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Approximation by Bézier-Baskakov-Jain type operators

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Abstract. In this paper, we introduce the Bézier variant of the α-Baskakov-Jain type operators. We explore the elements of the Lipschitz type space, propose a direct approximation theorem using the modulus of continuity, and assess the approximation rate for functions possessing derivatives of bounded variation. The use of computer graphics lends support to the validity of the theoretical components in this study.

1. Introduction

Baskakov type operators were introduced by Aral and Erbay [4] which is based on $\alpha \in [0,1]$ is defined as:

$$\mathscr{B}_n^{(\alpha)}(\xi; w) = \sum_{i=0}^{\infty} b_{n,i}^{(\alpha)}(w)\xi\left(\frac{i}{n}\right), w \in [0,\infty), \tag{1}$$

where

$$b_{n,i}^{(\alpha)} = \frac{w^{i-1}}{(1+w)^{n+i-1}}\left[\frac{\alpha w}{(1+w)}\binom{n+i-1}{i} - (1-\alpha)(1+w)\binom{n+i-3}{i-2} + (1-\alpha)w\binom{n+i-1}{i}\right].$$

The operators $\mathscr{B}_n^{(\alpha)}(\xi; w)$ reduce to the Baskakov operators [5] for $\alpha = 1$.

Gupta [17] introduced a general family of Durrmeyer type operators and calculated direct properties for these operators. Srivastava et al. [35] delved into the idea of statistical probability and statistical convergence for sequences of random variables and of real numbers, which are defined over a Banach space, employing deferred Nörlund summability mean. In their work, Kajla et al. [27] presented a generalized version of the operators (1) known as Durrmeyer type operators. They further analyze and determine the uniform convergence properties of these operators. Gupta and Srivastava [20] considered a general family of operators that possess the property of preserving linear functions. Braha et al. [7] examined the

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Baskakov-Schurer-Szász-Stancu operators, proving a Korovkin-type theorem, a Grüss-Voronovskaya-type theorem, and determining the convergence rate. Mohiuddine et al. [30] introduced Baskakov-Durrmeyer type operators that are parameter-dependent and quantitatively analyzed their approximation properties. Recently, in another study, Mohiuddine et al. [31] presented a variant of the operators (1) known as the Stancu-Kantorovich variant. They also established several direct results regarding these operators. Linear positive operators have been the focus of investigation by several researchers, as highlighted in references such as [1, 3, 14–16, 22, 23, 28, 32, 33, 41–51] and related literature. In numerous applications spanning fractional calculus, the demand arises for fractional derivatives of varying kinds (see references [12, 37–40]).

Let $\tau > 0$, $\gamma > 0$ and $\alpha \in [0,1]$, where $C_\gamma[0,\infty) := \xi \in C[0,\infty) : |\xi(v)| \leq N_\xi e^{\gamma v},$ for some $N_\xi > 0$, Kajla et al.[26] constructed $\alpha-$Baskakov-Jain type operators is defined as:

$$\mathscr{G}_{n,\tau}^{(\alpha)}(\xi;w) = \sum_{i=0}^{\infty} b_{n,i}^{(\alpha)}(w) \int_{o}^{\infty} b_{n,i}^{\tau}(v)\xi(v)dv, \tag{2}$$

where $b_{n,i}^{\tau}(v) = \dfrac{\tau}{B(i+1,\frac{n}{\tau})} \dfrac{(\tau v)^i}{(1+\tau v)^{\frac{n}{\tau}+i+1}}$ and $b_{n,i}^{(\alpha)}(w)$ is defined as above.

The defined form of the operators $\mathscr{G}_{n,\tau}^{(\alpha)}(\xi;w)$ for $\theta \geq 1$ is demonstrated in the Bézier variant,

$$\mathscr{G}_{n,\tau,\theta}^{\alpha}(\xi;w) = \sum_{i=0}^{\infty} Q_{n,i,\theta}^{\alpha} \int_{0}^{\infty} b_{n,i}^{\tau}(v)\xi(v)dv, \tag{3}$$

where $Q_{n,i,\theta}^{\alpha}(w) = (J_{n,i}^{\alpha}(w))^{\theta} - (J_{n,i+1}^{\alpha}(w))^{\theta}$, and $J_{n,i}^{\alpha}(w) = \sum_{j=i}^{\infty} b_{n,j}^{(\alpha)}(w)$ with Baskakov basis function. In another approach, we can also write the operators (3) as

$$\mathscr{G}_{n,\tau,\theta}^{\alpha}(\xi;w) = \int_{0}^{\infty} P_{n,\tau,\theta}^{\alpha}(w,v)\xi(v)dv, \quad w \in [0,\infty), \tag{4}$$

where

$$P_{n,\tau,\theta}^{\alpha}(w,v) = \sum_{i=0}^{\infty} Q_{n,i,\theta}^{\alpha} b_{n,i}^{\tau}(v).$$

For $\theta = 1$, the operators $\mathscr{G}_{n,\tau,\theta}^{\alpha}\xi$ reduce to $\mathscr{G}_{n,\tau}^{(\alpha)}\xi$.

The primary emphasis of this paper is on presenting the Bézier variant of $\alpha-$Baskakov-Jain type operators (3). After that we use Ditzian-Totik modulus of smoothness for calculating direct approximation theorem. Furthermore, we examine the convergence rate for these operators (3) for differential functions with bounded variation derivatives on finite sub-intervals of $(0,1)$.

2. Auxiliary Results

Let $\mathbb{N}$ represent the set of positive integers, and let $\mathbb{N}_0$ denote the set obtained by including zero in $\mathbb{N}$. Let C be a positive constant independent of w and n, though C may vary across different occurrences. Using this, we derive some auxiliary results for the operators in (3).

We calculate uniform approximation by using the test functions $e_m(w) = w^m$, for $m \in \mathbb{N}_0$ for this linear positive operators. We recall some results which is defined in [26].

Lemma 2.1. *[26] The moments of the operators $\mathscr{G}_{n,\tau}^{(\alpha)}(\xi;w)$ for $w \in [0,\infty)$ are as follows:*

(i) $\mathscr{G}_{n,\tau}^{(\alpha)}(e_0;w) = 1$;

(ii) $\mathscr{G}_{n,\tau}^{(\alpha)}(e_1;w) = \dfrac{w(n+2\alpha-2)}{(n-\tau)} + \dfrac{1}{(n-\tau)}$;

(iii) $\mathscr{G}_{n,\tau}^{(\alpha)}(e_2;w) = \dfrac{nw^2(-3+n+4\alpha)}{(n-2\tau)(n-\tau)} + \dfrac{w(4n+10(-1+\alpha))}{(n-2\tau)(n-\tau)} + \dfrac{2}{(n-2\tau)(n-\tau)}$.

Lemma 2.2. *[26] For any $n \in \mathbb{N}, n > \tau$, the central moments up to the second order for $\alpha-$Baskakov-Jain type operators (2) are given by:*

$$\mathscr{G}_{n,\tau}^{(\alpha)}((v-w);w) = \frac{w(\tau+2\alpha-2)}{(n-\tau)} + \frac{1}{(n-\tau)}; \mathscr{G}_{n,\tau}^{(\alpha)}((v-w)^2;w) = \frac{w^2(n+\tau(-8+2\tau+n+8\alpha))}{(n-2\tau)(n-\tau)} + \frac{2w(-5+2\tau+n+5\alpha)}{(n-2\tau)(n-\tau)} + \frac{2}{(n-2\tau)(n-\tau)}.$$

Lemma 2.3. *Consider ξ as a continuous function with real values which is bounded on $[0,\infty)$, and $\|\xi\| = \sup_{w\in[0,\infty)} |\xi(w)|$, then $|\mathscr{G}_{n,\tau}^{(\alpha)}(\xi)| \le \|\xi\|$.*

Lemma 2.4. *Consider ξ as a continuous function with real values which is bounded on $[0,\infty)$, then $|g_{n,\tau,\theta}^{\alpha}(\xi)| \le \theta\|\xi\|$.*

Proof. By the well known property $|c^\beta - d^\beta| \le \beta|c-d|$, with $\beta \ge 1, 0 \le c, d \le 1$ and properties of $Q_{n,i,\theta}^{\alpha}(w)$, we get

$$(J_{n,i}^{\alpha}(w))^\theta - (J_{n,i+1}^{\alpha}(w))^\theta \le \theta(J_{n,i}^{\alpha}(w) - J_{n,i+1}^{\alpha}(w)) = \theta b_{n,i}^{(\alpha)}(w). \tag{5}$$

Therefore, based on the definition of $g_{n,\tau,\theta}^{\alpha}(\xi)$ operators and taking into account Lemma 2.2, we derive $|\mathscr{G}_{n,\tau,\theta}^{\alpha}(\xi)| \le \theta|g_{n,\tau}^{\alpha}(\xi)| \le \theta\|\xi\|$. □

Lemma 2.5. *Let $w \in (0,\infty)$, then for sufficiently large n and $\theta \ge 1$, we get*

i) $\zeta_{n,\tau,\theta}^{\alpha}(w,y) = \displaystyle\int_0^y P_{n,\tau,\theta}^{\alpha}(w,v)dv \le \frac{\theta\lambda_\tau(\alpha)}{n}\frac{\phi^2(w)}{(w-y)^2}, \quad 0 \le y < w,$

ii) $1 - \zeta_{n,\tau,\theta}^{\alpha}(w,z) = \displaystyle\int_z^\infty P_{n,\tau,\theta}^{\alpha}(w,v)dv \le \frac{\theta\lambda_\tau(\alpha)}{n}\frac{\phi^2(w)}{(z-w)^2}, \quad w < z < \infty,$

where $\lambda_\tau(\alpha)$ is a positive constant which is depending on τ and α.

Proof.

i) Using (Eq.(2.2) from [26]) and Lemma 2.4, we have

$$\begin{aligned}\zeta_{n,\tau,\theta}^{\alpha}(w,y) &= \int_0^y P_{n,\tau,\theta}^{\alpha}(w,v)dt \le \int_0^y \left(\frac{w-t}{w-y}\right)^2 P_{n,\tau,\theta}^{\alpha}(w,v)dv \\ &\le \mathscr{G}_{n,\tau,\theta}^{\alpha}((v-w)^2;z)\,(z-y)^{-2} \le \theta\mathscr{G}_{n,\tau}^{(\alpha)}((v-w)^2;w)(w-y)^{-2} \\ &\le \frac{\theta\lambda_\tau(\alpha)}{n}\frac{\phi^2(w)}{(w-y)^2}, \quad 0 \le y < w.\end{aligned}$$

ii) The second relation can be illustrated in a similar manner. □

3. Local Approximation

Let us now consider the Lipschitz-type space in two parameters for $a_1 \ge 0$ and $a_2 > 0$,

$$Lip_M^{(a_1,a_2)}(\beta) := \left\{\xi \in C_B[0,\infty) : |\xi(v) - \xi(w)| \le \frac{|v-w|^\beta}{(v + a_1w^2 + a_2w)^{\beta/2}} : w \in (0,\infty)\right\}, \tag{6}$$

where $\beta \in (0,1]$ and M is a positive constant.

Theorem 3.1. *Let $\xi \in Lip_M^{(a_1,a_2)}(\beta)$. Then, for every $w > 0$, we have*

$$|\mathcal{G}_{n,\tau,\theta}^{\alpha}(\xi;w) - \xi(w)| \leq M\left(\theta \frac{\mu_{n,\tau,\alpha}^{(2)}(w)}{(a_1w^2 + a_2)}\right)^{\beta/2},$$

where $\mu_{n,\tau,\alpha}^{(2)}(w) = \mathcal{G}_{n,\tau,\theta}^{\alpha}((v-w)^2;w)$.

Proof. For $\beta = 1$, we prove the theorem first, we can also write

$$\begin{aligned} |\mathcal{G}_{n,\tau,\theta}^{\alpha}(\xi;w) - \xi(w)| &\leq \mathcal{G}_{n,\tau,\theta}^{\alpha}(|\xi(v) - \xi(w)|;w) \\ &\leq M\mathcal{G}_{n,\tau,\theta}^{\alpha}\left(\frac{|v-w|}{\sqrt{v + a_1w^2 + a_2}};w\right). \end{aligned}$$

Applying Lemma 2.2 and (5) and using the cauchy-schwarz inequality, the fact that $\frac{1}{\sqrt{1 + a_1w^2 + a_2w}} < \frac{1}{\sqrt{a_1w^2 + a_2w}}$, the above inequality implies that

$$\begin{aligned} |\mathcal{G}_{n,\tau,\theta}^{\alpha}(\xi;w) - \xi(w)| &\leq M\frac{1}{\sqrt{a_1w^2 + a_2w}}\mathcal{G}_{n,\tau,\theta}^{\alpha}(|v-w|;w) \\ &\leq M\frac{1}{\sqrt{a_1w^2 + a_2w}}\left(\mathcal{G}_{n,\tau,\theta}^{\alpha}((v-w)^2;w)\right)^{1/2} \\ &\leq M\left(\sqrt{\theta\frac{\mu_{n,\tau,\alpha}^{(2)}(w)}{(a_1w^2 + a_2w)}}\right). \end{aligned}$$

Thus, the result holds for $\beta = 1$. Now, let $0 < \beta < 1$, then by Lemma 2.2, (5) and using the Hölder inequality with $p = \frac{1}{\beta}$ and $q = \frac{1}{1-\beta}$, we get

$$\begin{aligned} |\mathcal{G}_{n,\tau,\theta}^{\alpha}(\xi;w) - \xi(w)| &\leq \sum_{i=0}^{\infty} Q_{n,i,\alpha}^{\theta} \int_0^{\infty} b_{n,i}^{\tau}(w)|\xi(v) - \xi(w)|dv \\ &\leq \left\{\sum_{i=0}^{\infty} Q_{n,i,\alpha}^{\theta}\left(\int_0^{\infty} b_{n,i}^{\tau}(w)|\xi(v) - \xi(w)|dv\right)^{1/\beta}\right\}^{\beta} \\ &\leq \left\{\sum_{i=0}^{\infty} Q_{n,i,\alpha}^{\theta}\int_0^{\infty} b_{n,i}^{\tau}(w)|\xi(v) - \xi(w)|^{1/\beta}dv\right\}^{\beta} \\ M &\leq \left\{\sum_{i=0}^{\infty} Q_{n,i,\alpha}^{\theta}\int_0^{\infty} b_{n,i}^{\tau}(w)\frac{|v-w|}{\sqrt{v + a_1w^2 + a_2w}}dv\right\}^{\beta} \\ &\leq \frac{M}{(a_1w^2 + a_2w)^{\beta/2}}\left\{\sum_{i=0}^{\infty} Q_{n,i,\alpha}^{\theta}\int_0^{\infty} b_{n,i}^{\tau}(w)|v-w|dv\right\}^{\beta} \\ &\leq \frac{M}{(a_1w^2 + a_2w)^{\beta/2}}\left(\mathcal{G}_{n,\tau,\theta}^{\alpha}((v-w)^2;w)\right)^{\beta/2} \\ &\leq M\left(\theta\frac{\mu_{n,\tau,\alpha}^{(2)}(w)}{(a_1w^2 + a_2)}\right)^{\beta/2}. \end{aligned}$$

Hence, the proof of theorem 3.1 is completed. □

Next, for $\xi \in C_B[0,\infty)$, using the Lipschitz-type maximal function of order ρ, we obtain the local direct estimate of the operators defined in (3) which is introduced by Lenze [29] as

$$\tilde{\omega}_{\rho}(\xi, w) = \sup_{v\neq w,\ v\in[0,\infty)} \frac{|\xi(v)-\xi(w)|}{|v-w|^{\rho}}, w \in (0,\infty) \text{ and } \rho \in (0,1]. \tag{7}$$

Theorem 3.2. *Let* $0<\rho<1$ *and* $\xi \in C_B[0,\infty)$. *Then, for all* $w\in[0,\infty)$, *we have*

$$|\mathscr{G}^{\alpha}_{n,\tau,\theta}(\xi;w)-\xi(w)| \le \tilde{\omega}_{\rho}(\xi,w)(\theta\mu^{(2)}_{n,\tau,\alpha}(w))^{\frac{\rho}{2}}.$$

Proof. From the results of (7), we get

$$|\mathscr{G}^{\alpha}_{n,\tau,\theta}(\xi;w)-\xi(w)| \le \tilde{\omega}_{\rho}(\xi,w)\mathscr{G}^{\alpha}_{n,\tau,\theta}(|v-w|^{\rho};w).$$

From (5) and Lemma 2.2 and applying the Hölder's inequality with $p=\frac{2}{\rho}$ and $\frac{1}{q}=1-\frac{1}{p}$, we have

$$\begin{aligned} |\mathscr{G}^{\alpha}_{n,\tau,\theta}(\xi;w)-\xi(w)| &\le \tilde{\omega}_{\rho}(\xi,w)\mathscr{G}^{\alpha}_{n,\tau,\theta}(|v-w|^{2};w)^{\rho/2} \\ &= \tilde{\omega}_{\rho}(\xi,w)(\theta\mu^{(2)}_{n,\tau,\alpha}(w))^{\frac{\rho}{2}}. \end{aligned}$$

Hence, the proof of theorem 3.2 is completed. □

Consider $H_2[0,\infty)$ as the set of functions ξ defined on $[0,\infty)$ that satisfy the condition that $|\xi(w)| \le M_{\xi}(1+w^2)$, where M_{ξ} is a constant which depends only on ξ i.e we follows [2, 4–6, 26, 29, 36]. $C_2[0,\infty)$ represents the subspace of $H_2[0,\infty)$ consisting of continuous functions. If $\lim_{w\to\infty} |\xi(w)|(1+w^2)^{-1}$ and $\xi \in C_2[0,\infty)$ finitely exists, i.e $\xi \in C_2^*[0,\infty)$. Then norm on $\xi \in C_2^*[0,\infty)$ is given by

$$\|\xi\|_2 := \sup_{w\in[0,\infty)} \frac{|\xi(w)|}{1+w^2}$$

on the closed interval $[b,c]$, the usual modulus of continuity of ξ for any $c>0$ is define as

$$\omega_c(\xi;\delta) = \sup_{|v-w|\le\delta, v,w\in[b,c]} |\xi(v)-\xi(w)|.$$

Theorem 3.3. *Let* $\xi \in C_2[0,\infty)$ *and for every* $w\in[0,c]$, $c>0$ *and* $n\in\mathbb{N}$, $\omega_{c+1}(\xi;\delta)$ *represents as the modulus of continuity is defined on the finite interval* $[0,c+1]$ *which is a subset of* $[0,\infty)$. *Then, we get*

$$|\mathscr{G}^{\alpha}_{n,\tau,\theta}(\xi;w)-\xi(w)| \le 4\theta M_{\xi}(1+c^2)\mu^{(2)}_{n,\tau,\alpha}(w) + 2\omega_{c+1}\Big(\xi;\sqrt{\theta\mu^{(2)}_{n,\tau,\alpha}(w)}\Big).$$

Proof. From [21], for $w\in[b,c]$ and $v\in[0,\infty)$, we have

$$|\xi(v)-\xi(w)| \le 4M_{\xi}(1+c^2)(v-w)^2 + \Big(1+\frac{|v-w|}{\delta}\Big)\omega_{c+1}(v;w),\ \delta>0.$$

Using (5)and by Cauchy-Schwarz inequality and applying $\mathscr{G}^{\alpha}_{n,\tau,\theta}(\cdot;w)$ to the above inequality, we get

$$\begin{aligned} |\mathscr{G}^{\alpha}_{n,\tau,\theta}(\xi;w)-\xi(w)| &\le 4\theta M_{\xi}(1+c^2)\mathscr{G}^{\alpha}_{n,\tau,\theta}((v-w)^2;w) + \omega_{c+1}(v;w)\Big(1+\frac{1}{\delta}\mathscr{G}^{\alpha}_{n,\tau,\theta}(|v-w|;w) \\ &\le 4\theta M_{\xi}(1+c^2)\mu^{(2)}_{n,\tau,\alpha}(w) + 2\omega_{c+1}\Big(\xi;\sqrt{\theta\mu^{(2)}_{n,\tau,\alpha}(w)}\Big), \end{aligned}$$

choosing $\delta = \sqrt{\theta\mu^{(2)}_{n,\tau,\alpha}(w)}$, we get the desired result. □

4. Rate of Convergence

Consider $\gamma \geq 0$ and $\xi \in DBV_\gamma(0,\infty)$ to represent the set of all differentiable functions defined on $(0,\infty)$ and possessing derivatives ξ' of bounded variation on each finite sub-interval and $|\xi(v)| \leq Mv^\gamma$, some $M > 0$ and for all $v > 0$. The functions $\xi \in DBV_\gamma(0,\infty)$ is represented as

$$\xi(w) = \int_0^w g(v)dv + \xi(0).$$

On any finite sub-interval of $(0,\infty)$, g demonstrates bounded variation.

Theorem 4.1. *Let $\xi \in DBV_\gamma(0,\infty)$, $\bigvee_c^d(\xi'_w)$ be the total variation of ξ'_w on $[c,d] \subset (0,\infty)$ and $\theta \geq 1$. Then, for sufficiently large n and every $w \in (0,\infty)$, we get*

$$\begin{aligned}\left|\mathscr{G}^\alpha_{n,\tau,\theta}(\xi;w) - \xi(w)\right| &\leq \frac{\sqrt{\theta}}{\theta+1}\left|\xi'(w+) + \theta\xi'(w-)\right|\sqrt{\frac{\lambda_\tau(\alpha)}{n}}\phi(w) + \sqrt{\frac{\lambda_\tau(\alpha)}{n}}\,\phi(w)\frac{\theta^{3/2}}{\theta+1}\left|\xi'(w+) - \xi'(w-)\right| \\ &+\frac{\theta\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\ ]}\bigvee_{w-w/i}^{w}(\xi'_w) + \frac{w}{\sqrt{n}}\bigvee_{w-w/\sqrt{n}}^{w}(\xi'_w) \\ &+\frac{\theta\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\ ]}\bigvee_{w}^{w+w/i}(\xi'_w) + \frac{w}{\sqrt{n}}\bigvee_{w}^{w+w/\sqrt{n}}(\xi'_w),\end{aligned}$$

where $\lambda_\tau(\alpha)$ is a positive constant which is depending on τ,α and the auxiliary function ξ'_w is also define as

$$\xi'_w(v) = \begin{cases} \xi'(v) - \xi'(w-), & 0 \leq v < w \\ 0, & v = w \\ \xi'(v) - \xi'(w+) & w < v \leq 1. \end{cases}$$

Proof. Since $\int_0^\infty P^\alpha_{n,\tau,\theta}(w,v)dv = \mathscr{G}^\alpha_{n,\tau,\theta}(e_0;w) = 1$, we have

$$\mathscr{G}^\alpha_{n,\tau,\theta}(\xi;w) - \xi(w) = \int_0^\infty (\xi(v) - \xi(w))\,P^\alpha_{n,\tau,\theta}(w,v)dv = \int_0^\infty \left(\int_w^v \xi'(t)dt\right)P^\alpha_{n,\tau,\theta}(w,v)dv. \tag{8}$$

Using the value of the function ξ'_w, for any $\xi \in DBV_\gamma(0,\infty)$, we have

$$\begin{aligned}\xi'(v) = \frac{1}{\theta+1}\Big(\xi'(w+) + \theta\xi'(w-)\Big) + \xi'_w(v) + \frac{1}{2}\Big(\xi'(w+) - \xi'(w-)\Big)\Big(\mathrm{sgn}(v-w) + \frac{\theta-1}{\theta+1}\Big) \\ + \delta_w(v)\Big(\xi'(w) - \frac{1}{2}\Big(\xi'(w+) + \xi'(w-)\Big)\Big),\end{aligned} \tag{9}$$

where

$$\delta_w(v) = \begin{cases} 1\ , & w = v \\ 0\ , & w \neq v. \end{cases}$$

It is clear that

$$\int_0^\infty P^\alpha_{n,\tau,\theta}(w,v)\int_w^v \Big(\xi'(w) - \frac{1}{2}\Big(\xi'(w+) + \xi'(w-)\Big)\Big)\delta_w(t)dtdv = 0.$$

Using the operators (4), then direct calculation lead us to

$$\begin{aligned} E_1 &= \int_0^\infty \Big(\int_w^v \frac{1}{\theta+1}\Big(\xi'(w+) + \theta\xi'(w-)\Big)dt\Big)P^\alpha_{n,\tau,\theta}(w,v)dv \\ &= \frac{1}{\theta+1}\Big|\xi'(w+) + \theta\xi'(w-)\Big|\int_0^\infty |v-w| P^\alpha_{n,\tau,\theta}(w,v)dv \\ &\le \frac{1}{\theta+1}\Big(\xi'(w+) + \theta\xi'(w-)\Big)\Big(\mathcal{G}^\alpha_{n,\tau,\theta}((e_1-w)^2;w)\Big)^{1/2} \le \frac{\sqrt{\theta}}{\theta+1}\Big|\xi'(w+) + \theta\xi'(w-)\Big|\sqrt{\frac{\lambda_\tau(\alpha)}{n}}\phi(w) \end{aligned} \tag{10}$$

and

$$\begin{aligned} E_2 &= \int_0^\infty \Big(\int_w^v \frac{1}{2}\Big(\xi'(w+) - \xi'(w-)\Big)\Big(\text{sgn}(t-w) + \frac{\theta-1}{\theta+1}\Big)dt\Big)P^\alpha_{n,\tau,\theta}(w,v)dv \\ &\le \frac{\theta}{\theta+1}\Big|\xi'(w+) - \xi'(w-)\Big|\int_0^\infty |v-w|\, P^\alpha_{n,\tau,\theta}(w,v)dv = \frac{\theta}{\theta+1}\Big|\xi'(w+) - \xi'(w-)\Big|\mathcal{G}^\alpha_{n,\tau,\theta}\left(|v-w|\,;w\right) \\ &\le \frac{\theta}{\theta+1}\Big|\xi'(w+) - \xi'(w-)\Big|\Big(\mathcal{G}^\alpha_{n,\tau,\theta}\left((e_1-w)^2;w\right)\Big)^{1/2} \le \frac{\theta^{3/2}}{\theta+1}\Big|\xi'(w+) - \xi'(w-)\Big|\sqrt{\frac{\lambda_\tau(\alpha)}{n}}\phi(w). \end{aligned} \tag{11}$$

By using the relations (8)–(11), we obtain the following estimate

$$\begin{aligned} &\Big|\mathcal{G}^\alpha_{n,\tau,\theta}(\xi;w) - \xi(w)\Big| \le \Big|A^\alpha_{n,\tau,\theta}(\xi'_w,w) + B^\alpha_{n,\tau,\theta}(\xi'_w,w)\Big| + \frac{\sqrt{\theta}}{\theta+1}\Big|\xi'(w+) + \theta\xi'(w-)\Big|\sqrt{\frac{\lambda_\tau(\alpha)}{n}}\phi(w) \\ &+\frac{\theta^{3/2}}{\theta+1}\left|\xi'(w+) - \xi'(w-)\right|\sqrt{\frac{\lambda_\tau(\alpha)}{n}}\,\phi(w), \end{aligned} \tag{12}$$

where

$$A^\alpha_{n,\tau,\theta}(\xi'_w,w) = \int_0^w \Big(\int_w^v \xi'_w(t)dt\Big)P^\alpha_{n,\tau,\theta}(w,v)dv$$

and

$$B^\alpha_{n,\tau,\theta}(\xi'_w,w) = \int_w^\infty \Big(\int_w^v \xi'_w(t)dt\Big)P^\alpha_{n,\tau,\theta}(w,v)dv.$$

For getting the final result of the theorem, we have to estimate the terms $A^\alpha_{n,\tau,\theta}(\xi'_w,w)$ and $B^\alpha_{n,\tau,\theta}(\xi'_w,w)$. Since $\int_c^d d_v\zeta^\alpha_{n,\tau,\theta}(w,v) \le 1$, for all $[c,d] \subseteq (0,\infty)$, applying Lemma 2.5 and using the method of integration by parts, with $y = w - (w/\sqrt{n})$, we get

$$\begin{aligned} \Big|A^\alpha_{n,\tau,\theta}(\xi'_w,w)\Big| &= \Big|\int_0^w \Big(\int_w^v \xi'_w(t)dt\Big)d_v\zeta^\alpha_{n,\tau,\theta}(w,v)\Big| = \Big|\int_0^w \zeta^\alpha_{n,\tau,\theta}(w,v)\xi'_w(v)dv\Big| \\ &\le \Big(\int_0^y + \int_y^w \Big)\Big|\xi'_w(v)\Big|\Big|\zeta^\alpha_{n,\tau,\theta}(w,v)\Big|dv \\ &\le \theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_0^y \bigvee_v^w(\xi'_w)(w-v)^{-2}dt + \int_y^w \bigvee_v^w(\xi'_w)dv \\ &\le \theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_0^y \bigvee_v^w(\xi'_w)(w-v)^{-2}dv + \frac{w}{\sqrt{n}}\bigvee_{w-w/\sqrt{n}}^w(\xi'_w). \end{aligned}$$

Taking $t = w/(w-v)$, we get

$$\theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_0^{w-w/\sqrt{n}}(w-v)^{-2}\bigvee_v^w(\xi'_w)dv = \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\int_1^{\sqrt{n}}\bigvee_{w-w/t}^{w}(\xi'_w)dt$$

$$\leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\int_i^{i+1}\bigvee_{w-w/t}^{w}(\xi'_w)dt \leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\bigvee_{w-w/i}^{w}(\xi'_w).$$

Hence we get the following estimation

$$\left|A^\alpha_{n,\tau,\theta}(\xi'_w,w)\right| \leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\bigvee_{w-w/i}^{w}(\xi'_w) + \frac{w}{\sqrt{n}}\bigvee_{w-w/\sqrt{n}}^{w}(\xi'_w). \tag{13}$$

Applying Lemma 2.5 and using again the integration by parts with $z = w + w/\sqrt{n}$, we get

$$\begin{aligned}
|B^\alpha_{n,\tau,\theta}(\xi'_w,w)| &= \left|\int_w^\infty\left(\int_w^v \xi'_w(t)dt\right)P^\alpha_{n,\tau,\theta}(w,v)dv\right| \qquad (14)\\
&= \left|\int_w^z\left(\int_w^v \xi'_w(t)dt\right)d_v(1-\zeta^\alpha_{n,\tau,\theta}(w,v)) + \int_z^\infty\left(\int_w^v \xi'_w(v)dv\right)d_v(1-\zeta^\alpha_{n,\tau,\theta}(w,v))\right|\\
&= \left|\left[\left(\int_w^v \xi'_w(t)dt\right)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))\right]_w^z - \int_w^z f'_w(v)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))dv\right.\\
&\left.+\int_z^\infty\left(\int_w^v \xi'_w(t)dt\right)d_v(1-\zeta^\alpha_{n,\tau,\theta}(w,v))\right|\\
&= \left|\left(\int_w^v \xi'_w(t)dt\right)(1-\zeta^\alpha_{n,\tau,\theta}(w,z)) - \int_w^z \xi'_w(v)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))dv\right.\\
&\left.+\left[\left(\int_w^v \xi'_w(t)dt\right)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))\right]_z^\infty - \int_z^\infty \xi'_w(v)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))dv\right|\\
&= \left|\int_w^z \xi'_w(v)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))dv + \int_z^\infty \xi'_w(v)(1-\zeta^\alpha_{n,\tau,\theta}(w,v))dv\right|\\
&< \theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_z^\infty\bigvee_w^v(\xi')_w(v-w)^{-2}dt + \int_w^z\bigvee_w^v(\xi'_w)dv\\
&\leq \theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_{w+w/\sqrt{n}}^\infty\bigvee_w^v(\xi'_w)(v-w)^{-2}dv + \frac{w}{\sqrt{n}}\bigvee_w^{w+w/\sqrt{n}}(\xi'_w).
\end{aligned}$$

Taking $t = w/(v-w)$, we get

$$\theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{n}\int_{w+w/\sqrt{n}}^\infty\bigvee_w^v(\xi'_w)(v-w)^{-2}dv = \theta\frac{\lambda_\tau(\alpha)\phi^2(w)}{wn}\int_0^{\sqrt{n}}\bigvee_w^{w+w/t}(\xi'_w)dt \tag{15}$$

$$\leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\int_i^{i+1}\bigvee_w^{w+w/t}(\xi'_w)dt \leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\bigvee_w^{w+w/i}(\xi'_w).$$

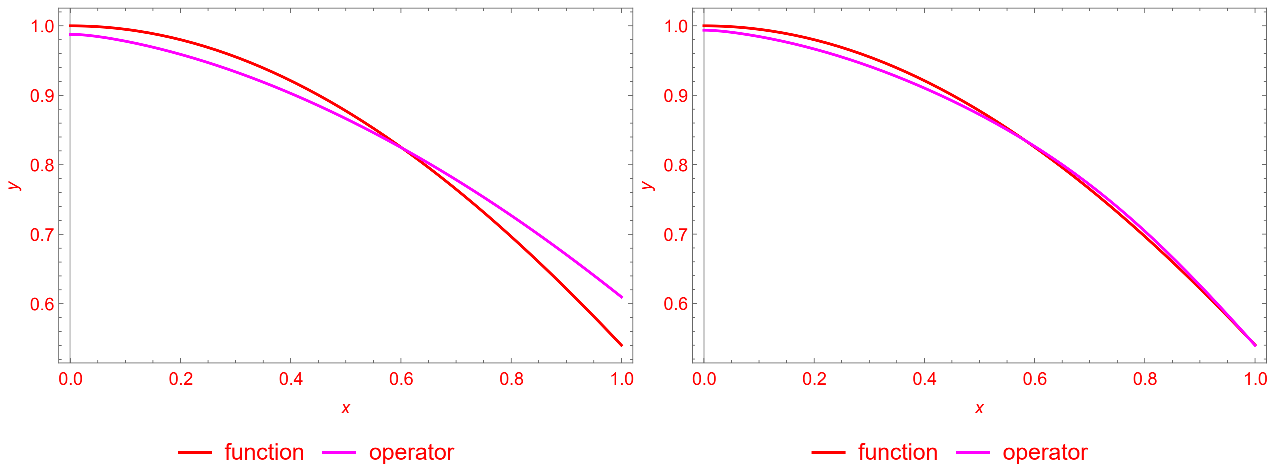
The estimation presented below is obtained by utilizing the equations (14)-(15)

$$|B^\alpha_{n,\tau,\theta}(\xi'_w,w)| \leq \theta\frac{\lambda_\tau(\alpha)(1+w)}{n}\sum_{i=1}^{[\sqrt{n}\,]}\bigvee_w^{w+w/i}(\xi'_w) + \frac{w}{\sqrt{n}}\bigvee_w^{w+w/\sqrt{n}}(\xi'_w). \tag{16}$$

The desired result can be obtained by utilizing the relations (12), (13), and (16). □

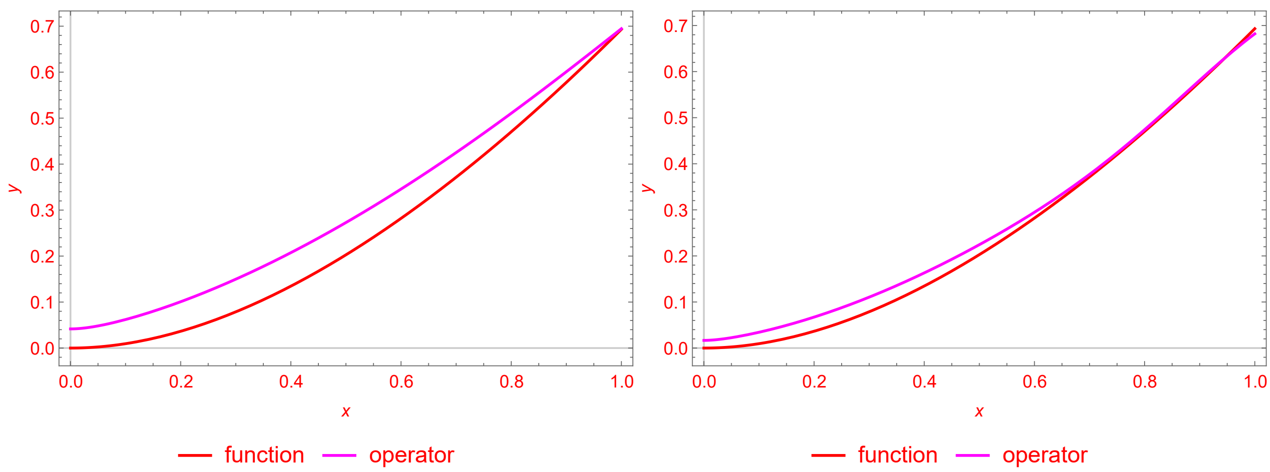
5. Graphical Analysis

In this section, convergence behaviour of this operators is seen by using some numerical results. We consider errors of convergence and convergence rate for this operators to the basic function $\xi(w) = \cos w$ in Figure 1 and Figure 3, respectively. We can also see the effectiveness of this operators by considering $\xi(w) = w \log(1 + w)$. In Figure 1 (A), we choose $\tau = 1/120, \theta = 2, n = 9$ and $\alpha = 0.9$ to see convergence of $B^{\alpha}_{n,\tau,\theta}$, where $w \in [0, 1]$. We consider $\theta = 2, n = 13, \alpha = 0.9$ and $\tau = 1/4$ in Figure 1 (B). In Figure 2 (A), we choose $\tau = 1/2, n = 7, \theta = 2$ and $\alpha = 0.9$ to see convergence of $B^{\alpha}_{m,\tau,\theta}$, where $w \in [0, 1]$. We choose $\tau = 1/2, n = 9, \theta = 2$ and $\alpha = 0.9$ in Figure 2(B). In Figure 3 and Figure 4, we obtain related errors of approximations for the proposed operators with certain values of n, α, τ and θ. Obviously we can conclude that, the proposed operators $B^{\alpha}_{n,\tau,\theta}$ also have less errors for very small values of n.



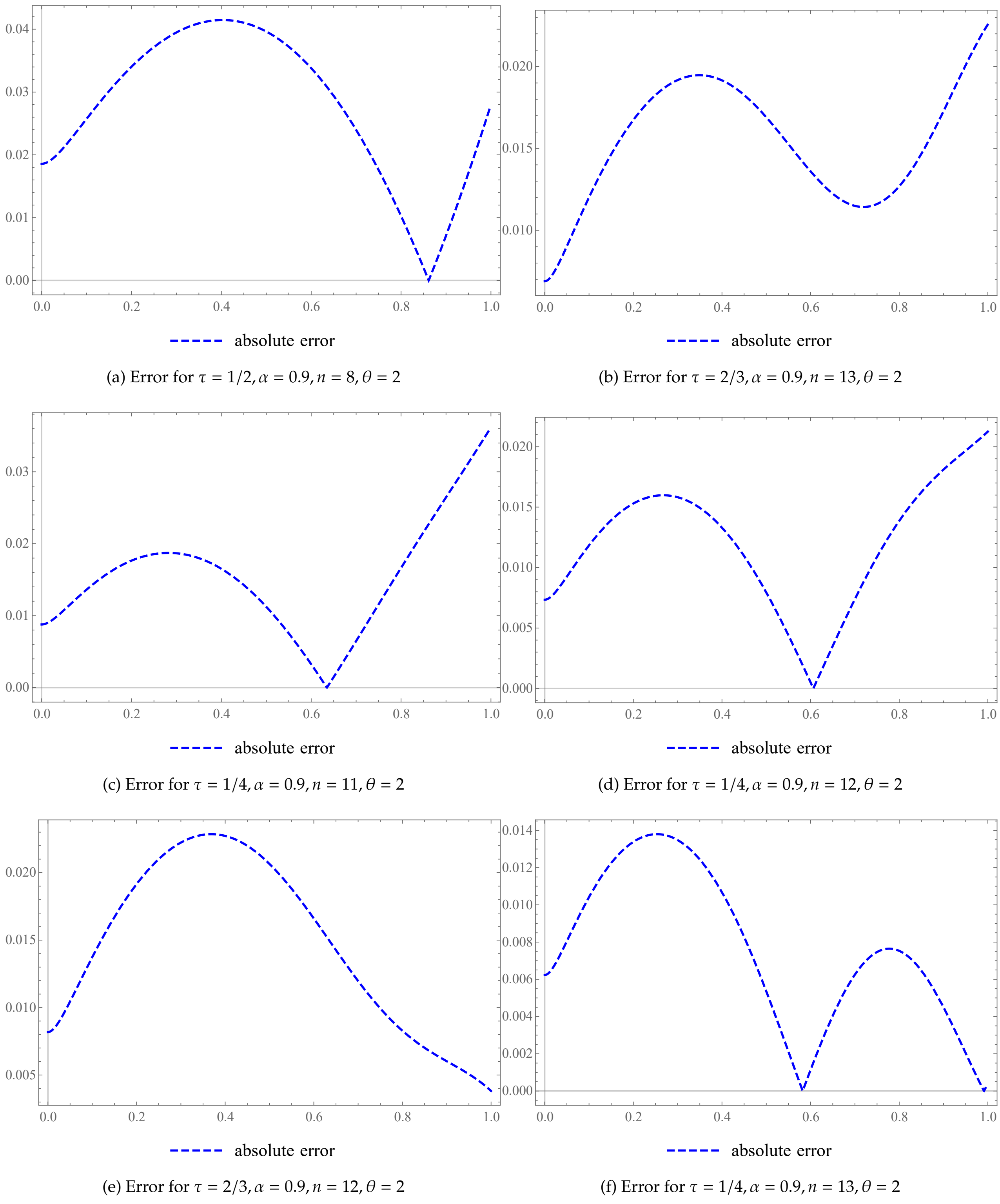
(a) Approximation by $\mathscr{G}^{0.9}_{9,1/120,2}$ (b) Approximation by $\mathscr{G}^{0.9}_{13,1/4,2}$

Figure 1: Approximation by $\mathscr{G}^{\alpha}_{n,\tau,\theta}$ operators for certain parameters



(a) Approximation by $\mathscr{G}^{0.9}_{7,1/2,2}$ (b) Approximation by $\mathscr{G}^{0.9}_{9,1/2,2}$

Figure 2: Approximation by $\mathscr{G}^{\alpha}_{n,\tau,\theta}$ operators for certain parameters



(a) Error for $\tau = 1/2, \alpha = 0.9, n = 8, \theta = 2$

(b) Error for $\tau = 2/3, \alpha = 0.9, n = 13, \theta = 2$

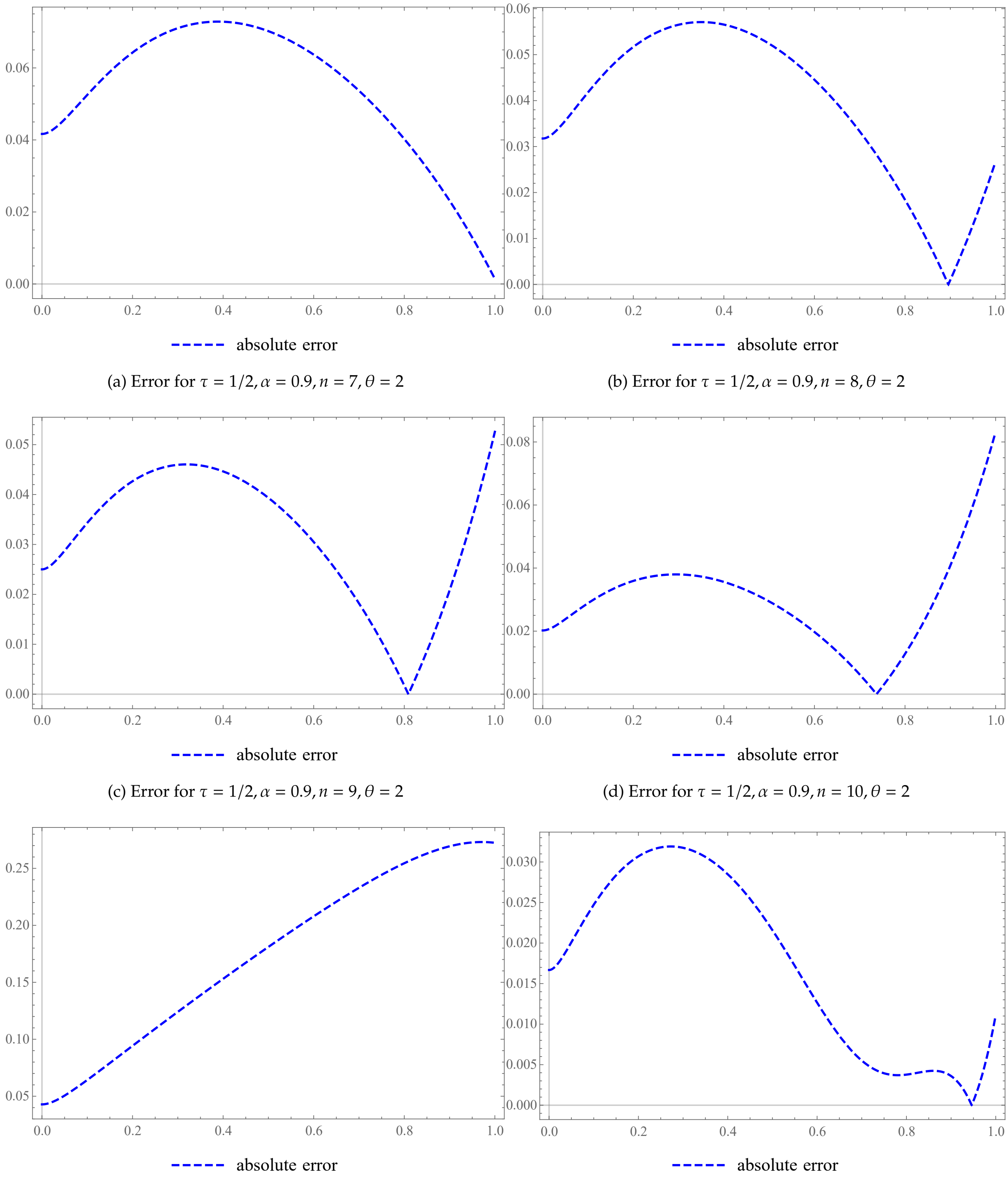
(c) Error for $\tau = 1/4, \alpha = 0.9, n = 11, \theta = 2$

(d) Error for $\tau = 1/4, \alpha = 0.9, n = 12, \theta = 2$

(e) Error for $\tau = 2/3, \alpha = 0.9, n = 12, \theta = 2$

(f) Error for $\tau = 1/4, \alpha = 0.9, n = 13, \theta = 2$

Figure 3: Error of approximation by $\mathscr{G}^{\alpha}_{n,\tau,\theta}$ operators for certain parameters



(a) Error for $\tau = 1/2, \alpha = 0.9, n = 7, \theta = 2$

(b) Error for $\tau = 1/2, \alpha = 0.9, n = 8, \theta = 2$

(c) Error for $\tau = 1/2, \alpha = 0.9, n = 9, \theta = 2$

(d) Error for $\tau = 1/2, \alpha = 0.9, n = 10, \theta = 2$

(e) Error for $\tau = 2, \alpha = 0.9, n = 9, \theta = 2$

(f) Error for $\tau = 1/2, \alpha = 0.9, n = 11, \theta = 2$

Figure 4: Error of approximation by $\mathscr{G}^{\alpha}_{n,\tau,\theta}$ operators for certain parameters

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