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Sobolev-type theorem for commutators of variable Marcinkiewicz fractional in grand variable Herz-Morrey spaces

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Abstract. Let $\mathbb{S}^{n-1}$ denote unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero. The variable Marcinkiewicz fractional integral operator is defined as

$$[b,\mu_\Phi]^m_\zeta(f)(z_1) = \left(\int\limits_0^\infty \left| \int\limits_{|z_1-z_2|\le s} \frac{\Phi(z_1-z_2)[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}} f(z_2)dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

The commutators of Marcinkiewicz fractional operator of variable order is shown to be bounded on the grand variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)$.

1. Introduction

We consider the Riesz potential operator

$$I^{\zeta(z_1)}f(z_1) = \int_{\mathbb{R}^n} \frac{f(z_2)}{|z_1-z_2|^{n-\zeta(z_1)}} dz_2, \qquad 0 < \zeta(z_1) < n. \tag{1}$$

Note that the $\zeta(z_1)$ is the order of the Riesz potential operator which is variable. Nowadays there is a vast boom of research related to both the study of of the Herz spaces themselves and the operator theory in these spaces. This is caused under the influence of some possible applications in modeling with non-standard local growth (in differential equations , fluid mechanics, elasticity theory, see for example [1, 2]). Boundedness of the various operators has been intensively studied in the last years, and one the remarkable results was on the boundedness of the Hardy-Littlewood maximal operator in these spaces. We refer, for example, to the papers [3–16], see also references therein.

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In [17] authors considered variable potential operators $I^{\zeta(x)}$ to prove a Sobolev-type theorem for the potential operator from the Lebesgue space $L^{p(\cdot)}$ into the weighted Lebesgue space $L^{q(\cdot)}_{w}$ in $\mathbb{R}^n$, under the conditions that $p(x)$ is satisfying the logarithmic condition locally and at infinity. It was not supposed that $p(x)$ is constant at infinity but also assumed that $p(x)$ took its minimal value at infinity.

In [3], remarkable results were proved for boundedness on homogeneous Herz spaces $K^{\alpha(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ and non-homogeneous Herz spaces $\dot{K}^{\alpha(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ with variable exponents $\alpha(\cdot)$ and $v(\cdot)$. Meanwhile, they also proved boundedness for variable fractional integrals on Herz spaces with variable exponent. In [18] authors considered the Herz Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{u,v(\cdot)}(\mathbb{R}^n)$ with variable exponent and investigated mapping properties for the fractional Hardy type operators $\mathcal{H}_{\beta(\cdot)}$ and $\mathcal{H}^{*}_{\beta(\cdot)}$ in these spaces..

Grand variable Herz-Morrey spaces were introduced in [19] and proved boundedness for Riesz potential operator in these spaces. Inspired by the concept, in this article we will demonstrate the boundedness of commutators of Marcinkiewicz fractional integral operator of variable order on grand variable Herz-Morrey spaces.

We divided this article into different sections. Apart from introduction, a section is dedicated to basic lemmas and definitions. One section is for Sobolev type theorem for commutators of Marcinkiewicz fractional integral operator of variable order in grand variable Herz-Morrey spaces.

Definition 1.1. *Consider a measurable set $\mathcal{A}$ in $\mathbb{R}^n$ and a measurable function $p(\cdot) : \mathcal{A} \to [1, \infty)$. Then*

(a) Now Lebesgue space with variable exponent $L^{p(\cdot)}(\mathcal{A})$ is defined by

$$L^{p(\cdot)}(\mathcal{A}) = \left\{ f \textit{ is measurable} : \int_{\mathcal{A}} \left(\frac{|f(z)|}{\Gamma} \right)^{p(z)} dz < \infty \textit{ where } \Gamma \textit{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(\mathcal{A})$ can be defined as,

$$\|f\|_{L^{p(\cdot)}(\mathcal{A})} = \inf \left\{ \Gamma > 0 : \int_{H} \left(\frac{|f(z)|}{\Gamma} \right)^{p(z)} dz \leq 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathcal{A})$ can be defined as

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathcal{A}) := \left\{ f : f \in L^{p(\cdot)}(K) \textit{ for all compact subsets } K \subset \mathcal{A} \right\}.$$

We use these notations in this paper:

(i) Suppose that $f \in L^{1}_{\mathrm{loc}}(\mathcal{A})$ now the Hardy-Littlewood maximal operator $\mathfrak{M}$ can be given as

$$\mathfrak{M}f(z) := \sup_{0<r} \frac{1}{r^n} \int\limits_{B(i,r)} |f(z)| dz \quad (z \in \mathcal{A}),$$

where $B(i, r) := \{x \in \mathcal{A} : |i - x| < r\}$.

(ii) Let $p(\cdot)$ be a measurable function and we suppose that

$$1 \leq p_{-}(\mathcal{A}) \leq p_{+}(\mathcal{A}) < \infty, \tag{2}$$

where

$$p_{-} := \operatorname*{ess\,inf}_{x \in \mathcal{A}} p(x), \quad p_{+} := \operatorname*{ess\,sup}_{x \in \mathcal{A}} p(x).$$

The set $\mathcal{P}(\mathcal{A})$ consists of all $p(\cdot)$ such that $p_{-} > 1$ and $p_{+} < \infty$.

(iii) Let $p \in \mathcal{P}(\mathcal{A})$ then $\mathfrak{B}^{\log} = \mathfrak{B}^{\log}(\mathcal{A})$ is the class of functions satisfying (2) and log-condition given by,

$$|p(x_1) - p(x_2)| \leq \frac{C(p)}{-\ln|x_1 - x_2|}, \quad |x_1 - x_2| \leq \frac{1}{2}, \quad x_1, x_2 \in \mathcal{A}. \tag{3}$$

(iv) For an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, $\mathfrak{B}_\infty(\mathcal{A})$ is the subset of $\mathcal{P}(\mathcal{A})$ and values are in $[1, \infty)$ satisfying following condition

$$|p(x) - p_\infty| \leq \frac{C}{\ln(e + |x|)}, \tag{4}$$

where $p_\infty \in (1, \infty)$. $\mathfrak{B}_{0,\infty}(\mathcal{A})$ is the subset of $\mathcal{P}(\mathcal{A})$ satisfying the condition (4) and also the following condition

$$|p(x) - p_0| \leq \frac{C}{\ln|x|}, |x| \leq \frac{1}{2}, \tag{5}$$

for homogeneous Herz spaces.

(v) $p(\cdot) \in \mathcal{A}$ then $\mathcal{B}(\mathcal{A})$ is the collection of $p(\cdot) \in \mathcal{P}(\mathcal{A})$ where $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathcal{A})$.

(vi) $\mathcal{S}_t = \mathcal{S}(0, 2^t) = \{x \in \mathbb{R}^n : |x| < 2^t\}$ for all $l \in \mathbb{Z}$. $F_t = \mathcal{S}_t \setminus \mathcal{S}_{t-1}$, $\chi_{F_t} = \chi_t$ where χ denote the characteristic function.

C is a constant, its value can be changes from line to line.

2. Preliminaries

We are assuming that order of Riesz potential operator $\zeta(x)$ is not continuous rather we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

1. $\zeta_0 := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} \zeta(x) > 0$,
2. $\operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)\zeta(x) < n$,
3. $\operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(\infty)\zeta(x) < n$.

The following proposition is th one of the main requirement to prove our main results. The given proposition was proved in [17] and commonly known as Sobolev theorem for Riesz potential operator in Lebesgue spaces under the some necessary assumptions on exponent.

Proposition 2.1. *Suppose that $p(\cdot) \in \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_{0,\infty}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$, and assume*

$$1 < p(\infty) \leq p(x) \leq p_+ < \infty.$$

Let $\zeta(x)$ is satisfying the above conditions 1, 2 and 3. Then we have following weighted Sobolev-type estimate for the fractional operator $I^{\zeta(z)}$,

$$\left\|(1 + |z|)^{-\lambda(z)} I^{\zeta(z)}(f)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \left\|f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

is the Sobolev exponent.

$$\lambda(z) = C\zeta(z)\left(1 - \frac{\zeta(z)}{n}\right) \leq C\frac{n}{4}.$$

Where C is denoting the Dini-Lipschitz constant from the inequality (4) in which $a(\cdot)$ replaced by $p(\cdot)$.

Remark 2.2.

(i) If $\zeta(z)$ is satisfying the condition (4): $|\zeta(z)-\zeta_\infty| \leq \frac{C_\infty}{\ln(e+|z|)}$ *for* $x \in \mathbb{R}^n$. *Then* $(1+|z|)^{-\lambda(z)}$ *is equivalent to the weight* $(1+|z|)^{-\lambda_\infty}$.

(ii) One can replace the variable order of Riesz potential operator $\zeta(x)$ by $\zeta(y)$ in the case of potentials over bounded domain, such potentials differ unessential if the function $\zeta(x)$ is satisfying the smoothness logarithmic condition as (3), *since*

$$C_1|z_1-z_2|^{n-\zeta(z_2)}| \leq |z_1-z_2|^{n-\zeta(z_1)} \leq C_2|z_1-z_2|^{n-\zeta(z_2)}.$$

Lemma 2.3. *[24] Let $D>1$ and $p \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0}s^{\frac{n}{p(0)}} \leq \|\chi_{R_{s,D_s}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq t_0 s^{\frac{n}{p(0)}}, \text{ for } 0<s\leq 1 \tag{6}$$

and

$$\frac{1}{t_\infty}s^{\frac{n}{p_\infty}} \leq \|\chi_{R_{s,D_s}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq t_\infty s^{\frac{n}{p_\infty}}, \text{ for } s\geq 1, \tag{7}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ is depending on D but not depending on s.

Lemma 2.4. *[20][Generalized Hölder's inequality] Consider a measurable subset H such that $H \subseteq \mathbb{R}^n$, and $1 \leq p_-(H) \leq p_+(H) \leq \infty$. Then*

$$\|fg\|_{L^{r(\cdot)}(H)} \leq \|f\|_{L^{p(\cdot)}(H)}\|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f \in L^{p(\cdot)}(H)$, $g \in L^{q(\cdot)}(H)$ and $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for every $z \in H$.

3. Grand variable Herz-Morrey spaces

Grand variable Herz-Morrey spaces is defined in this section.

Definition 3.1. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $u \in [1,\infty)$, $p:\mathbb{R}^n \to [1,\infty)$, $\theta>0$, $0\leq\beta<\infty$. We define the homogeneous grand variable Herz-Morrey spaces can be defined by the norm:*

$$M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n) = \left\{g \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} = \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{l_0}\|2^{k\alpha(\cdot)}g\chi_l\|^{u(1+\psi)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{u(1+\psi)}}.$$

Non-homogeneous grand variable Herz-Morrey spaces can be defined in the similar way.

Definition 3.2 (BMO space). *A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|Q|}\int_Q |u(y)-u_Q|dy$ is bounded. Mathematically,*

$$\|u\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |u(y)-u_Q|dy < \infty.$$

Let $\mathbb{S}^{n-1}$ is denoting the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. $\Phi \in L^r(\mathbb{S}^{n-1})$ is a function of degree zero which is a homogeneous such that

$$\int_{\mathbb{S}^{n-1}} \Phi(z_2')d\Phi(z_2') = 0, \tag{8}$$

where $z'_2 = z_2/|z_2|$ and z_2 is not zero. The Marcinkiewicz integral is introduced as:

$$[b, \mu_\Phi]^m_\zeta(f)(z_1) = \left(\int_0^\infty |F_{\Phi,s}(f)(z_1)|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Phi,s}(f)(z_1) = \int_{|z_1-z_2|\le s} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-\zeta(z_1)-1}} f(z_2)dz_2.$$

The variable Marcinkiewicz fractional integral are defined as:

$$[b, \mu_\Phi]^m_\zeta(f)(z_1) = \left(\int_0^\infty \left| \int_{z_1-z_2|\le s} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-\zeta(z_1)-1}} f(z_2)dz_2\right|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}}.$$

Lemma 3.3 ([23]). *Let k is a positive integers. Then for all $b \in BMO(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ for $j > i$,*

$$C^{-1}\|b\|^k_{BMO(\mathbb{R}^n)} \le \sup_{D:ball} \frac{1}{\|\chi_D\|_{L^{p(\cdot)}}} \|(b-b_D)^k \chi_D\|_{L^{p(\cdot)}} \tag{9}$$

$$\le C\|b\|^k_{BMO(\mathbb{R}^n)}, \tag{10}$$

$$\|(b-b_{D_i})^k \chi_{D_j}\|_{L^{p(\cdot)}} \le C(j-i)^k \|b\|^k_{BMO(\mathbb{R}^n)} \|\chi_{D_j}\|_{L^{p(\cdot)}}. \tag{11}$$

Lemma 3.4. *[22] If $a > 0$, $s \in [1, \infty]$, $0 < d \le s$ and $-m + (m-1)ds < u < \infty$, then*

$$\left(\int_{|z_2|\le a|z_1|} |z_2|^u |\Phi(z_1-z_2)|^d dz_2\right)^{1/d} \le |z_1|^{(u+m)/d} \|\Phi\|_{L^s(\mathbb{S}^{m-1})}.$$

4. Sobolev type theorem for grand variable Herz-Morrey spaces

Theorem 4.1. *Let $0 < v \le 1$, $\alpha(\cdot), q_1(\cdot) \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q_1^- \le q_1^+ < \infty$, $1 \le u < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be homogeneous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q_1'^-$. Let α be such that :*

(i) $-\frac{n}{q_1(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q_1'(0)} - v - \frac{n}{s}$

(ii) $-\frac{n}{q_{1\infty}} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q'_{1\infty}} - v - \frac{n}{s}$,

$$\left\|(1+|z_1|)^{-\lambda(z_1)} [b, \mu_\Phi]^m_\zeta (g)\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)} \le C \left\|g\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $g \in M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)$, and $g(z_1) = \sum_{l=-\infty}^{\infty} g(z_1)\chi_l(z_1) = \sum_{l=-\infty}^{\infty} g_l(z_1)$, we have,

$$\begin{aligned}\left\|(1+|z_1|)^{-\lambda(z_1)}[b, \mu_\Phi]^m_\zeta(g)\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)} &= \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \\ &\times \left(\psi^\theta \sum_{k=-\infty}^{l_0} \|2^{k\alpha(\cdot)}\chi_k(1+|z_1|)^{-\lambda(z_1)}[b, \mu_\Phi]^m_\zeta(g)\|^{u(1+\psi)}_{L^{q_2(\cdot)}}\right)^{\frac{1}{u(1+\psi)}} \\ \le \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} &\left(\psi^\theta \sum_{k=-\infty}^{l_0} \left(\sum_{l=-\infty}^{\infty} \|2^{k\alpha(\cdot)}\chi_k(1+|z_1|)^{-\lambda(z_1)}[b, \mu_\Phi]^m_\zeta(g)(\chi_l)\|^{u(1+\psi)}_{L^{q_2(\cdot)}}\right)\right)^{\frac{1}{u(1+\psi)}}\end{aligned}$$

$$\leq \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{l_0}\left(\sum_{l=-\infty}^{k}\|2^{k\alpha(\cdot)}\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{l_0}\left(\sum_{l=k+1}^{\infty}\|2^{k\alpha(\cdot)}\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$=: E_1 + E_2.$$

For E_1, splitting E_1 by using Minkowski's inequality we have

$$E_1 \leq \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\left(\sum_{l=-\infty}^{k}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{k}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$=: E_{11} + E_{12}.$$

For estimating E_{11}, we use the facts that, for each $k \in \mathbb{Z}$ and $l \leq k$ and a.e. $z_1 \in F_k$, $z_2 \in F_l$, we know that $|z_1 - z_2| \approx |z_1| \approx 2^k$,

$$\left|\mu_\Phi(g\chi_l)(z_1)\right| \leq \left(\int_o^{|z_1|}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_{|z_1|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$

$$=: I_{11} + I_{12}.$$

By the virtue of the mean value theorem we obtain,

$$\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right| \leq \frac{|z_2|}{|z_1-z_2|^3}. \tag{12}$$

For I_{11}, by using Minkowski's inequaltiy, generalized Hölder's inequality, and inequality 12 we have

$$I_{11} \leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left(\int_{|z_1-z_2|}^{|z_1|}\frac{dt}{t^3}\right)^{1/2}dz_2$$

$$\leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right|^{1/2}dz_2$$

$$\leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left|\frac{|z_2}{|z_1-z_2|^3}\right|^{1/2}dz_2$$

$$\leq \frac{2^{l/2}}{|z_1|^{n+\frac{1}{2}}.|z_1|^{-\zeta(z_1)}}\int_{F_l}|\Phi(z_1-z_2)|[b(z_1)-b(z_2)]^m|g(z_2)|dz_2$$

$$\leq 2^{(l-k)/2}2^{-kn}|z_1|^{\zeta(z_1)}\|g_l\|_{L^{q_1(\cdot)}}\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.$$
$$\leq 2^{(l-k)/2}2^{-kn}|z_1|^{\zeta(z_1)}\Big\{|b(z_1)-b_{D_l}|^m\int_{F_l}|\Phi(z_1-z_2)||g_l(z_2)|dz_2$$
$$+\int_{F_l}|b(z_2)-b_{D_l}|^m|\Phi(z_1-z_2)||g_l(z_2)|dz_2\Big\}$$
$$\leq 2^{(l-k)/2}2^{-kn}|z_1|^{\zeta(z_1)}\|g_l(z_2)\|_{L^{q_1(\cdot)}}\Big(|b(z_1)-b_{D_l}|^m\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}$$
$$+\|(b(\cdot)-b_{D_l})^m(\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\Big).$$

Similarly, we can consider I_{12}, we have

$$I_{12}\leq\int_{\mathbb{R}^n}\frac{|\Phi(z-1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g_l(z_2)|\left(\int_{|z_1|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dz_2$$
$$\leq\int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g_l(z_2)|dz_2$$
$$\leq |z_1|^{-n}|z_1|^{\zeta(z_1)}\int_{F_l}|\Phi(z_1-z_2)||g(z_2)|dz_2$$
$$\leq 2^{-kn}|z_1|^{\zeta(z_1)}\|g_l(z_2)\|_{L^{q_1(\cdot)}}\Big\{|b(z_1)-b_{D_l}|^m\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}+\|(b(\cdot)-b_{D_l})^m(\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\Big\}.$$

We define $q_1(\cdot)$ by the relation $\frac{1}{q_1'(x)}=\frac{1}{q_1(x)}+\frac{1}{s}$. By using Lemma (3.4) and generalized Hölder's inequality we have

$$\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\leq\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_l(\cdot)\|_{L^{q_1(\cdot)}}$$
$$\leq 2^{-lv}\left(\int_{2^{l-1}<|z_2|<2^l}|\Phi(z_1-z_2)|^s|z_2|^{sv}dz_2\right)^{1/s}\|\chi_l\|_{L^{q_1(\cdot)}}$$
$$\leq 2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}.$$

Similarly, by using Lemma (3.3) we have

$$\|(b(\cdot)-b_{D_l})^m(\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}$$
$$\leq\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}\|(b(\cdot)-b_{D_l})^m\chi_l(\cdot)\|_{L^{q_1(\cdot)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1(\cdot)}}\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}.$$

It is known, see e.g. [18] that

$$I^{\zeta(\cdot)}((b(z_1)-b_{D_l})^m\chi_k)(z_1)\geq I^{\zeta(\cdot)}(\chi_k)(z_1).(\chi_k)(z_1)$$
$$=\int_{F_k}\frac{|b(z_1)-b_{D_l}|^m}{|z_1-z_2|^{\zeta(z_1)-n}}dz_2.\chi_{D_k}(z_1)$$
$$\geq C|b(z_1)-b_{D_l}|^m|z_1|^{\zeta(z_1)}.\chi_k(z_1).$$

Consequently, by proposition (2.1) we have

$$\left\|(b(z_1)-b_{D_l})^m|z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{L^{q_2(\cdot)}}$$

$$\leq \left\|(1+|z_1|)^{-\lambda(z_1)} I^{\zeta(\cdot)}((b(z_1)-b_{D_l})^m \chi_{D_k})(z_1)\right\|_{L^{q_2(\cdot)}}$$
$$\leq \left\|(b(z_1)-b_{D_l})^m \chi_k)(z_1)\right\|_{L^{q_1(\cdot)}}.$$

As a result we get

$$\begin{aligned}
&\left\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g\chi_l)\right\|_{L^{q_2(\cdot)}}\\
&\leq C2^{-kn}\|g_l\|_{L^{q_1(\cdot)}}\Big\{\left\|(b(z_1)-b_{D_l})^m|z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{L^{q_2(\cdot)}} 2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\||z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}}\Big\}\\
&\leq C2^{-kn}\|g_l\|_{L^{q_1(\cdot)}}\Big\{(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\Big\}\\
&\leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-kn}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_l\|_{L^{q_1(\cdot)}}\|g_l\|_{L^{q_1(\cdot)}}.
\end{aligned}$$

Applying results to E_{11} we can get

$$\begin{aligned}
E_{11} \leq C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg[\psi^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\\
\times\Big(\sum_{l=-\infty}^{k}2^{(l-k)(n/q_1'(0)-v-\frac{n}{s})}(k-l)^m\|g_l\|_{L^{q_1(\cdot)}}\Big)^{u(1+\psi)}\Bigg]^{\frac{1}{u(1+\psi)}}\\
\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg[\psi^\theta\sum_{k=-\infty}^{-1}\Big(\sum_{l=-\infty}^{k}2^{\alpha(0)l}\|g_l\|_{L^{q_1(\cdot)}}2^{b(l-k)}(k-l)^m\Big)^{u(1+\psi)}\Bigg]^{\frac{1}{u(1+\psi)}}.
\end{aligned}$$

Let $b_1=\frac{n}{q_1'(0)}-v-\frac{n}{s}-\alpha(0)>0$, applying Hölders inequality, $2^{-u(1+\psi)}<2^{-u}$ and Fubini's theorem for series and we get,

$$\begin{aligned}
E_{11} \leq& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg[\psi^\theta\sum_{k=-\infty}^{-1}\Bigg(\sum_{l=-\infty}^{k}2^{\alpha(0)u(1+\psi)l}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}2^{b_1u(1+\psi)(l-k)/2}(k-l)^{mu(1+\psi)}\\
&\times\sum_{l=-\infty}^{k}2^{bu(1+\psi)'(l-k)/2}(k-l)^{mu'(1+\psi)/2}\Bigg)^{\frac{u(1+\psi)}{u(1+\psi)'}}\Bigg]^{\frac{1}{u(1+\psi)}}\\
\leq& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k}2^{\alpha(0)u(1+\psi)l}(k-l)^{mu(1+\psi)}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}2^{b_1u(1+\psi)(l-k)/2}\Bigg)^{\frac{1}{u(1+\psi)}}\\
\leq& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{l=-\infty}^{-1}2^{\alpha(0)u(1+\psi)l}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\sum_{k=l}^{-1}2^{b_1u(1+\psi)(l-k)/2}(k-l)^{mu(1+\psi)}\Bigg)^{\frac{1}{u(1+\psi)}}\\
\leq& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{l=-\infty}^{-1}2^{\alpha(0)u(1+\psi)l}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\sum_{k=l}^{-1}2^{b_1p(l-k)/2}(k-l)^{m}\Bigg)^{\frac{1}{u(1+\psi)}}\\
\leq& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{l=-\infty}^{-1}2^{\alpha(0)u(1+\psi)l}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\Bigg)^{\frac{1}{u(1+\psi)}}
\end{aligned}$$

$$\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{l=-\infty}^{l_0}\|2^{\alpha(\cdot)l}g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}}.$$

Now for E_{12} using Minkowski's inequality we have

$$E_{12}\leq\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=0}^{k}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$=: A_1+A_2.$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $q_1'(0)$ with $q_{1\infty}'$ and by the use $b_2:=\frac{n}{q_{1\infty}'}-v-\frac{n}{s}-\alpha_\infty>0$. For A_1 we have

$$\|[b,\mu_\Phi]_\zeta^m(g\chi_l)\chi_k\|_{L^{q_2(\cdot)}}$$

$$\leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}^m2^{-kn}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_l\|_{L^{q_1(\cdot)}}\|g_l\|_{L^{q_1(\cdot)}}$$

$$\leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}^m2^{l(\frac{n}{q_1(0)}-v)}2^{k(v+\frac{n}{s}-\frac{n}{q_{1\infty}'})}\|g_l\|_{L^{q_1(\cdot)}}.$$

Now by using the fact $-\frac{n}{q_{1\infty}'}+v+\frac{n}{s}+\alpha_\infty<0$ we have,

$$A_1\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{ka_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}I^m_{\zeta(\cdot),b}g_l\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{ka_\infty u(1+\psi)}\right.$$

$$\left.\times\left(\sum_{l=-\infty}^{-1}(k-l)^m2^{l(\frac{n}{q_1(0)}-v)}2^{k(v+\frac{n}{s}-\frac{n}{q_{1\infty}'})}\|g_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\right.$$

$$\left.\times\left(\sum_{l=-\infty}^{-1}(k-l)^m2^{l(\frac{n}{q_1(0)}-v-la_\infty)}2^{k(v+\frac{n}{s}-\frac{n}{q_{1\infty}'}+ka_\infty)}2^{la_\infty}\|g_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\left(\sum_{l=-\infty}^{-1}2^{a_\infty lu(1+\psi)}((k-l)^m2^{l(\frac{n}{q_1(0)}-v-la_\infty)}2^{k(v+\frac{n}{s}-\frac{n}{q_{1\infty}'}+ka_\infty)})^{u(1+\psi)/2}\right.\right.$$

$$\left.\left.\times\|g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\left(\sum_{l=-\infty}^{-1}((k-l)^m2^{l(\frac{n}{q_1(0)}-v-la_\infty)}2^{k(v+\frac{n}{s}-\frac{n}{q_{1\infty}'}+ka_\infty)})^{u(1+\psi)'/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)}}\right)\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\sum_{l=-\infty}^{-1}2^{a_\infty lu(1+\psi)}\|g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\right.$$

$$\times\left((k-l)^m 2^{l(\frac{n}{q_1(0)}-v-la_\infty)}2^{k(v+\frac{n}{s}-\frac{n}{q'_{1\infty}}+ka_\infty)}\right)^{u(1+\psi)/2}\Bigg)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{l=-\infty}^{-1}2^{a_\infty lu(1+\psi)}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}$$

$$\times\sum_{k=l}^{\infty}((k-l)^m 2^{l(\frac{n}{q_1(0)}-v-la_\infty)}2^{k(v+\frac{n}{s}-\frac{n}{q'_{1\infty}}+ka_\infty)}\Bigg)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{l=-\infty}^{-1}2^{a_\infty lu(1+\psi)}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Now we estimate E_2, for each $k\in\mathbb{Z}$ and $l\geq k+1$ and a.e. $z_1\in F_k$, $z_2\in F_l$, we know that $|z_1-z_2|\approx|z_2|\approx 2^l$, we consider

$$|\mu_\Phi(g\chi_l)(z_1)|\leq\left(\int_o^{|z_2|}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_{|z_2|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$

$$=: I_{31}+I_{32}.$$

By using the similar arguments as used in I_{11}, we obtain

$$I_{31}\leq 2^{(l-k)/2}2^{-ln}|z_1|^{\zeta(z_1)}\|g_l(z_2)\|_{L^{q_1(\cdot)}}\left(|b(z_1)-b_{D_l}|^m\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q'_1(\cdot)}}+\|(b(\cdot)-b_{D_l})^m(\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q'_1(\cdot)}}\right).$$

By using same arguments of I_{12}, we obtain

$$I_{32}\leq 2^{-ln}|z_1|^{\zeta(z_1)}\|g\chi_l(z_2)\|_{L^{q_1(\cdot)}}\left\{|b(z_1)-b_{D_l}|^m\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q'_1(\cdot)}}+\|(b(\cdot)-b_{D_l})^m(\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q'_1(\cdot)}}\right\}.$$

$$\begin{aligned}
&\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]^m_\zeta(g_l)\|_{L^{q_2(\cdot)}}\\
&\leq C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{\left\|(b(z_1)-b_{D_l})^m|z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{L^{q_2(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{D_l}\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\||z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}}\Big\}\\
&\leq C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\Big\}\\
&\leq C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&\leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_l\|_{L^{q_1(\cdot)}}\|g_l\|_{L^{q_1(\cdot)}}\\
&\leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}\|g_l\|_{L^{q_1(\cdot)}}.
\end{aligned}$$

Now splitting E_2 we have

$$E_2 \leq \max\Bigg\{ \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{l_0}\left(\sum_{l=k+1}^{\infty}\|2^{k\alpha(\cdot)}\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}},$$

$$\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(-)u(1+\psi)}\left(\sum_{l=k+1}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=k+1}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$=:E_{21}+E_{22}.$$

For E_{22} we have

$$E_{22} \leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=k+1}^{\infty}2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}(k-l)^m\|g_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\|g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\left(\sum_{l=k+1}^{\infty}2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}(k-l)^m\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

where $d=\frac{n}{q_{1\infty}}+v+\frac{n}{s}+a_\infty>0$. Then we use Hölder's theorem for series and $2^{-u(1+\psi)}<2^{-u}$ to obtain

$$E_{22} \leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\Bigg[\psi^{\theta}\sum_{k=0}^{l_0}\left(\sum_{l=k+1}^{\infty}2^{l\alpha_\infty u(1+\psi)}\|g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}(k-l)^{mu(1+\psi)'/2}2^{du(1+\psi)(k-l)/2}\right)$$
$$\times\left(\sum_{l=k+1}^{\infty}(k-l)^{mu(1+\psi)'/2}2^{du(1+\psi)'(k-l)/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\Bigg]^{\frac{1}{u(1+\psi)}}$$

$$\leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\sum_{l=k+1}^{\infty}2^{l\alpha_\infty u(1+\psi)}\|g_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}(k-l)^{mu(1+\psi)'/2}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\sum_{l=k+1}^{\infty}\sum_{j=-\infty}^{l}2^{j\alpha_\infty u(1+\psi)}\|g_j\|_{L^{q_1(\cdot)}}^{u(1+\psi)}(k-l)^{mu(1+\psi)'/2}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}\sum_{l=k+1}^{\infty}(k-l)^{mu(1+\psi)'/2}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}}$$

$$\leq C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}}.$$

Now for E_{21} using Minkowski's inequality we have

$$E_{21} \leq \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\left(\sum_{l=k+1}^{-1}\left\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\right\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+\sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\left(\sum_{l=0}^{\infty}\left\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]_\zeta^m(g_l)\right\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$=:B_1 + B_2.$

The estimate for B_1 follows in a similar manner to E_{22} with $q_{1\infty}$ replaced by $q_1(0)$ and using the fact that $\frac{n}{q_1(0)} + v + \frac{n}{s} + \alpha(0) > 0$. For B_2 we have

$$\begin{aligned}
&\left\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]^m_\zeta(g\chi_l)\right\|_{L^{q_2(\cdot)}}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{\left\|(b(z_1)-b_{D_l})^m|z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{L^{q_2(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\||z_1|^{\zeta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}}\Big\}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\Big\}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&\le C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}2^{ln/q_{1\infty}}2^{kn/q_1(0)}\|g_l\|_{L^{q_1(\cdot)}}\\
&\le C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g_l\|_{L^{q_1(\cdot)}}.
\end{aligned}$$

$$\begin{aligned}
B_2 \le& C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\left(\sum_{l=0}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}I^m_{\zeta(\cdot),b}(g_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\right.\\
&\left.\times\left(\sum_{l=0}^{\infty}(k-l)^m2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}\right.\\
&\left.\times\left(\sum_{l=0}^{\infty}2^{la(0)}(k-l)^m2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0)}\|g_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=0}^{\infty}2^{la(0)(u(1+\psi))}((k-l)^m2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0)})^{u(1+\psi)/2}\right.\right.\\
&\left.\left.\times\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\left(\sum_{l=0}^{\infty}((k-l)^m2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0)})^{u(1+\psi)'/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\right)\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}\sum_{l=0}^{\infty}2^{la(0)(u(1+\psi))}\|g_l\|^{u(1+\psi)/2}_{L^{q_1(\cdot)}}\right.\\
&\left.\times\left((k-l)^m2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{l=0}^{\infty}2^{la(0)(u(1+\psi))}\|g_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\sum_{k=-\infty}^{l}\right.
\end{aligned}$$

$$\times\Big((k-l)^m 2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0))}\Big)^{u(1+\psi)/2}\Bigg)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\Bigg(\psi^\theta\sum_{l=0}^{\infty}\sum_{j=-\infty}^{l}2^{ja(0)(u(1+\psi))}\|g_j\|^{u(1+\psi)}_{q_1(\cdot)}\sum_{k=-\infty}^{l}$$

$$\times\Big((k-l)^m 2^{-l(v+\frac{n}{s}+\frac{n}{q_{1\infty}}-la(0))}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+ka(0))}\Big)^{u(1+\psi)/2}\Bigg)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1 and E_2 yields

$$\left\|(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]^m_\zeta(g)\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)}\leq C\left\|g\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)},$$

which ends the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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