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On an inverse boundary value problem for the sixth-order Boussinesq equation with integral conditions

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Abstract. This work investigates the inverse boundary value problem for a sixth-order Boussinesq equation, where the time-dependent coefficients are unknown. The primary objective is to determine these unknown coefficients and solve the corresponding problem. The original problem is transformed into an equivalent auxiliary problem. By utilizing the contraction mapping method, we prove the existence and uniqueness of the solution for the auxiliary problem. Consequently, we establish the existence and uniqueness of a classical solution to the initial problem.

1. Introduction

In many practical applications, problems arise when determining the coefficients or the right-hand side of differential equations based on known solution data. These problems are known as inverse boundary value problems of mathematical physics. Such problems emerge in diverse fields including seismology, mineral exploration, biology, medicine, and industrial quality control, positioning them as key problems in contemporary mathematics.

Inverse problems represent a rapidly growing area of modern mathematics, with recent advancements leading to their widespread application across various scientific disciplines. Numerous studies have explored different inverse problems for various types of partial differential equations. Notable contributions include the work of A.N. Tikhonov [21], M.M. Lavrent'ev [9, 10], A.M. Denisov [2], M.I. Ivanchov [6], among others. Their work has significantly influenced subsequent developments in this field.

The sixth-order Boussinesq equation with double dispersion, which models wave motion on water with surface stress, was examined by Schneider and Wayne in [19]. Numerous boundary value problems associated with Boussinesq-type equations have been extensively studied in the literature [3, 5, 11, 16–18, 22–24]. Additionally, inverse problems for various classes of partial differential equations have received

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significant attention in several works [1, 7, 8, 13, 14]. Specifically, inverse problems for the sixth-order Boussinesq equation have been considered in [4, 25].

In this paper, we establish the existence and uniqueness of the solution of the inverse boundary value problem for the double-dispersive sixth-order Boussinesq equation.

2. Problem statement and its reduction to equivalent problem

Let $T > 0$ be a fixed constant, and define the domain $\Omega_T := \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$. We consider the one-dimensional inverse problem of determining the unknown set of functions $\{v(x,t), \alpha(t), \beta(t)\}$ for the following double-dispersive sixth-order Boussinesq equation [20]:

$$v_{tt}(x,t) - v_{xx}(x,t) - v_{ttxx}(x,t) + v_{xxxx}(x,t) + v_{ttxxxx}(x,t) = \alpha(t)v(x,t) + \beta(t)h(x,t) + r(x,t), \tag{1}$$

subject to the nonlocal initial conditions:

$$\begin{aligned} v(x,0) &= \int_0^T Q_1(t)v(x,t)dt + \eta(x), \quad x \in [0,1], \\ v_t(x,0) &= \int_0^T Q_2(t)v(x,t)dt + \xi(x), \quad x \in [0,1], \end{aligned} \tag{2}$$

and boundary conditions:

$$v_x(0,t) = 0, \quad v_x(1,t) = 0, \quad v_{xxx}(0,t) = 0, \quad t \in [0,T], \tag{3}$$

as well as the nonlocal integral condition:

$$\int_0^1 v(x,t)dx = 0, \quad t \in [0,T], \tag{4}$$

and over-determination conditions:

$$v(x_j,t) = g_j(t), \quad j = 1,2;\ x_1 \ne x_2, \quad t \in [0,T], \tag{5}$$

where $x_j \in [0,1]$ for $j = 1,2$ are fixed points, and the functions $h(x,t), r(x,t), \eta(x), \xi(x), Q_j(t)$ for $j = 1,2$, and $g_j(t)$ for $j = 1,2$ are assumed to be sufficiently smooth for $x \in [0,1]$ and $t \in [0,T]$.

Definition 2.1. *The triple $\{v(x,t), \alpha(t), \beta(t)\}$ is said to be a classical solution to the problem* (1)–(5) *if the functions $v(x,t) \in \widetilde{C}^{4,2}(\Omega_T)$ and $\alpha(t), \beta(t) \in C[0,T]$ satisfy equation* (1) *in the region Ω_T, the conditions* (2) *on the interval $[0,1]$, and the conditions* (3)–(5) *on the interval $[0,T]$, where*

$$\widetilde{C}^{4,2}(\Omega_T) = \left\{ v(x,t) : v(x,t) \in C^2(\Omega_T), v_{xxxx}(x,t), v_{ttxxxx}(x,t) \in C(\Omega_T) \right\}.$$

To investigate the problem (1)–(5), we first consider the following auxiliary problem:

$$z''(t) = \alpha(t)z(t), \quad t \in [0,T], \tag{6}$$

$$z(0) = \int_0^T Q_1(t)z(t)dt, \quad z'(0) = \int_0^T Q_2(t)z(t)dt, \tag{7}$$

where $\alpha(t), Q_1(t), Q_2(t) \in C[0,T]$ are given functions, and $z = z(t)$ is the desired solution. Here, by the solution of the problem given by (6) and (7), we mean a function $z(t) \in C^2[0,T]$ that satisfies the conditions (6) and (7) in the classical sense.

Lemma 2.2. ([15]). *Assume that $\alpha(t), Q_1(t), Q_2(t) \in C[0,T]$, and $\|\alpha(t)\|_{C[0,T]} \le R$, where R is a constant. Additionally, suppose the following condition holds:*

$$\left(T\,\|Q_2(t)\|_{C[0,T]} + \|Q_1(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1.$$

Then, the problem given by (6) *and* (7) *has a unique trivial solution.*

Next, in conjunction with the inverse boundary value problem (1)–(5), we consider the following auxiliary inverse boundary value problem: the objective is to determine a triple of functions $\{v(x,t), \alpha(t), \beta(t)\}$, where $v(x,t) \in \widetilde{C}^{4,2}(\Omega_T)$ and $\alpha(t), \beta(t) \in C[0,T]$, that satisfy the relations (1)–(3) along with the following conditions:

$$v_{xxx}(1,t) = 0, \quad t \in [0,T], \tag{8}$$

$$g_j''(t) - v_{xx}(x_j,t) - v_{ttxx}(x_j,t) + v_{xxxx}(x_j,t) + v_{ttxxxx}(x_j,t) = \alpha(t)g_j(t) + \beta(t)h(x_j,t) + r(x_j,t),$$
$$j = 1,2, \quad t \in [0,T]. \tag{9}$$

This formulation allows us to proceed with the analysis of the original problem as follows:

Theorem 2.3. *Suppose that $\eta(x), \xi(x) \in C^3[0,1]$, $\eta'''(1) = 0$, $\xi'''(1) = 0$ and $Q_j(t) \in C[0,T]$ for $j = 1,2$. Additionally, let $g_j(t) \in C^2[0,T]$ for $j = 1,2$, and suppose $g(t) \equiv g_1(t)h(x_2,t) - g_2(t)h(x_1,t) \neq 0$ for $t \in [0,T]$. Assume that $h(x,t), r(x,t) \in C(\Omega_T)$, and the following conditions hold:*

$$\int_0^1 h(x,t)dx = 0, \quad \int_0^1 r(x,t)dx = 0, \quad t \in [0,T], \tag{10}$$

$$D \equiv \begin{vmatrix} 1 - \int_0^T Q_1(t)\cos t\,dt & -\int_0^T Q_1(t)\sin t\,dt \\ -\int_0^T Q_2(t)\cos t\,dt & 1 - \int_0^T Q_2(t)\sin t\,dt \end{vmatrix} \neq 0, \tag{11}$$

$$\int_0^1 \eta(x)dx = 0, \quad \int_0^1 \xi(x)dx = 0, \tag{12}$$

$$g_j(0) = \int_0^T Q_1(t)g_j(t)dt + \eta(x_j), \quad g_j'(0) = \int_0^T Q_2(t)g_j(t)dt + \xi(x_j), \quad j = 1,2. \tag{13}$$

Then, the following statements are valid:
(i) Every classical solution $\{v(x,t), \alpha(t), \beta(t)\}$ of the problem (1)–(5) *is also a solution of the problem* (1)–(3), (8), (9).
(ii) Conversely, every solution $\{v(x,t), \alpha(t), \beta(t)\}$ of the problem (1)–(3), (8), (9), *that satisfies the condition:*

$$\left(T\,\|Q_2(t)\|_{C[0,T]} + \|Q_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|\alpha(t)\|_{C[0,T]}\right)T < 1,$$

is a classical solution of the problem (1)–(5).

Proof. Let $\{v(x,t), \alpha(t), \beta(t)\}$ be any classical solution to the problem (1)–(5). By integrating both sides of equation (1) with respect to x from 0 to 1, we find:

$$\frac{d^2}{dt^2}\int_0^1 v(x,t)dx - (v_x(1,t) - v_x(0,t)) - (v_{ttx}(1,t) - v_{ttx}(0,t)) + (v_{xxx}(1,t) - v_{xxx}(0,t))$$
$$+ (v_{ttxxx}(1,t) - v_{ttxxx}(0,t)) = \alpha(t)\int_0^1 v(x,t)dx + \beta(t)\int_0^1 h(x,t)dx + \int_0^1 r(x,t)dx, \quad t \in [0,T]. \tag{14}$$

Taking into account the conditions (10) and considering (3) and (4), we obtain:

$$v_{ttxxx}(1,t) + v_{xxx}(1,t) = 0, \quad t \in [0,T]. \tag{15}$$

By applying (2) and using $\eta'''(1) = 0$ and $\xi'''(1) = 0$, we obtain:

$$v_{xxx}(1,0) - \int_0^T Q_1(t)v_{xxx}(1,t)dt = \eta'''(1) = 0, \quad v_{txxx}(1,0) - \int_0^T Q_2(t)v_{xxx}(1,t)dt = \xi'''(1) = 0. \tag{16}$$

When the condition $D \neq 0$ holds, the problem given by (15) and (16) has only a trivial solution, which implies $v_{xxx}(1,t) = 0$ for $t \in [0,T]$, i.e. condition (8) is fulfilled.

Next, setting $x = x_j$ in equation (1), we find:

$$v_{tt}(x_j,t) - v_{xx}(x_j,t) - v_{ttxx}(x_j,t) + v_{xxxx}(x_j,t) + v_{ttxxxx}(x_j,t) = \alpha(t)v(x_j,t) + \beta(t)h(x_j,t) + r(x_j,t),$$
$$j = 1,2, \quad t \in [0,T]. \tag{17}$$

Considering that $g_j(t) \in C^2[0,T]$ for $j = 1,2$ and differentiating (5) twice yields:

$$v_{tt}(x_j,t) = g_j''(t), \quad j = 1,2, \quad t \in [0,T]. \tag{18}$$

From (17) and considering (5) and (18), we conclude that the relation (9) holds. Now, assuming that $\{v(x,t), \alpha(t), \beta(t)\}$ is a solution of the problem (1)–(3), (8), (9). Then, from (14), taking into account the conditions (10) and using the relations (3) and (8), we obtain:

$$\frac{d^2}{dt^2}\int_0^1 v(x,t)dx = \alpha(t)\int_0^1 v(x,t)dx, \quad t \in [0,T]. \tag{19}$$

Furthermore, from (2) and (12), it is straightforward to see that:

$$\int_0^1 v(x,0)dx - \int_0^T Q_1(t)\left(\int_0^1 v(x,t)dx\right)dt = \int_0^1 \left(v(x,0) - \int_0^T Q_1(t)v(x,t)dt\right)dx = \int_0^1 \eta(x)dx = 0, \tag{20}$$

$$\int_0^1 v_t(x,0)dx - \int_0^T Q_2(t)\left(\int_0^1 v(x,t)dx\right)dt = \int_0^1 \left(v_t(x,0) - \int_0^T Q_2(t)v(x,t)dt\right)dx = \int_0^1 \xi(x)dx = 0. \tag{21}$$

Since, by Lemma 2.2, the problem given by (19)–(21) admits only a trivial solution, it follows that:

$$\int_0^1 v(x,t)dx = 0, \quad t \in [0,T],$$

i.e. condition (4) is satisfied. Next, from equations (9) and (17), we obtain:

$$\frac{d^2}{dt^2}\left(v(x_j,t) - g_j(t)\right) = \alpha(t)\left(v(x_j,t) - g_j(t)\right), \quad j = 1,2, \quad t \in [0,T]. \tag{22}$$

By utilizing (2) and the compatibility conditions (13), we obtain:

$$v(x_j,0) - g_j(0) - \int_0^T Q_1(t)\left(v(x_j,t) - g_j(t)\right)dt = v(x_j,0) - \int_0^T Q_1(t)v(x_j,t)dt$$
$$-\left(g_j(0) - \int_0^T Q_1(t)g_j(t)dt\right) = \eta(x_j) - \left(g_j(0) - \int_0^T Q_1(t)g_j(t)dt\right) = 0, \quad j = 1,2, \tag{23}$$

$$v_t(x_j,0) - g_j'(0) - \int_0^T Q_2(t)\left(v(x_j,t) - g_j(t)\right)dt = v_t(x_j,0) - \int_0^T Q_2(t)v(x_j,t)dt$$
$$-\left(g_j'(0) - \int_0^T Q_2(t)g_j(t)dt\right) = \xi(x_j) - \left(g_j'(0) - \int_0^T Q_2(t)g_j(t)dt\right) = 0, \quad j = 1,2. \tag{24}$$

By Lemma 2.2 and the relations (22)–(24), we conclude that conditions (5) are fulfilled. This completes the proof of the theorem. □

3. Existence and uniqueness of the classical solution

We aim to determine the first component $v(x,t)$ of the classical solution $\{v(x,t), \alpha(t), \beta(t)\}$ of the problem (1)–(3), (8), (9) in the following form:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)\cos(\mu_n x), \tag{25}$$

where

$$v_n(t) = \gamma_n \int_0^1 v(x,t)\cos(\mu_n x)dx, \quad \mu_n = n\pi, \quad n = 0,1,2,\dots,$$

and

$$\gamma_n = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n = 1,2,\dots. \end{cases}$$

By applying the formal scheme of the Fourier method to equations (1) and (2), we obtain:

$$\left(1+\mu_n^2+\mu_n^4\right)v_n''(t) + \left(\mu_n^2+\mu_n^4\right)v_n(t) = G_n(t;v,\alpha,\beta), \quad n = 0,1,2,\dots; \quad 0 \le t \le T, \tag{26}$$

$$v_n(0) = \eta_n + \int_0^T Q_1(t)v_n(t)dt, \quad v_n'(0) = \xi_n + \int_0^T Q_2(t)v_n(t)dt, \quad n = 0,1,2,\dots, \tag{27}$$

where

$$G_n(t;v,\alpha,\beta) = r_n(t) + \alpha(t)v_n(t) + \beta(t)h_n(t), \quad n = 0,1,2,\dots,$$

$$r_n(t) = \gamma_n \int_0^1 r(x,t)\cos(\mu_n x)dx, \quad h_n(t) = \gamma_n \int_0^1 h(x,t)\cos(\mu_n x)dx, \quad n = 0,1,2,\dots,$$

$$\eta_n = \gamma_n \int_0^1 \eta(x)\cos(\mu_n x)dx, \quad \xi_n = \gamma_n \int_0^1 \xi(x)\cos(\mu_n x)dx, \quad n = 0,1,2,\dots.$$

Solving the problem given by (26) and (27), we obtain:

$$v_0(t) = \eta_0 + \int_0^T Q_1(t)v_0(t)dt + t\left(\xi_0 + \int_0^T Q_2(t)v_0(t)dt\right) + \int_0^t (t-\tau)G_0(\tau;v,\alpha,\beta)d\tau, \tag{28}$$

$$\begin{aligned} v_n(t) = &\left(\eta_n + \int_0^T Q_1(t)v_n(t)dt\right)\cos(\omega_n t) + \frac{1}{\omega_n}\left(\xi_n + \int_0^T Q_2(t)v_n(t)dt\right)\sin(\omega_n t) \\ &+ \frac{1}{\omega_n\left(1+\mu_n^2+\mu_n^4\right)}\int_0^t G_n(\tau;v,\alpha,\beta)\sin(\omega_n(t-\tau))d\tau, \quad n = 1,2,\dots; \quad 0 \le t \le T, \end{aligned} \tag{29}$$

where

$$\omega_n = \sqrt{\frac{\mu_n^2+\mu_n^4}{1+\mu_n^2+\mu_n^4}}, \quad n = 0,1,2,\dots.$$

To determine the first component of the classical solution of the problem (1)–(3), (8), (9), we substitute the expressions $v_n(t)$ for $n = 0, 1, 2, \ldots$ into (25) and obtain:

$$\begin{aligned}
v(x,t) =& \left(\eta_0 + \int_0^T Q_1(t)v_0(t)dt\right) + t\left(\xi_0 + \int_0^T Q_2(t)v_0(t)dt\right) + \int_0^t (t-\tau)G_0(\tau; v, \alpha, \beta)d\tau \\
&+ \sum_{n=1}^{\infty} \left\{ \left(\eta_n + \int_0^T Q_1(t)v_n(t)dt\right)\cos(\omega_n t) + \frac{1}{\omega_n}\left(\xi_n + \int_0^T Q_2(t)v_n(t)dt\right)\sin(\omega_n t) \right. \\
&\left. + \frac{1}{\omega_n\left(1+\mu_n^2+\mu_n^4\right)} \int_0^t G_n(\tau; v, \alpha, \beta)\sin(\omega_n(t-\tau))d\tau \right\} \cos(\mu_n x). \qquad (30)
\end{aligned}$$

It follows from (9) and (25) that:

$$\begin{aligned}
\alpha(t) =& [g(t)]^{-1}\{(g_1''(t) - r(x_1,t))h(x_2,t) - (g_2''(t) - r(x_2,t))h(x_1,t) \\
&+ \sum_{n=1}^{\infty} (h(x_2,t)\cos(\mu_n x_1) - h(x_1,t)\cos(\mu_n x_2))(\mu_n^2 + \mu_n^4)(v_n''(t) + v_n(t))\}, \qquad (31)
\end{aligned}$$

$$\begin{aligned}
\beta(t) =& [g(t)]^{-1}\{(g_2''(t) - r(x_2,t))g_1(t) - (g_1''(t) - r(x_1,t))g_2(t) \\
&+ \sum_{n=1}^{\infty} (g_1(t)\cos(\mu_n x_2) - g_2(t)\cos(\mu_n x_1))(\mu_n^2 + \mu_n^4)(v_n''(t) + v_n(t))\}. \qquad (32)
\end{aligned}$$

From (26) and (29), we obtain:

$$\begin{aligned}
(\mu_n^2+\mu_n^4)(v_n''(t) + v_n(t)) =& -v_n''(t) + G_n(t; v, \alpha, \beta) = \frac{\mu_n^2+\mu_n^4}{1+\mu_n^2+\mu_n^4}v_n(t) + \left(1 - \frac{1}{1+\mu_n^2+\mu_n^4}\right)G_n(t; v, \alpha, \beta) \\
=& \frac{\mu_n^2+\mu_n^4}{1+\mu_n^2+\mu_n^4}v_n(t) + \frac{\mu_n^2+\mu_n^4}{1+\mu_n^2+\mu_n^4}G_n(t; v, \alpha, \beta) = \omega_n^2 v_n(t) + \omega_n^2 G_n(t; v, \alpha, \beta) \\
=& \omega_n^2 \left[\left(\eta_n + \int_0^T Q_1(t)v_n(t)dt\right)\cos(\omega_n t) + \frac{1}{\omega_n}\left(\xi_n + \int_0^T Q_2(t)v_n(t)dt\right)\sin(\omega_n t) \right. \\
&\left. + \frac{1}{\omega_n\left(1+\mu_n^2+\mu_n^4\right)} \int_0^t G_n(\tau; v, \alpha, \beta)\sin(\omega_n(t-\tau))d\tau + G_n(t; v, \alpha, \beta) \right], \quad n = 1, 2, \ldots. \qquad (33)
\end{aligned}$$

Substituting equation (33) into (31) and (32), we obtain the equations for the second and third components of the solution to the problem (1)–(3), (8), (9):

$$\begin{aligned}
\alpha(t) =& [g(t)]^{-1}\{(g_1''(t) - r(x_1,t))h(x_2,t) - (g_2''(t) - r(x_2,t))h(x_1,t) \\
&+ \sum_{n=1}^{\infty} (h(x_2,t)\cos(\mu_n x_1) - h(x_1,t)\cos(\mu_n x_2))\omega_n^2 \\
&\times \left[\left(\eta_n + \int_0^T Q_1(t)v_n(t)dt\right)\cos(\omega_n t) + \frac{1}{\omega_n}\left(\xi_n + \int_0^T Q_2(t)v_n(t)dt\right)\sin(\omega_n t) \right. \\
&\left.\left. + \frac{1}{\omega_n\left(1+\mu_n^2+\mu_n^4\right)} \int_0^t G_n(\tau; v, \alpha, \beta)\sin(\omega_n(t-\tau))d\tau + G_n(t; v, \alpha, \beta) \right] \right\}, \qquad (34)
\end{aligned}$$

$$\begin{aligned}\beta(t) =&[g(t)]^{-1}\{(g_2''(t) - r(x_2,t))g_1(t) - (g_1''(t) - r(x_1,t))g_2(t)\\ &+\sum_{n=1}^{\infty}(g_1(t)\cos(\mu_n x_2) - g_2(t)\cos(\mu_n x_1))\omega_n^2\\ &\times\Bigg[\left(\eta_n + \int_0^T Q_1(t)v_n(t)dt\right)\cos(\omega_n t) + \frac{1}{\omega_n}\left(\xi_n + \int_0^T Q_2(t)v_n(t)dt\right)\sin(\omega_n t)\\ &\quad+\frac{1}{\omega_n\left(1+\mu_n^2+\mu_n^4\right)}\int_0^t G_n(\tau;v,\alpha,\beta)\sin(\omega_n(t-\tau))d\tau + G_n(t;v,\alpha,\beta)\Bigg]\Bigg\}.\end{aligned} \tag{35}$$

Thus, the solution of the problem (1)–(3), (8), (9) reduces to the solution of the system (30), (34), (35) with respect to the unknown functions $v(x,t)$, $\alpha(t)$ and $\beta(t)$.

Lemma 3.1. *If* $\{v(x,t),\alpha(t),\beta(t)\}$ *is any solution to the problem* (1)–(3), (8), (9), *then the functions*

$$v_n(t) = \gamma_n\int_0^1 v(x,t)\cos(\mu_n x)dx, \quad n = 0,1,2,\ldots,$$

satisfy the system given by (28) *and* (29) *in* $C[0,T]$.

Proof. Let $\{v(x,t),\alpha(t),\beta(t)\}$ be any solution of the problem (1)–(3), (8), (9). By multiplying both sides of equation (1) by the function $\gamma_n\cos(\mu_n x)$ for $n = 0,1,2,\ldots$, integrating the result with respect to x from 0 to 1, and using the following relations:

$$\gamma_n\int_0^1 v_{tt}(x,t)\cos(\mu_n x)dx = \frac{d^2}{dt^2}\left(\gamma_n\int_0^1 v(x,t)\cos(\mu_n x)dx\right) = v_n''(t), \quad n = 0,1,2,\ldots,$$

$$\gamma_n\int_0^1 v_{xx}(x,t)\cos(\mu_n x)dx = -\mu_n^2\left(\gamma_n\int_0^1 v(x,t)\cos(\mu_n x)dx\right) = -\mu_n^2 v_n(t), \quad n = 0,1,2,\ldots,$$

$$\gamma_n\int_0^1 v_{ttxx}(x,t)\cos(\mu_n x)dx = -\mu_n^2\left(\gamma_n\int_0^1 v_{tt}(x,t)\cos(\mu_n x)dx\right) = -\mu_n^2 v_n''(t), \quad n = 0,1,2,\ldots,$$

$$\gamma_n\int_0^1 v_{xxxx}(x,t)\cos(\mu_n x)dx = \mu_n^4\left(\gamma_n\int_0^1 v(x,t)\cos(\mu_n x)dx\right) = \mu_n^4 v_n(t), \quad n = 0,1,2,\ldots,$$

$$\gamma_n\int_0^1 v_{ttxxxx}(x,t)\cos(\mu_n x)dx = \mu_n^4\left(\gamma_n\int_0^1 v_{tt}(x,t)\cos(\mu_n x)dx\right) = \mu_n^4 v_n''(t), \quad n = 0,1,2,\ldots,$$

we obtain that equation (26) is satisfied. Similarly, from equation (2), we establish that condition (27) holds. Therefore, the functions $v_n(t)$ for $n = 0,1,2,\ldots$ are solutions of the problem (26) and (27). Hence, it follows directly that the functions $v_n(t)$ for $n = 0,1,2,\ldots$ satisfy on $[0,T]$ the system (28) and (29). Thus, the lemma is proved. □

It follows from Lemma 3.1 that:

Corollary 3.2. *Let the system* (30), (34), (35) *have a unique solution. Then, the problem* (1)–(3), (8), (9) *cannot have more than one solution, i.e. if the problem* (1)–(3), (8), (9) *has a solution, it must be unique.*

To further analyze the problem (1)–(3), (8), (9), we introduce the following functional spaces. Denote by $B_{2,T}^5$ [12] the set of all functions of the form:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)\cos(\mu_n x), \quad \mu_n = n\pi,$$

considered in the region Ω_T, where each $v_n(t)$ for $n = 0, 1, 2, \dots$ is continuous over the interval $[0, T]$ and satisfies the following condition:

$$\|v_0(t)\|_{C[0,T]} + \left\{ \sum_{n=1}^{\infty} \left(\mu_n^5 \|v_n(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty.$$

The norm in this set is defined as:

$$\|v(x,t)\|_{B_{2,T}^5} = \|v_0(t)\|_{C[0,T]} + \left\{ \sum_{n=1}^{\infty} \left(\mu_n^5 \|v_n(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}}.$$

It is known that $B_{2,T}^5$ is a Banach space. Denote by E_T^5 the space $B_{2,T}^5 \times C[0,T] \times C[0,T]$ of vector-functions $w(x,t) = \{v(x,t), \alpha(t), \beta(t)\}$ with the norm:

$$\|w(x,t)\|_{E_T^5} = \|v(x,t)\|_{B_{2,T}^5} + \|\alpha(t)\|_{C[0,T]} + \left\|\beta(t)\right\|_{C[0,T]}.$$

Obviously, E_T^5 is also a Banach space. Now consider the operator:

$$\Psi(v, \alpha, \beta) = \{\Psi_1(v,\alpha,\beta), \Psi_2(v,\alpha,\beta), \Psi_3(v,\alpha,\beta)\}, \tag{36}$$

in the space E_T^5, where

$$\Psi_1(v,\alpha,\beta) = \widetilde{v}(x,t) \equiv \sum_{n=0}^{\infty} \widetilde{v}_n(t)\cos(\mu_n x), \quad \Psi_2(v,\alpha,\beta) = \widetilde{\alpha}(t), \quad \Psi_3(v,\alpha,\beta) = \widetilde{\beta}(t),$$

and the functions $\widetilde{v}_0(t), \widetilde{v}_n(t)$ for $n = 1, 2, \dots$, $\widetilde{\alpha}(t)$ and $\widetilde{\beta}(t)$ are equal to the right-hand sides of equations (28), (29), (34) and (35), respectively.

It is straightforward to observe that:

$$\frac{1}{\sqrt{3}} < \omega_n < \sqrt{2}, \quad \frac{1}{\sqrt{2}} < \frac{1}{\omega_n} < \sqrt{3}.$$

Taking these relations into consideration and using (28), (29), (34) and (35), we have the following estimates:

$$\begin{aligned} \left\|\widetilde{v}_0(t)\right\|_{C[0,T]} \le{}& \left|\eta_0\right| + T\left|\xi_0\right| + T\left(\|Q_1(t)\|_{C[0,T]} + T\|Q_2(t)\|_{C[0,T]}\right)\|v_0(t)\|_{C[0,T]} + T\sqrt{T}\left(\int_0^T |r_0(\tau)|^2\, d\tau\right)^{\frac{1}{2}} \\ &+ T^2\|\alpha(t)\|_{C[0,T]}\|v_0(t)\|_{C[0,T]} + T\sqrt{T}\left\|\beta(t)\right\|_{C[0,T]}\left(\int_0^T |h_0(\tau)|^2\, d\tau\right)^{\frac{1}{2}}, \end{aligned} \tag{37}$$

$$\begin{aligned} \left(\sum_{n=1}^{\infty}\left(\mu_n^5\left\|\widetilde{v}_n(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le{}& \sqrt{7}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\left|\eta_n\right|\right)^2\right)^{\frac{1}{2}} + \sqrt{21}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\left|\xi_n\right|\right)^2\right)^{\frac{1}{2}} \\ &+ \sqrt{7}T\left(\|Q_1(t)\|_{C[0,T]} + \sqrt{3}\|Q_2(t)\|_{C[0,T]}\right)\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &+ \sqrt{21T}\left(\int_0^T \sum_{n=1}^{\infty}\left(\mu_n\left|r_n(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{21}T\|\alpha(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &+ \sqrt{21T}\left\|\beta(t)\right\|_{C[0,T]}\left(\int_0^T \sum_{n=1}^{\infty}\left(\mu_n\left|h_n(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{38}$$

$$\left\|\widetilde{\alpha}(t)\right\|_{C[0,T]} \le \left\|[g(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| \left(g_1''(t) - r(x_1,t)\right) h(x_2,t) - \left(g_2''(t) - r(x_2,t)\right) h(x_1,t) \right\|_{C[0,T]} \right.$$
$$+ \left(\sum_{n=1}^{\infty} \mu_n^{-2} \right)^{\frac{1}{2}} \left\| |h(x_2,t)| + |h(x_1,t)| \right\|_{C[0,T]} \left[\left(2 \sum_{n=1}^{\infty} \left(\mu_n^5 \left|\eta_n\right| \right)^2 \right)^{\frac{1}{2}} \right.$$
$$+ 2T \left\|Q_1(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{2} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left|\xi_n\right| \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2}T \left\|Q_2(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2T} \left(\int_0^T \sum_{n=1}^{\infty} \left(\mu_n \left|r_n(\tau)\right| \right)^2 d\tau \right)^{\frac{1}{2}} + \sqrt{2}T \left\|\alpha(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2T} \left\|\beta(t)\right\|_{C[0,T]} \left(\int_0^T \sum_{n=1}^{\infty} \left(\mu_n \left|h_n(\tau)\right| \right)^2 d\tau \right)^{\frac{1}{2}} + 2 \left(\sum_{n=1}^{\infty} \left(\mu_n \left\|r_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$\left. \left. + 2 \left\|\alpha(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + 2 \left\|\beta(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n \left\|h_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \right] \right\}, \tag{39}$$

$$\left\|\widetilde{\beta}(t)\right\|_{C[0,T]} \le \left\|[g(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| \left(g_2''(t) - r(x_2,t)\right) g_1(t) - \left(g_1''(t) - r(x_1,t)\right) g_2(t) \right\|_{C[0,T]} \right.$$
$$+ \left(\sum_{n=1}^{\infty} \mu_n^{-2} \right)^{\frac{1}{2}} \left\| \left|g_1(t)\right| + \left|g_2(t)\right| \right\|_{C[0,T]} \left[\left(2 \sum_{n=1}^{\infty} \left(\mu_n^5 \left|\eta_n\right| \right)^2 \right)^{\frac{1}{2}} \right.$$
$$+ 2T \left\|Q_1(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{2} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left|\xi_n\right| \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2}T \left\|Q_2(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2T} \left(\int_0^T \sum_{n=1}^{\infty} \left(\mu_n \left|r_n(\tau)\right| \right)^2 d\tau \right)^{\frac{1}{2}} + \sqrt{2}T \left\|\alpha(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$+ \sqrt{2T} \left\|\beta(t)\right\|_{C[0,T]} \left(\int_0^T \sum_{n=1}^{\infty} \left(\mu_n \left|h_n(\tau)\right| \right)^2 d\tau \right)^{\frac{1}{2}} + 2 \left(\sum_{n=1}^{\infty} \left(\mu_n \left\|r_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}$$
$$\left. \left. + 2 \left\|\alpha(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n^5 \left\|v_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + 2 \left\|\beta(t)\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\mu_n \left\|h_n(t)\right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \right] \right\}. \tag{40}$$

Suppose that the data for the problem (1)–(3), (8), (9) satisfy the following assumptions:
(I) $\eta(x) \in C^4[0,1],\ \eta^{(5)}(x) \in L_2(0,1),\ \eta'(0) = \eta'(1) = \eta'''(0) = \eta'''(1) = 0;$
(II) $\xi(x) \in C^4[0,1],\ \xi^{(5)}(x) \in L_2(0,1),\ \xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0;$
(III) $r(x,t) \in C(\Omega_T),\ r_x(x,t) \in L_2(\Omega_T);$
(IV) $h(x,t) \in C(\Omega_T),\ h_x(x,t) \in L_2(\Omega_T);$
(V) $g_j(t) \in C^2[0,T]$ for $j = 1,2,\ g(t) \equiv g_1(t)h(x_2,t) - g_2(t)h(x_1,t) \neq 0,\ 0 \le t \le T.$

Under these assumptions, from (37)–(40), we can deduce the following estimates:

$$\begin{aligned}\left\|\widetilde{v}_0(t)\right\|_{C[0,T]} \le& \left\|\eta(x)\right\|_{L_2(0,1)} + T\left\|\xi(x)\right\|_{L_2(0,1)} + T\sqrt{T}\left\|r(x,t)\right\|_{L_2(\Omega_T)} \\ &+ T\left(\|Q_1(t)\|_{C[0,T]} + T\|Q_2(t)\|_{C[0,T]} + T\|\alpha(t)\|_{C[0,T]}\right)\|v_0(t)\|_{C[0,T]} \\ &+ T\sqrt{T}\left\|\beta(t)\right\|_{C[0,T]}\|h(x,t)\|_{L_2(\Omega_T)},\end{aligned} \tag{41}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\left(\mu_n^5\left\|\widetilde{v}_n(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le \sqrt{7}\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{21}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{21T}\|r_x(x,t)\|_{L_2(\Omega_T)} \\ &+ \sqrt{7}T\left(\|Q_1(t)\|_{C[0,T]} + \sqrt{3}\|Q_2(t)\|_{C[0,T]} + \sqrt{3}\|\alpha(t)\|_{C[0,T]}\right)\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &+ \sqrt{21T}\left\|\beta(t)\right\|_{C[0,T]}\|h_x(x,t)\|_{L_2(\Omega_T)},\end{aligned} \tag{42}$$

$$\begin{aligned}&\left\|\widetilde{\alpha}(t)\right\|_{C[0,T]} \le \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(g_1''(t) - r(x_1,t)\right)h(x_2,t) - \left(g_2''(t) - r(x_2,t)\right)h(x_1,t)\right\|_{C[0,T]}\right. \\ &+\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\||h(x_2,t)| + |h(x_1,t)|\right\|_{C[0,T]}\left[2\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2T}\|r_x(x,t)\|_{L_2(\Omega_T)}\right. \\ &+ 2\left\|\|r_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} + \sqrt{2T}\left\|\beta(t)\right\|_{C[0,T]}\|h_x(x,t)\|_{L_2(\Omega_T)} + 2\left\|\beta(t)\right\|_{C[0,T]}\left\|\|h_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} \\ &+ T\left(2\|Q_1(t)\|_{C[0,T]} + \sqrt{2}\|Q_2(t)\|_{C[0,T]} + \sqrt{2}\|\alpha(t)\|_{C[0,T]}\right)\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\left.\left.+2\|\alpha(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\},\end{aligned} \tag{43}$$

$$\begin{aligned}&\left\|\widetilde{\beta}(t)\right\|_{C[0,T]} \le \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(g_2''(t) - r(x_2,t)\right)g_1(t) - \left(g_1''(t) - r(x_1,t)\right)g_2(t)\right\|_{C[0,T]}\right. \\ &+\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|g_1(t)\right| + \left|g_2(t)\right|\right\|_{C[0,T]}\left[2\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2T}\|r_x(x,t)\|_{L_2(\Omega_T)}\right. \\ &+ 2\left\|\|r_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} + \sqrt{2T}\left\|\beta(t)\right\|_{C[0,T]}\|h_x(x,t)\|_{L_2(\Omega_T)} + 2\left\|\beta(t)\right\|_{C[0,T]}\left\|\|h_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} \\ &+ T\left(2\|Q_1(t)\|_{C[0,T]} + \sqrt{2}\|Q_2(t)\|_{C[0,T]} + \sqrt{2}\|\alpha(t)\|_{C[0,T]}\right)\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\left.\left.+2\|\alpha(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\|v_n(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\}.\end{aligned} \tag{44}$$

It follows from (41) and (42) that:

$$\left\|\widetilde{v}(x,t)\right\|_{B_{2,T}^5} \le E_1(T) + F_1(T)\|\alpha(t)\|_{C[0,T]}\|v(x,t)\|_{B_{2,T}^5} + G_1(T)\|v(x,t)\|_{B_{2,T}^5} + H_1(T)\left\|\beta(t)\right\|_{C[0,T]}, \tag{45}$$

where

$$E_1(T) = \left\|\eta(x)\right\|_{L_2(0,1)} + T\left\|\xi(x)\right\|_{L_2(0,1)} + T\sqrt{T}\left\|r(x,t)\right\|_{L_2(\Omega_T)} + \sqrt{7}\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)}$$
$$+ \sqrt{21}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{21T}\left\|r_x(x,t)\right\|_{L_2(\Omega_T)},$$
$$F_1(T) = T^2 + \sqrt{21}T, \quad G_1(T) = T\left(1+\sqrt{7}\right)\left\|Q_1(t)\right\|_{C[0,T]} + T\left(T+\sqrt{21}\right)\left\|Q_2(t)\right\|_{C[0,T]},$$
$$H_1(T) = T\sqrt{T}\left\|h(x,t)\right\|_{L_2(\Omega_T)} + \sqrt{21T}\left\|h_x(x,t)\right\|_{L_2(\Omega_T)}.$$

Moreover, from (43) and (44), we can also express the following inequalities:

$$\left\|\widetilde{\alpha}(t)\right\|_{C[0,T]} \le E_2(T) + F_2(T)\left\|\alpha(t)\right\|_{C[0,T]}\left\|v(x,t)\right\|_{B^5_{2,T}} + G_2(T)\left\|v(x,t)\right\|_{B^5_{2,T}} + H_2(T)\left\|\beta(t)\right\|_{C[0,T]}, \tag{46}$$

$$\left\|\widetilde{\beta}(t)\right\|_{C[0,T]} \le E_3(T) + F_3(T)\left\|\alpha(t)\right\|_{C[0,T]}\left\|v(x,t)\right\|_{B^5_{2,T}} + G_3(T)\left\|v(x,t)\right\|_{B^5_{2,T}} + H_3(T)\left\|\beta(t)\right\|_{C[0,T]}, \tag{47}$$

where

$$E_2(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(g_1''(t) - r(x_1,t)\right)h(x_2,t) - \left(g_2''(t) - r(x_2,t)\right)h(x_1,t)\right\|_{C[0,T]}\right.$$
$$+\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|h(x_2,t)\right| + \left|h(x_1,t)\right|\right\|_{C[0,T]}\left[2\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)}\right.$$
$$\left.\left. + \sqrt{2T}\left\|r_x(x,t)\right\|_{L_2(\Omega_T)} + 2\left\|\left\|r_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right]\right\},$$

$$F_2(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|h(x_2,t)\right| + \left|h(x_1,t)\right|\right\|_{C[0,T]}\left(\sqrt{2}T + 2\right),$$

$$G_2(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}} T\left\|\left|h(x_2,t)\right| + \left|h(x_1,t)\right|\right\|_{C[0,T]}\left(2\left\|Q_1(t)\right\|_{C[0,T]} + \sqrt{2}\left\|Q_2(t)\right\|_{C[0,T]}\right),$$

$$H_2(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|h(x_2,t)\right| + \left|h(x_1,t)\right|\right\|_{C[0,T]}\left(\sqrt{2T}\left\|h_x(x,t)\right\|_{L_2(\Omega_T)}\right.$$
$$\left. + 2\left\|\left\|h_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right),$$

$$E_3(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(g_2''(t) - r(x_2,t)\right)g_1(t) - \left(g_1''(t) - r(x_1,t)\right)g_2(t)\right\|_{C[0,T]}\right.$$
$$+\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|g_1(t)\right| + \left|g_2(t)\right|\right\|_{C[0,T]}\left[2\left\|\eta^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{2}\left\|\xi^{(5)}(x)\right\|_{L_2(0,1)}\right.$$
$$\left.\left. + \sqrt{2T}\left\|r_x(x,t)\right\|_{L_2(\Omega_T)} + 2\left\|\left\|r_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right]\right\},$$

$$F_3(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\mu_n^{-2}\right)^{\frac{1}{2}}\left\|\left|g_1(t)\right| + \left|g_2(t)\right|\right\|_{C[0,T]}\left(\sqrt{2}T + 2\right),$$

$$G_3(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \mu_n^{-2}\right)^{\frac{1}{2}} T \left\| |g_1(t)| + |g_2(t)| \right\|_{C[0,T]} \left(2 \|Q_1(t)\|_{C[0,T]} + \sqrt{2} \|Q_2(t)\|_{C[0,T]}\right),$$

$$H_3(T) = \left\|[g(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \mu_n^{-2}\right)^{\frac{1}{2}} \left\| |g_1(t)| + |g_2(t)| \right\|_{C[0,T]} \Big(\sqrt{2T} \|h_x(x,t)\|_{L_2(\Omega_T)} + 2 \left\| \|h_x(x,t)\|_{C[0,T]} \right\|_{L_2(0,1)} \Big).$$

From the inequalities (45)–(47), we deduce the following estimate:

$$\left\|\widetilde{v}(x,t)\right\|_{B^5_{2,T}} + \left\|\widetilde{\alpha}(t)\right\|_{C[0,T]} + \left\|\widetilde{\beta}(t)\right\|_{C[0,T]} \le E(T) + F(T) \|\alpha(t)\|_{C[0,T]} \|v(x,t)\|_{B^5_{2,T}} + G(T) \|v(x,t)\|_{B^5_{2,T}} + H(T) \left\|\beta(t)\right\|_{C[0,T]}, \tag{48}$$

where

$$\begin{aligned} &E(T) = E_1(T) + E_2(T) + E_3(T), \quad F(T) = F_1(T) + F_2(T) + F_3(T), \\ &G(T) = G_1(T) + G_2(T) + G_3(T), \quad H(T) = H_1(T) + H_2(T) + H_3(T). \end{aligned} \tag{49}$$

This leads us to the following theorem:

Theorem 3.3. *Assume that* (I)–(V) *hold, along with the inequality:*

$$(F(T)(E(T) + 2) + G(T) + H(T))(E(T) + 2) < 1, \tag{50}$$

then, the problem (1)–(3), (8), (9) *has a unique solution in the ball* $K = K_R\left(\|w\|_{E^5_T} \le R \le E(T) + 2\right)$ *of the space* E^5_T.

Remark 3.4. *Inequality* (50) *holds for sufficiently small values of* $\left\|[g(t)]^{-1}\right\|_{C[0,T]} + T$.

Proof. Consider the following operator equation in the space E^5_T:

$$w = \Psi w, \tag{51}$$

where $w = \{v, \alpha, \beta\}$ and the components $\Psi_j(v, \alpha, \beta)$ for $j = 1, 2, 3$ of the operator $\Psi(v, \alpha, \beta)$ are determined by the right-hand sides of equations (30), (34) and (35). Consider the operator $\Psi(v, \alpha, \beta)$ in the ball $K = K_R$ from E^5_T. By analogy with (48), for any $w_1, w_2, w_3 \in K_R$, the following estimates hold:

$$\begin{aligned} &\|\Psi w\|_{E^5_T} \le E(T) + F(T) \|\alpha(t)\|_{C[0,T]} \|v(x,t)\|_{B^5_{2,T}} + G(T) \|v(x,t)\|_{B^5_{2,T}} + H(T) \left\|\beta(t)\right\|_{C[0,T]} \\ &\le E(T) + F(T)R^2 + G(T)R + H(T)R \le E(T) + (F(T)(E(T) + 2) + G(T) + H(T))(E(T) + 2), \end{aligned} \tag{52}$$

$$\begin{aligned} \|\Psi w_1 - \Psi w_2\|_{E^5_T} \le& F(T)R\Big(\|v_1(x,t) - v_2(x,t)\|_{B^5_{2,T}} + \|\alpha_1(t) - \alpha_2(t)\|_{C[0,T]}\Big) \\ &+ G(T) \|v_1(x,t) - v_2(x,t)\|_{B^5_{2,T}} + H(T) \left\|\beta_1(t) - \beta_2(t)\right\|_{C[0,T]}. \end{aligned} \tag{53}$$

It follows from (50), (52) and (53) that the operator Ψ acts in the ball $K = K_R$, and satisfies the conditions of the contraction mapping principle. Therefore, the operator Ψ has a unique fixed point $w = \{v, \alpha, \beta\}$ in the ball $K = K_R$ which is a solution of equation (51); i.e. a unique solution to the system (30), (34), (35).

Then, the function $v(x,t)$, as an element of space $B^5_{2,T}$, is continuous and has continuous derivatives $v_x(x,t)$, $v_{xx}(x,t)$, $v_{xxx}(x,t)$ and $v_{xxxx}(x,t)$ in the region Ω_T.

Equation (26) provides the estimate:

$$\left(\sum_{n=1}^{\infty}\left(\mu_n^5\left\|v_n''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{2}\left(\sum_{n=1}^{\infty}\left(\mu_n^5\left\|v_n(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \sqrt{2}\left\|\left\|r_x(x,t)+\alpha(t)v_x(x,t)+\beta(t)h_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}. \tag{54}$$

From the last relation, we obtain the continuity of the functions $v_{tt}(x,t)$, $v_{ttx}(x,t)$, $v_{ttxx}(x,t)$, $v_{ttxxx}(x,t)$ and $v_{ttxxxx}(x,t)$ in the region Ω_T.

It is straightforward to verify that equation (1) and the conditions (2), (3), (8), (9) are satisfied in the usual sense. Therefore, $\{v(x,t),\alpha(t),\beta(t)\}$ is a solution of the problem (1)–(3), (8), (9), and, by Lemma 3.1, it is unique in the ball $K = K_R$. Thus, the proof is complete. □

In summary, from Theorem 2.3 and Theorem 3.3, straightforward reasoning implies the unique solvability of the original problem (1)–(5).

Theorem 3.5. *Suppose that all assumptions of Theorem 3.3 hold, along with the conditions:*

$$D \ne 0, \quad \int_0^1 \eta(x)dx = 0, \quad \int_0^1 \xi(x)dx = 0, \quad \int_0^1 h(x,t)dx = 0, \quad \int_0^1 r(x,t)dx = 0,$$

$$g_j(0) = \int_0^T Q_1(t)g_j(t)dt + \eta(x_j), \quad g_j'(0) = \int_0^T Q_2(t)g_j(t)dt + \xi(x_j), \quad j = 1,2,$$

$$\left(T\left\|Q_2(t)\right\|_{C[0,T]} + \left\|Q_1(t)\right\|_{C[0,T]} + \frac{T}{2}\left(E(T)+2\right)\right)T < 1.$$

Then, the problem (1)–(5) *has a unique classical solution in the ball* $K = K_R\left(\|w\|_{E_T^5} \le E(T)+2\right)$ *of the space* E_T^5.

4. Conclusions

This paper has established the solvability of the inverse boundary value problem for a sixth-order Boussinesq equation with unknown time-dependent coefficients. The primary focus has been to determine these unknown coefficients and the solution of the problem. By defining a classical solution and transforming the original problem into an auxiliary equivalent form, the contraction mapping principle has been applied to prove the existence and uniqueness of the solution for the transformed problem. Utilizing the equivalence between the original and auxiliary problems, the existence and uniqueness of a classical solution for the original boundary value problem have been proven. These results offer deeper insights into the mathematical structure of high-order equations and their broader applicability in related fields.

Data availability
No data sets are generated or analysed during the current study.

Declarations
Conflict of interest The authors declare that they have no competing interests.

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