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A note on distributional chaos for $\mathbb{Z}^d$-actions

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Abstract. In this paper, we study the relationship between the notions of distributional chaos and specification property defined for multidimensional time discrete dynamical systems. Essentially, we prove that a $\mathbb{Z}^d$-action on a compact metric space with weak specification property and with a pair of distal points admits a dense distributionally scrambled set of type 1.

1. Introduction

Studying the chaotic and unpredictable behavior of a system is one of the central ideas in the theory of dynamical systems. In 1975, Li and Yorke gave the first mathematical definition of chaos for continuous self-maps defined on an interval [6]. Since then, various definitions of chaos have been established and studied. Also, many extensions of Li-Yorke chaos are studied based on the size of the set of Li-Yorke chaotic pairs, for example, generic chaos, ϵ-Li-Yorke chaos, dense chaos, and others. Positive topological entropy of a system is also considered to depict the chaotic behavior of a system. It is known that, for continuous maps on intervals, positive topological entropy implies dense Li-Yorke chaos. However, there are continuous maps on compact metric space which are Li-Yorke chaotic but have zero topological entropy [10]. Thus, positive entropy is stronger than Li-Yorke chaos. In 1994, Schweizer and Smítal introduced the notion of chaos based on the probabilistic measure of proximity of pairs, popularly termed as distributional chaos [10]. Distributional chaos is considered an important generalization of Li-Yorke chaos because it is equivalent to positive topological entropy for continuous maps on compact intervals. In [1], the authors considered two generalizations of distributional chaos and thus introduced three nonequivalent variants of distributional chaos, namely $DC1$, $DC2$ and $DC3$.

Another important notion of chaos is specification, which was introduced by Bowen in 1971 for Axiom A diffeomorphisms [2]. Informally, by specification property, we mean that any finite set of orbital fragments that are appropriately spaced apart in time can be traced by a single orbit. Many researchers have studied the relationship between the notion of specification property and distributional chaos. Sklar and Smítal have proved that a continuous map defined on a compact metric space with the specification property is distributionally chaotic of type 3 [15]. Oprocha and Štefánková proved that a continuous map acting on compact metric space with a weaker form of specification property and with a pair of distal points is distributionally chaotic in a very strong sense [8].

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Dynamics has traditionally been the study of the iterates of a single transformation, modelling the time evolution of a physical system. However, many physical and mathematical systems have other symmetries, which lead to the study of joint action of several commuting transformations. The study of joint actions is much more than a routine generalization of a single transformation. Various authors have studied the dynamics for such systems. Ruelle studied the duality of entropy and pressure for $\mathbb{Z}^d$-actions having expansiveness and specification [9]. Oprocha proved the Spectral Decomposition Theorem for multidimensional time discrete dynamical system [7]. Kim and Lee have proved the Spectral Decomposition Theorem for the set of k-type nonwandering points of $\mathbb{Z}^2$-actions [5]. In [12, 13], the authors introduced and studied the notion of k-type Devaney chaos for $\mathbb{Z}^d$-actions. The notion of k-type collective sensitivity for $\mathbb{Z}^d$-actions is defined and studied in [14]. Shah introduced the concept of generators for $\mathbb{Z}^d$-actions and studied the properties of expansive $\mathbb{Z}^d$-actions in [11]. The dynamics of the multidimensional time discrete dynamical system induced by a continuous map is studied in [4].

Hunter introduced the notion of distributional chaos for continuous $\mathbb{Z}^d$-action defined on a compact metric space and proved that a multidimensional subshift with the weak specification property is uniformly distributionally chaotic of type 1 [3]. In this paper, we relate the notions of distributional chaos and specification property for a continuous $\mathbb{Z}^d$-action defined on a compact metric space. The paper is organized in the following manner: In Section 2, we discuss the preliminaries required for the development of the paper. In Section 3, we prove that a continuous $\mathbb{Z}^d$-action on a compact metric space with weak specification property and with a pair of distal points is distributionally chaotic of type 1, and admits a dense distributionally scrambled set of type 1. This extends the result due to Oprocha and Štefánková for the multidimensional time discrete dynamical system

2. Preliminaries and notations

Throughout, by a dynamical system we mean a pair (X, T), where X is a compact metric space with metric ρ and T is a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, i.e., $T : \mathbb{Z}^d \times X \to X$ satisfies the following three conditions:

1. $T(n, \cdot)$ is a homeomorphism on X, for any $n \in \mathbb{Z}^d$,
2. $T(0, x) = x$ for any $x \in X$,
3. $T(m, T(n, \cdot)) = T(m + n, \cdot)$, for any $m, n \in \mathbb{Z}^d$.

We denote $T(n, x)$ by $T^n(x)$, for $n \in \mathbb{Z}^d$. Note that, the distance between two subsets A and B of $\mathbb{Z}^d$ is given by $\varrho(A, B) = \min\limits_{a\in A, b\in B} ||a - b||_\infty$, where $||a - b||_\infty = \max\limits_{0\leq i\leq d} |a_i - b_i|$ denotes the supremum norm in $\mathbb{Z}^d$.

A point $x \in X$ is said to be *periodic* if the set $\{T^n(x) : n \in \mathbb{Z}^d\}$ is finite. We call a pair x, y in X *proximal* if for any $\epsilon > 0$ there exists $n \in \mathbb{Z}^d$ such that $\rho(T^n(x), T^n(y)) < \epsilon$. A pair x, y in X is said to be *distal* if it is not proximal.

2.1. *Specification property for $\mathbb{Z}^d$-actions*

Definition 2.1. ([3]) A $\mathbb{Z}^d$-action T is said to have the *strong specification property* (briefly *SSP*) if for every $\epsilon > 0$ there exists a positive integer N_ϵ such that for every finite collection of sets $Q_1, Q_2, \ldots, Q_k$ in $\mathbb{Z}^d$ satisfying $\varrho(Q_i, Q_j) \geq N_\epsilon$, for $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$, any finite collection of points $x_1, x_2, \ldots, x_k$ in X, and any subgroup $\Lambda \subset \mathbb{Z}^d$ with $\varrho(Q_i + q, Q_j) \geq N_\epsilon$, for $i, j \in \{1, 2, \ldots, k\}$ and $q \in \Lambda \setminus \{0\}$, there exists $y \in X$ such that $\rho(T^t(y), T^t(x_j)) < \epsilon$, for all $t \in Q_j$, $j \in \{1, 2, \ldots, k\}$, and $T^q(y) = y$, for every $q \in \Lambda$.

If the periodicity condition given in terms of subgroup $\Lambda \subset \mathbb{Z}^d$ in above definition is omitted, T is said to have *specification property* (briefly *SP*). For the special case $k = 2$, the notion of specification property is termed as weak specification property and can be reformulated in the following (equivalent) way:

Definition 2.2. A $\mathbb{Z}^d$-action T is said to have the *weak specification property* (briefly *WSP*) if for every $\epsilon > 0$ and every positive integer k there exists a positive integer $N = N(\epsilon, k)$ such that for every finite collection of sets $Q_1, Q_2, \ldots, Q_k$ in $\mathbb{Z}^d$ satisfying $\varrho(Q_i, Q_j) \geq N$, for $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$, any pair of points u, v in X, there exists $y \in X$ such that $\rho(T^t(y), T^t(u)) < \epsilon$, for all $t \in Q_j$, $j = 2i + 1 \leq k$, and $\rho(T^t(y), T^t(v)) < \epsilon$, for all $t \in Q_j$, $j = 2i \leq k$.

For any $\epsilon > 0$, if $\rho(T^t(x), T^t(y)) < \epsilon$, for all $t \in Q_i \subset \mathbb{Z}^d$, we say that x *ϵ-traces* y in Q_i. Note that, the definition of *WSP* can be easily reformulated using any finite sequence of points $x_1, x_2, \ldots, x_l$ instead of just a pair of two points u, v.

2.2. Distributional chaos for $\mathbb{Z}^d$-actions

Let (X, ρ) be a metric space and let T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$. For $n \in \mathbb{N}$, points $x, y \in X$, and $t \in \mathbb{R}$, define

$$\Phi^{(n)}_{xy}(t) = \frac{1}{\#\Lambda(n)} \#\{m \in \Lambda(n) | \rho(T^m(x), T^m(y)) < t\},$$

where $\Lambda(n) = \{-(n-1), -(n-2), \ldots, 0, \ldots, (n-2), (n-1)\}^d \subset \mathbb{Z}^d$, and $\#A$ denotes the cardinality of the set A. Let

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi^{(n)}_{xy}(t), \text{ and } \qquad \Phi^*_{xy}(t) = \limsup_{n\to\infty} \Phi^{(n)}_{xy}(t).$$

These non-decreasing functions are called the lower and upper distribution functions for T, respectively.

Definition 2.3. ([3]) A pair of points $x, y \in X$ is called

1. *distributionally chaotic of type 1 (DC1)* if $\Phi_{xy}(t) = 0$, for some $t > 0$, and $\Phi^*_{xy}(t) = 1$, for all $t > 0$,
2. *distributionally chaotic of type 2 (DC2)* if $\Phi_{xy}(t) < \Phi^*_{xy}(t)$, for some $t > 0$, and $\Phi^*_{xy}(t) = 1$, for all $t > 0$,
3. *distributionally chaotic of type 3 (DC3)* if $\Phi_{xy}(t) < \Phi^*_{xy}(t)$, for some $t > 0$.

A set containing at least two points is called a *distributionally scrambled set of type k* for T if any pair of its distinct points is distributionally chaotic of type k, where $k \in \{1, 2, 3\}$.

Definition 2.4. A $\mathbb{Z}^d$-action T is said to be *distributionally chaotic of type k*, where $k \in \{1, 2, 3\}$, if there exists an uncountable distributionally scrambled set of type k for T.

In order to avoid confusion, we denote the distribution function for T by $\Phi^{(n)}_{xy}(T, t)$. Recall that, for dynamical systems (X, T_1) and (Y, T_2), the $\mathbb{Z}^d$-actions $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$ for all $n \in \mathbb{Z}^d$, and the map h is called the *topological conjugacy* between T_1 and T_2.

Proposition 2.5. *Let (X, T_1) and (Y, T_2) be two dynamical systems, where X and Y are compact metric spaces with metrics ρ_1 and ρ_2, respectively. If T_1 and T_2 are topologically conjugate then T_1 is DCk implies T_2 is DCk, $k \in \{1, 2\}$.*

Proof. Since T_1 and T_2 are topologically conjugate, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$ for all $n \in \mathbb{Z}^d$. By uniform continuity of h, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_1(x_1, x_2) < \delta$ implies $\rho_2(h(x_1), h(x_2)) < \epsilon$. For any $x_1, x_2 \in X$, let $y_1 = h(x_1)$ and $y_2 = h(x_2)$. Then

$$\begin{aligned}
\Phi^{(n)}_{x_1x_2}(T_1, \delta) &= \frac{1}{\#\Lambda(n)} \#\{m \in \Lambda(n) | \rho_1(T_1^m(x_1), T_1^m(x_2)) < \delta\} \\
&\leq \frac{1}{\#\Lambda(n)} \#\{m \in \Lambda(n) | \rho_2(h(T_1^m(x_1)), h(T_1^m(x_2))) < \epsilon\} \\
&= \frac{1}{\#\Lambda(n)} \#\{m \in \Lambda(n) | \rho_2(T_2^m(h(x_1)), T_2^m(h(x_2))) < \epsilon\} \\
&= \frac{1}{\#\Lambda(n)} \#\{m \in \Lambda(n) | \rho_2(T_2^m(y_1), T_2^m(y_2)) < \epsilon\} \\
&= \Phi^{(n)}_{y_1y_2}(T_2, \epsilon)
\end{aligned}$$

Similarly by uniform continuity of h^{-1}, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\Phi^{(n)}_{y_1y_2}(T_2, \delta) \leq \Phi^{(n)}_{x_1x_2}(T_1, \epsilon)$.

Since T_1 is $DC1$, X has an uncountable subset D which is distributionally scrambled set of type 1 for T_1. Then $h(D)$ is an uncountable subset of Y. Moreover, if $y_1, y_2 \in h(D)$, then $y_1 = h(x_1)$ and $y_2 = h(x_2)$ where $x_1, x_2 \in D$. Since D is distributionally scrambled set of type 1, $\Phi^*_{x_1x_2}(T_1, t) = 1$, for all $t > 0$, and $\Phi_{x_1x_2}(T_1, \epsilon) = 0$, for some $\epsilon > 0$. This implies that $\Phi^*_{y_1y_2}(T_2, t) = 1$, for all $t > 0$, and $\Phi_{y_1y_2}(T_2, \delta) = 0$, for some $\delta > 0$. Thus, $h(D)$ is uncountable distributionally scrambled set of type 1 for T_2. Hence T_2 is $DC1$. On similar lines, we can prove that T_1 is $DC2$ implies T_2 is $DC2$. □

Remark 2.6. In reference to the example given in [1], we can say that $DC3$ is not preserved under topological conjugacy.

3. Main results

In this section, we establish the relation between weak specification property and distributional chaos for $\mathbb{Z}^d$-actions on compact metric spaces.

Theorem 3.1. *Let (X, ρ) be a compact metric space without isolated points and T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then T is distributionally chaotic of type 1.*

Proof. Let u, $v \in X$ be a distal pair and let $G \subset X$ be an open ball with center at g and of radius $\epsilon > 0$. Set $\epsilon_i = \frac{\epsilon}{2^i}$, $i \geq 1$. Let $M_i = M(\epsilon_i, 2^i)$, as in the definition of *WSP*, for all $i \geq 1$. Also, let $a_1 < a_2 < a_3 < \ldots$ be an increasing sequence of positive integers such that $a_{i+1} - a_i > M_{i+1}$, for any $i \geq 1$, and $\lim_{i\to\infty} \frac{a_{i+1}-M_{i+1}}{a_i} = \infty$. Further, we define a subsequence $\{m(i)\}_{i=1}^{\infty}$ of the sequence of positive integers, with $m(1) > 1$, such that $m(i + 1) > m(i) + i$.

Step 1: Consider

$$Q^1_0 = \{t \in \mathbb{Z}^d | 0 \leq ||t|| \leq a_1\},$$
$$Q^1_1 = \{t \in \mathbb{Z}^d | a_{m(1)} \leq ||t|| \leq a_{m(1)+1} - M_2\},$$
$$Q^1_2 = \{t \in \mathbb{Z}^d | a_{m(2)} \leq ||t|| \leq a_{m(2)+1} - M_2\},$$
$$Q^1_3 = \{t \in \mathbb{Z}^d | a_{m(2)+1} \leq ||t|| \leq a_{m(2)+2} - M_2\}.$$

Since T has *WSP*, for ϵ_2 and the sequences of points g, u, v, u and g, u, v, v, we get points $x_u, x_v \in G$, respectively, so that the points x_u, x_v: ϵ_2-trace g in Q^1_0, ϵ_2-trace u in Q^1_1, ϵ_2-trace v in Q^1_2. Moreover, x_u ϵ_2-trace u in Q^1_3 and x_v ϵ_2-trace v in Q^1_3.

Since T^k, $k \in \mathbb{Z}^d$, are uniformly continuous, there are disjoint compact balls B_u and B_v in G centered at x_u and x_v, respectively, such that each $x \in B_\alpha$ ϵ_3-traces x_α in $Q^{(1)} = \{t \in \mathbb{Z}^d | 0 \leq ||t|| \leq a_{m(2)+2} - M_2\}$, where $\alpha \in \{u, v\}$. Without loss of generality we can assume that the diameter of either B_u or B_v is $\epsilon_{j(1)}$, where $j(1) \geq 3$.

Step 2: Again by *WSP*, for $\epsilon_{j(1)} > 0$ and for

$$Q^2_1 = \{t \in \mathbb{Z}^d | a_{m(3)} \leq ||t|| \leq a_{m(3)+1} - M_{j(1)}\},$$
$$Q^2_2 = \{t \in \mathbb{Z}^d | a_{m(4)} \leq ||t|| \leq a_{m(4)+1} - M_{j(1)}\},$$
$$Q^2_3 = \{t \in \mathbb{Z}^d | a_{m(4)+1} \leq ||t|| \leq a_{m(4)+2} - M_{j(1)}\},$$
$$Q^2_4 = \{t \in \mathbb{Z}^d | a_{m(4)+2} \leq ||t|| \leq a_{m(4)+3} - M_{j(1)}\}.$$

there are points $x_{uu}, x_{uv} \in B_u$ and $x_{vu}, x_{vv} \in B_v$, which $\epsilon_{j(1)}$-trace u in Q^2_1 and $\epsilon_{j(1)}$-trace v in Q^2_2. Additionally,

$$x_{uu}, x_{uv}\ \epsilon_{j(1)}\text{-trace } u, \text{ and } x_{vu}, x_{vv}\ \epsilon_{j(1)}\text{-trace } v \text{ in } Q^2_3,$$
$$x_{uu}, x_{vu}\ \epsilon_{j(1)}\text{-trace } u, \text{ and } x_{uv}, x_{vv}\ \epsilon_{j(1)}\text{-trace } v \text{ in } Q^2_4.$$

Again by uniform continuity, there are disjoint compact balls B_{uu}, B_{uv} in interior of B_u centered at x_{uu} and x_{uv}, respectively, and B_{vu}, B_{vv} in interior of B_v centered at x_{vu} and x_{vv}, respectively, such that each $x \in B_\alpha$ $\epsilon_{j(1)+1}$-traces x_α in $Q^{(2)} = \{t \in \mathbb{Z}^d | 0 \le ||t|| \le a_{m(4)+3} - M_{j(1)}\}$. Without loss of generality we can assume that the diameter of either B_α is $\epsilon_{j(2)}$, where $j(2) > j(1) + 1$.

Continuing like this, for each $k \in \mathbb{N}$ and any $\alpha \in \{u, v\}^k$, there is a compact ball B_α centered at x_α, and positive integers $j(1) < j(2) < \ldots < j(k)$ such that if x, y are distinct points in B_α, then x $\epsilon_{j(k)-1}$-traces y in $Q^{(k)} = \{t \in \mathbb{Z}^d | 0 \le ||t|| \le a_{m(2k)+k+1} - M_{j(k-1)}\}$. Also, if $\alpha, \beta \in \{u, v\}^k$ and $\alpha \ne \beta$ then $B_\alpha \cap B_\beta = \phi$ and $B_{\alpha u} \cup B_{\alpha v} \subset B_\alpha$.

Moreover, for any $x \in B_\alpha$, where $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$:
(i) The point x ϵ_i-traces u in Q_1^i, and ϵ_i-traces v in Q_2^i, $1 \le i < k$.
(ii) For $1 \le i \le k$, x ϵ_i-traces α_1 in Q_3^i, and ϵ_i-traces α_2 in Q_4^i for $2 \le i \le k$. In general, for any j, $1 \le j \le k$, x_α ϵ_i-traces α_j in Q_{j+2}^i, for $j \le i \le k$.

Take $S = \bigcap_{n=1}^\infty \bigcup_{\alpha \in \{u,v\}^n} B_\alpha$. Since $\operatorname{diam}(B_\alpha) \to 0$ as $n \to \infty$, $S \subset G$ is a Cantor set. Further, since each sequence in $\{u, v\}^{\mathbb{N}}$ corresponds to a distinct element in S, and since the set of all binary sequences is uncountable, it follows that S is uncountable. Note that, any $s \in S$ can be uniquely determined as $s_\alpha = \bigcap_{n=1}^\infty B_{\alpha_1\alpha_2\ldots\alpha_n}$ where $\alpha = \alpha_1\alpha_2 \ldots \alpha_n \ldots \in \{u, v\}^{\mathbb{N}}$.

Let x and y be distinct points in S. Then for any $\delta > 0$, there will be an m such that $\epsilon_i < \delta$, for all $i > m$. Using condition (i), we get that both x, y, ϵ_i-traces u in Q_1^i, and ϵ_i-traces v in Q_2^i, for all $i \ge 1$. Thus, $\rho(T^p(x), T^p(y)) < \frac{\epsilon_i}{2} < \delta$, for all $p \in Q_1^i \cup Q_2^i$, and for all $i > m$. Hence, $\Phi_{xy}^*(\delta) = 1$, for any $\delta > 0$.

Further $x, y \in S$ implies $x = x_\alpha$, $y = y_\beta$ for some $\alpha = \alpha_1\alpha_2 \ldots \alpha_n \ldots$ and $\beta = \beta_1\beta_2 \ldots \beta_n \ldots$ in $\{u, v\}^{\mathbb{N}}$. Since x and y are distinct, there exist a positive integer j such that $\alpha_j \ne \beta_j$. Without loss of generality we assume that $\alpha_j = u$ and $\beta_j = v$. Using condition (ii), we have that x ϵ_i-traces u in Q_{j+2}^i and y ϵ_i-traces v in Q_{j+2}^i, for $j \le i$. Since u, v is a distal pair, there exist a $\delta_0 > 0$ such that $\rho(T^n(u), T^n(v)) > \delta_0$, for all $n \in \mathbb{Z}^d$. Choose a positive integer m, such that $\epsilon_i < \frac{\delta_0}{3}$, for all $i > m$. Then $\rho(T^p(x), T^p(y)) > \frac{\delta_0}{3}$, for any $p \in Q_{j+2}^i$, and for all $j \le i$ and $i > m$. For if, $\rho(T^p(x), T^p(y)) < \frac{\delta_0}{3}$ for any $p \in Q_{j+2}^i$, for $j \le i$ and $i > m$, then $\rho(T^p(x), T^p(u)) < \epsilon_i$ and $\rho(T^p(y), T^p(v)) < \epsilon_i$ will imply that $\rho(T^p(u), T^p(v)) < \delta_0$, a contradiction. Hence $\Phi_{xy}(\delta_0) = 0$.

Thus, S is an uncountable distributional scrambled set of type 1. Hence T is distributionally chaotic of type 1. □

Theorem 3.2. *Let (X, ρ) be a compact metric space without isolated points and T be a continuous $\mathbb{Z}^d$-action on X, $d \in \mathbb{N}$, $d > 1$, having a distal pair. If T has weak specification property, then X has a dense $DC1$ scrambled set.*

Proof. Since (X, ρ) is a compact metric space, it is second countable. Let $\{G_i\}_{i=1}^\infty$ denote a countable base of topology of X consisting of open balls. For $G_1 = B(g_1, \epsilon^{(1)})$, using the arguments given in Theorem 3.1, we get a Cantor *DC1* scrambled set S_1 and a sequence $\mu_1 = \{m(1, i)\}_{i=1}^\infty$ such that $x_1 \in S_1$ $\epsilon_i^{(1)}$-traces u in $Q_1^{(1,i)} = \{t \in \mathbb{Z}^d | a_{m(1,2i-1)} \le ||t|| \le a_{m(1,2i-1)+1} - M_{j(1,i-1)}\}$, and $\epsilon_i^{(1)}$-traces v in $Q_2^{(1,i)} = \{t \in \mathbb{Z}^d | a_{m(1,2i)} \le ||t|| \le a_{m(1,2i)+1} - M_{j(1,i-1)}\}$, for all $i \ge 1$, where $j(1, 0) = 2$.

Again following the steps discussed in Theorem 3.1, for $G_2 = B(g_2, \epsilon^{(2)})$, and for the sequence $a_{m(1,1)} < a_{m(1,3)} < a_{m(1,5)} < \ldots$ of positive integers, we obtain a subsequence μ_2 of $\{m(1, 1), m(1, 3), m(1, 5), \ldots\}$ denoted by $\{m(2, i)\}_{i=1}^\infty$ and a Cantor *DC1* scrambled set S_2 such that $x_2 \in S_2$ $\epsilon_i^{(2)}$-traces u in $Q_1^{(2,i)} = \{t \in \mathbb{Z}^d | a_{m(2,2i-1)} \le ||t|| \le a_{m(2,2i-1)+1} - M_{j(2,i-1)}\}$, and $\epsilon_i^{(2)}$-traces v in $Q_2^{(2,i)} = \{t \in \mathbb{Z}^d | a_{m(2,2i)} \le ||t|| \le a_{m(2,2i)+1} - M_{j(2,i-1)}\}$, for all $i \ge 1$. Continuing this way, we can find a sequence $S_1, S_2, \ldots$ of Cantor sets such that $S = \bigcup_{n=1}^\infty S_n$ is dense in X.

We now prove that S is a *DC1* scrambled set. Clearly S_n is a *DC1* scrambled set for each $n \in \mathbb{N}$. Consider $y, z \in S$ such that $y \in S_p$ and $z \in S_q$, $p \ne q$. Assume that $p < q$. Then it follows that y ϵ_i-traces z in $Q_1^{(q,i)} = \{t \in \mathbb{Z}^d | a_{m(q,2i-1)} \le ||t|| \le a_{m(q,2i-1)+1} - M_{j(q,i-1)}\}$, for any $i \ge 1$, where $\epsilon_i = \max\{\epsilon_i^{(p)}, \epsilon_i^{(q)}\}$. Moreover y $\epsilon_i^{(p)}$-traces u and z $\epsilon_i^{(q)}$-traces v in $Q_2^{(q,i)} = \{t \in \mathbb{Z}^d | a_{m(q,2i)} \le ||t|| \le a_{m(q,2i)+1} - M_{j(q,i-1)}\}$, for any $i \ge 1$. Thus, we get that y and z are distributionally chaotic of type 1. Hence S is a *DC1* scrambled set. □

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