.
[8] E. Flandrin, T. Kaiser, R. Kužel, H. Li, Z. Ryjáček, *Neighborhood unions and extremal spanning trees*, Discrete Math. **308** (2008), 2343–2350.
[9] W. Gao, W. Wang, Y. Chen, *Tight bounds for the existence of path factors in network vulnerability parameter settings*, Int. J. Intell. Syst. **36** (2021), 1133–1158.
[10] W. Haemers, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. **227** (1995), 593–616.
[11] M. Kano, G. Katona, Z. Király, *Packing paths of length at least two*, Discrete Math. **283** (2004), 129–135.
[12] M. Kano, A. Saito, *Star-factors with large components*, Discrete Math. **312** (2012), 2005–2008.
[13] A. Kaneko, *Spanning trees with constraints on the leaf degree*, Discrete Appl. Math. **115** (2001), 73–76.
[14] A. Kaneko, M. Kano, K. Suzuki, *Spanning trees with leaf distance at least four*, J. Graph Theory **55** (2007), 83–90.
[15] S. Kim, S. O, J. Park, H. Ree, *An odd* $[1, b]$*-factor in regular graphs from eigenvalues*, Discrete Math. **343** (2020), 111906.
[16] A. Kyaw, *A sufficient condition for a graph to have a k-tree*, Graphs Combin. **17** (2001), 113–121.
[17] A. Kyaw, *Spanning trees with at most k leaves in* $K_{1,4}$*-free graphs*, Discrete Math. **311** (2011), 2135–2142.
[18] R. Johansson, *An El-Zahár type condition ensuring path-factors*, J. Graph Theory **28**(1) (1998), 39–42.
[19] M. Johnson, D. Paulusma, C. Wood, *Path factors and parallel knock-out schemes of almost claw-free graphs*, Discrete Math. **310** (2010), 1413–1423.
[20] G. Liu, J. Wang, (a, b, k)*-critical graphs*, Adv. Math. (China) **27**(6) (1998), 536–540.
[21] H. Liu, *Sun toughness and path-factor uniform graphs*, RAIRO Oper. Res. **56**(6) (2022), 4057–4062.
[22] H. Liu, X. Pan, *Independence number and minimum degree for path-factor critical uniform graphs*, Discrete Appl. Math. **359** (2024), 153–158.
[23] X. Lv, *A degree condition for graphs being fractional* (a, b, k)*-critical covered*, Filomat **37**(10) (2023), 3315–3320.
[24] H. Matsuda, *Fan-type results for the existence of* $[a, b]$*-factors*, Discrete Math. **306** (2006), 688–693.
[25] V. Nikiforov, *Merging the A- and Q-spectral theories*, Appl. Anal. Discrete Math. **11** (2017), 81–107.
[26] S. O, *Spectral radius and matchings in graphs*, Linear Algebra Appl. **614** (2021), 316–324.
[27] S. Win, *On a connection between the existence of k-trees and the toughness of a graph*, Graphs Combin. **5** (1989), 201–205.
[28] J. Wu, *A sufficient condition for the existence of fractional* (g, f, n)*-critical covered graphs*, Filomat **38**(6) (2024), 2177–2183.
[29] J. Wu, *Characterizing spanning trees via the size or the spectral radius of graphs*, Aequationes Math. **98**(6) (2024), 1441-1445.
[30] L. You, M. Yang, W. So, W. Xi, *On the spectrum of an equitable quotient matrix and its application*, Linear Algebra Appl. **577** (2019), 21–40.
[31] S. Zhou, *A neighborhood union condition for fractional* (a, b, k)*-critical covered graphs*, Discrete Appl. Math. **323** (2022), 343–348.
[32] S. Zhou, *Some results on path-factor critical avoidable graphs*, Discuss. Math. Graph Theory **43**(1) (2023), 233–244.
[33] S. Zhou, H. Liu, *Two sufficient conditions for odd* $[1, b]$*-factors in graphs*, Linear Algebra Appl. **661** (2023), 149–162.
[34] S. Zhou, Q. Pan, L. Xu, *Isolated toughness for fractional* $(2, b, k)$*-critical covered graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **24**(1) (2023), 11–18.
[35] S. Zhou, Q. Pan, Y. Xu, *A new result on orthogonal factorizations in networks*, Filomat **38**(20) (2024), 7235-7244.
[36] S. Zhou, Z. Sun, H. Liu, *Distance signless Laplacian spectral radius for the existence of path-factors in graphs*, Aequationes Math. **98**(3) (2024), 727–737.

[37] S. Zhou, Z. Sun, H. Liu, *$\mathcal{D}$-index and $\mathcal{Q}$-index for spanning trees with leaf degree at most k in graphs*, Discrete Math. **347**(5) (2024), 113927.
[38] S. Zhou, Z. Sun, Y. Zhang, *Spectral radius and k-factor-critical graphs*, J. Supercomput. **81**(3) (2025), 456.
[39] S. Zhou, J. Wu, *Spanning k-trees and distance spectral radius in graphs*, J. Supercomput. **80**(16) (2024), 23357–23366.
[40] S. Zhou, Y. Xu, Z. Sun, *Some results about star-factors in graphs*, Contrib. Discrete Math. **19**(3) (2024), 154–162.
[41] S. Zhou, Y. Zhang, H. Liu, *Some properties of (a, b, k)-critical graphs*, Filomat **38**(16) (2024), 5885–5894.
[42] S. Zhou, Y. Zhang, H. Liu, *Spanning k-trees and distance signless Laplacian spectral radius of graphs*, Discrete Appl. Math. **358** (2024), 358–365.
[43] S. Zhou, Y. Zhang, Z. Sun, *The A_α-spectral radius for path-factors in graphs*, Discrete Math. **347**(5) (2024), 113940.