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Corrigendum to "SIRS model for computer viruses propagation with time dependent delay and jumps" [Filomat 39:8 (2025), 2525-2556]

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Abstract. The authors correct part of Theorem 3.1 in the paper titled above [1] in order to avoid using inadequate inequalities and approximations while defining constant C. These corrections do not influence on the formulation of Theorem 3.1 and its conclusion. Also, parts of proof of Theorem 4.2 are corrected in order to avoid using inadequate inequalities. This corrections do not influence on conclusion of theorem. The authors apologize for any inconvenience that may have been caused.

The corrections in paper [1] are as follows:

- On page 2533, in line 14 it should be stated "Function $H : \mathbb{R}^d \times \mathbb{R}^d \times Y \to \mathbb{R}^d$, $Y \subset (0, +\infty)$".
- On page 2533, in the Assumption 2.1 text "for all $t \geq -\tau$" should be replaced with "for all $u \in Y$", while in Assumption 2.3 text "$k_1 \geq 0$ and $0 < k_2 \leq 1$" and "$k_1 \leq \delta'(t) \leq k_2$" should be replaced with "$k_1 > 0$" and "$\delta'(t) \leq 1 - k_1$", respectively.
- On page 2553, in Table 2, range of parameters η_1 and η_2 should be $(-1, 1)$, while range of parameter δ should be $(0, 1]$.
- In the proof of Theorem 3.1 on the pages 2535-2539 we made following changes:

◦ On page 2535, in line 28, expression "$V := V_1 + V_2 : \mathbb{R}^4_+ \to [0, \infty)$" should be replaced with $V := V_1 + V_2$, while at the end of page 2535 function $V_1(t)$ in (12) should be replaced with

$$
\begin{aligned}
V_1(t) &:= V_1(S(t), I_1(t), I_2(t), R(t)) \\
&= (S(t) + a_1 - a_1 \ln(S(t) + 1)) + (I_1(t) + 1 - \ln(I_1(t) + 1)) + (I_2(t) + 1 - \ln(I_2(t) + 1)) \\
&\quad + (R(t) + 1 - \ln R(t)),
\end{aligned}
$$

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for $t \in [-\tau, \tau_k)$. Also, on page 2538, line 7 in equation (19) should be replaced with above mentioned and defined function.

∘ In the first line on the page 2536, "$u + 1 - \ln u > 0$" should be replaced with "$u + 1 - \ln(u+1) > 0$".

∘ On the page 2536 equation (13) should be replaced with

$$\begin{aligned}
LV_1(t) &= \Big(1 - \frac{a_1}{S(t)+1}\Big)\Big(\Lambda - \beta_1 S(t) I_1(t-\delta_1(t)) - \beta_2 S(t) I_2(t-\delta_2(t)) - (\mu+\gamma)S(t) + \delta R(t)\Big)\\
&\quad + \Big(1 - \frac{1}{I_1(t)+1}\Big)\Big(\beta_1 S(t) I_1(t-\delta_1(t)) - \big(\mu+\varepsilon_1+\gamma_1\big)I_1(t)\Big)\\
&\quad + \Big(1 - \frac{1}{I_2(t)+1}\Big)\Big(\beta_2 S(t) I_2(t-\delta_2(t)) - \big(\mu+\varepsilon_2+\gamma_2\big)I_2(t)\Big)\\
&\quad + \Big(1 - \frac{1}{R(t)}\Big)\Big(\gamma S(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu+\delta)R(t)\Big) + \frac{a_1\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(S(t)+1)^2}\\
&\quad + \frac{a_1\sigma_2^2 S^2(t) I_2^2(t-\delta_2(t))}{2(S(t)+1)^2} + \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2} + \frac{\sigma_2^2 S^2(t) I_2^2(t-\delta_2(t))}{2(I_2(t)+1)^2}\\
&\quad + \int_Y \Big[S(t) - \Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-) + a_1\\
&\qquad - a_1 \ln\Big(S(t) - \Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-) + 1\Big)\\
&\qquad + I_1(t) + \eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big) + 1 - \ln\Big(I_1(t) + \eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big) + 1\Big)\\
&\qquad + I_2(t) + \eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big) + 1 - \ln\Big(I_2(t) + \eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big) + 1\Big)\\
&\qquad + R(t) + 1 - \ln R(t)\Big]\lambda(du)\\
&\quad - \int_Y \Big[S(t) + a_1 - a_1\ln(S(t)+1) + I_1(t) + 1 - \ln(I_1(t)+1) + I_2(t) + 1 - \ln(I_2(t)+1)\\
&\qquad + R(t) + 1 - \ln R(t)\Big]\lambda(du)\\
&\quad - \int_Y \Big[- \Big(1 - \frac{a_1}{S(t)+1}\Big)\Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)\\
&\qquad + \Big(1 - \frac{1}{I_1(t)+1}\Big)\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)\\
&\qquad + \Big(1 - \frac{1}{I_2(t)+1}\Big)\eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big)\Big]\lambda(du)\\
&:= L_1(V_1(t)) + L_2(V_1(t)),
\end{aligned}$$

for $t \in [-\tau, \tau_k)$.

∘ The same page, 2536, $L_1(\cdot)$ and $L_2(\cdot)$ should be replaced by following expressions

$$\begin{aligned}
L_1(V_1(t)) &\le \Big(\Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta\Big) - \mu N(t) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t)\\
&\quad - \frac{a_1}{S(t)+1}\Big(\Lambda + \delta R(t)\Big) - \frac{1}{R(t)}\Big(\gamma S(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t)\Big) + a_1\Big(\beta_1 I_1(t-\delta_1(t)) + \beta_2 I_2(t-\delta_2(t))\Big)\\
&\quad - \frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} - \frac{\beta_2 S(t) I_2(t-\delta_2(t))}{I_2(t)+1} + \frac{a_1}{2}\Big(\sigma_1^2 I_1^2(t-\delta_1(t)) + \sigma_2^2 I_2^2(t-\delta_2(t))\Big)\\
&\quad + \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2} + \frac{\sigma_2^2 S^2(t) I_2^2(t-\delta_2(t))}{2(I_2(t)+1)^2}
\end{aligned}$$

and

$$
\begin{aligned}
L_2(V_1(t)) = &-a_1 \int_Y \ln\left(1 - \frac{\left(\eta_1(u)I_1\left((t-\delta_1(t))^-\right) + \eta_2(u)I_2\left((t-\delta_2(t))^-\right)\right)S(t^-)}{S(t)+1}\right)\lambda(du) \\
&- \int_Y \ln\left(1 + \frac{\eta_1(u)S(t^-)I_1\left((t-\delta_1(t))^-\right)}{I_1(t)+1}\right)\lambda(du) - \int_Y \ln\left(1 + \frac{\eta_2(u)S(t^-)I_2\left((t-\delta_2(t))^-\right)}{I_2(t)+1}\right)\lambda(du) \\
&- a_1 \int_Y \frac{\left(\eta_1(u)I_1\left((t-\delta_1(t))^-\right) + \eta_2(u)I_2\left((t-\delta_2(t))^-\right)\right)S(t^-)}{S(t)+1}\lambda(du) \\
&+ \int_Y \frac{\eta_1(u)S(t^-)I_1\left((t-\delta_1(t))^-\right)}{I_1(t)+1}\lambda(du) + \int_Y \frac{\eta_2(u)S(t^-)I_2\left((t-\delta_2(t))^-\right)}{I_2(t)+1}\lambda(du),
\end{aligned}
$$

for $t \in [-\tau, \tau_k)$.

∘ At the beginning of page 2537, instead of (14) it should be

$$
\begin{aligned}
L_1(V_1(t)) \le &\left(\Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta\right) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) \\
&+ a_1\beta_2 I_2(t-\delta_2(t)) + \frac{\sigma_1^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_1^2(t-\delta_1(t)) + \frac{\sigma_2^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_2^2(t-\delta_2(t)),
\end{aligned}
$$

where it is used that $S(t) \le \frac{\Lambda}{\mu}$ and $\frac{1}{I_1(t)+1}, \frac{1}{I_2(t)+1} \le 1$, for $t \in [-\tau, \tau_k)$.

∘ In line 5, page 2537, expressions for x, y and z are replaced with $x = \frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))S(t^-)}{S(t)+1}$, $y = \frac{\eta_1(u)S(t^-)I_1((t-\delta_1(t))^-)}{I_1(t)+1}$ and $z = \frac{\eta_2(u)S(t^-)I_2((t-\delta_2(t))^-)}{I_2(t)+1}$.

∘ On page 2537, (15) should be replaced with

$$
\begin{aligned}
L_2(V_1(t)) \le &\, a_1 \int_Y \frac{\frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))^2S^2(t^-)}{(S(t)+1)^2}}{2\left(1 - \theta\frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))^2S^2(t^-)}{(S(t)+1)^2}\right)}\lambda(du) \\
&+ \int_Y \frac{\frac{\eta_1^2(u)S^2(t^-)I_1^2((t-\delta_1(t))^-)}{(I_1(t)+1)^2}}{2\left(1 - \theta\frac{\eta_1^2(u)S^2(t^-)I_1^2((t-\delta_1(t))^-)}{(I_1(t)+1)^2}\right)}\lambda(du) + \int_Y \frac{\frac{\eta_2^2(u)S^2(t^-)I_2^2((t-\delta_2(t))^-)}{(I_2(t)+1)^2}}{2\left(1 - \theta\frac{\eta_2^2(u)S^2(t^-)I_2^2((t-\delta_2(t))^-)}{(I_2(t)+1)^2}\right)}\lambda(du) \\
\le &\, h^2\frac{\Lambda^2}{\mu^2}\lambda(Y)\left(\frac{2a_1}{1 - 4\theta h^2\frac{\Lambda^2}{\mu^2}} + \frac{1}{1 - \theta h^2\frac{\Lambda^2}{\mu^2}}\right),
\end{aligned}
$$

where we used the fact that $S(t) \le \frac{\Lambda}{\mu}$ and $\frac{1}{I_1(t)+1}, \frac{1}{I_2(t)+1}, \frac{1}{S(t)+1} \le 1$, for $t \in [-\tau, \tau_k)$.

∘ On page 2537, (16) should be replaced with

$$
\begin{aligned}
LV_1(t) \le &\left(\Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta\right) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) \\
&+ a_1\beta_2 I_2(t-\delta_2(t)) + \frac{\sigma_1^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_1^2(t-\delta_1(t)) + \frac{\sigma_2^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_2^2(t-\delta_2(t)) \\
&+ h^2\frac{\Lambda^2}{\mu^2}\lambda(Y)\left(\frac{2a_1}{1 - 4\theta h^2\frac{\Lambda^2}{\mu^2}} + \frac{1}{1 - \theta h^2\frac{\Lambda^2}{\mu^2}}\right) \\
= &\, K_1 - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) + a_1\beta_2 I_2(t-\delta_2(t)) + \frac{\sigma_1^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_1^2(t-\delta_1(t)) \\
&+ \frac{\sigma_2^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right)I_2^2(t-\delta_2(t)),
\end{aligned}
$$

while line 16 on page 2537, where constant K_1 is defined, should be replaced with

$$K_1 := \Lambda + (3 + a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta + h^2 \frac{\Lambda^2}{\mu^2}\lambda(Y)\left(\frac{2a_1}{1 - 4\theta h^2 \frac{\Lambda^2}{\mu^2}} + \frac{1}{1 - \theta h^2 \frac{\Lambda^2}{\mu^2}}\right).$$

∘ On page 2538, first two lines should be replaced with

$$\begin{aligned} L(V_2(t)) &= a_2\Big(I_1(t) - (1 - \delta_1'(t))I_1(t - \delta_1(t))\Big) + a_3\Big(I_2(t) - (1 - \delta_2'(t))I_2(t - \delta_2(t))\Big) \\ &\quad + a_4\Big(I_1^2(t) - (1 - \delta_1'(t))I_1^2(t - \delta_1(t))\Big) + a_5\Big(I_2^2(t) - (1 - \delta_2'(t))I_2^2(t - \delta_2(t))\Big). \end{aligned}$$

On the same page instead of lines 10 and 11 there should be

$$\begin{aligned} LV(t) &= LV_1(t) + LV_2(t) \\ &\le K_1 + I_1(t)\Big(a_2 - \varepsilon_1\Big) + I_2(t)\Big(a_3 - \varepsilon_2\Big) + I_1(t - \delta_1(t))\Big(a_1\beta_1 - a_2k_1\Big) + I_2(t - \delta_2(t))\Big(a_1\beta_2 - a_3k_1\Big) \\ &\quad + a_4I_1^2(t) + a_5I_2^2(t) + I_1^2(t - \delta_1(t))\left(\frac{\sigma_1^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right) - a_4k_1\right) + I_2^2(t - \delta_2(t))\left(\frac{\sigma_2^2}{2}\left(a_1 + \frac{\Lambda^2}{\mu^2}\right) - a_5k_1\right), \end{aligned}$$

while in line 13 constants a_4 and a_5 should be replaced with $a_4 = \frac{\left(k_1 \max\left\{\frac{\varepsilon_1}{\beta_1}, \frac{\varepsilon_2}{\beta_2}\right\} + \frac{\Lambda^2}{\mu^2}\right)\frac{\sigma_1^2}{2}}{k_1}$ and $a_5 = \frac{\left(k_1 \max\left\{\frac{\varepsilon_1}{\beta_1}, \frac{\varepsilon_2}{\beta_2}\right\} + \frac{\Lambda^2}{\mu^2}\right)\frac{\sigma_2^2}{2}}{k_1}$.
∘ In (20) on page 2538, C should be defined as $C := K_1 + \frac{\Lambda^2}{\mu^2}(a_4 + a_5)$, where $I_1(t), I_2(t) \le \frac{\Lambda}{\mu}$, for $t \in [-\tau, \tau_k)$ is used.
∘ On pages 2538, last equality should be replaced with

$$\begin{aligned} dV(t) &= LV(t)dt + \left(1 - \frac{a_1}{S(t) + 1}\right)\Big(- \sigma_1S(t)I_1(t - \delta_1(t))dB_1(t) - \sigma_2S(t)I_2(t - \delta_2(t))dB_2(t)\Big) \\ &\quad + \left(1 - \frac{1}{I_1(t) + 1}\right)\sigma_1S(t)I_1(t - \delta_1(t))dB_1(t) + \left(1 - \frac{1}{I_2(t) + 1}\right)\sigma_2S(t)I_2(t - \delta_2(t))dB_2(t) \\ &+ \int_Y \Big[S(t) - \Big(\eta_1(u)I_1\Big((t - \delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t - \delta_2(t))^-\Big)\Big)S(t^-) + a_1 \\ &\quad - a_1 \ln\Big(S(t) - \Big(\eta_1(u)I_1\Big((t - \delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t - \delta_2(t))^-\Big)\Big)S(t^-) + 1\Big) \\ &\quad + I_1(t) + \eta_1(u)S(t^-)I_1\Big((t - \delta_1(t))^-\Big) + 1 - \ln\Big(I_1(t) + \eta_1(u)S(t^-)I_1\Big((t - \delta_1(t))^-\Big) + 1\Big) \\ &\quad + I_2(t) + \eta_2(u)S(t^-)I_2\Big((t - \delta_2(t))^-\Big) + 1 - \ln\Big(I_2(t) + \eta_2(u)S(t^-)I_2\Big((t - \delta_2(t))^-\Big) + 1\Big) \\ &\quad + R(t) + 1 - \ln R(t)\Big]\tilde{N}(dt, du) \\ &- \int_Y \Big[S(t) + a_1 - a_1 \ln(S(t) + 1) + I_1(t) + 1 - \ln(I_1(t) + 1) + I_2(t) + 1 - \ln(I_2(t) + 1) + R(t) + 1 - \ln R(t)\Big]\tilde{N}(dt, du), \end{aligned}$$

while at the beginning of page 2539, first three lines should be replaced with following expression

$$\begin{aligned} dV(t) &\le LV(t)dt + \left(a_1\sigma_1I_1(t - \delta_1(t)) - \frac{\sigma_1S(t)I_1(t - \delta_1(t))}{I_1(t) + 1}\right)dB_1(t) \\ &\quad + \left(a_1\sigma_2I_2(t - \delta_2(t)) - \frac{\sigma_2S(t)I_2(t - \delta_2(t))}{I_2(t) + 1}\right)dB_2(t) \\ &\quad - a_1 \int_Y \ln\left(1 - \frac{\Big(\eta_1(u)I_1\Big((t - \delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t - \delta_2(t))^-\Big)\Big)S(t^-)}{S(t) + 1}\right)\tilde{N}(dt, du) \\ &\quad - \int_Y \left[\ln\left(1 + \frac{\eta_1(u)S(t^-)I_1\Big((t - \delta_1(t))^-\Big)}{I_1(t) + 1}\right) + \ln\left(1 + \frac{\eta_2(u)S(t^-)I_2\Big((t - \delta_2(t))^-\Big)}{I_2(t) + 1}\right)\right]\tilde{N}(dt, du). \end{aligned}$$

Therefore, inequality (21) on page 2539 should be replaced with

$$\begin{aligned}
dV(t) \leq Cdt &+ \left(a_1\sigma_1 I_1(t-\delta_1(t)) - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}\right)dB_1(t)\\
&+ \left(a_1\sigma_2 I_2(t-\delta_2(t)) - \frac{\sigma_2 S(t) I_2(t-\delta_2(t))}{I_2(t)+1}\right)dB_2(t)\\
&- a_1\int_Y \ln\left(1 - \frac{\Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)}{S(t)+1}\right)\tilde{N}(dt,du)\\
&- \int_Y \left[\ln\left(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\right) + \ln\left(1 + \frac{\eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\right)\right]\tilde{N}(dt,du).
\end{aligned}$$

∘ On page 2539 in lines 13, 14 and 16 expressions "$k + 1 - \ln k$" and "$\frac{1}{k} + 1 - \ln\frac{1}{k}$" should be replaced with "$k + 1 - \ln(k+1)$" and "$\frac{1}{k} + 1 - \ln\left(\frac{1}{k} + 1\right)$", respectively.
The rest of the proof and conclusion stay the same.

- In the proof of Theorem 4.2 on pages 2539-2546 we made following changes:

∘ On page 2540 lines 1, 8 and 15 should be replaced with "If following condition holds", while on the same page Case (1.1) - lines 2-3, Case (2.1) - lines 9-10 and Case (3.1) - lines 16-17 should be erased.
∘ On page 2540, line 20 should be replaced with

$$\begin{aligned}
&{}''\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} < \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\} \quad a.s. \quad i.e.\\
&\mathcal{R}_0^{1,2} := \frac{1}{\min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}}\left(\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2}\right) < 1,''
\end{aligned}$$

∘ On pages 2541-2542 Case (1.1) should be erased.
∘ On page 2542 lines 17-25 should be replaced by following:

1. Applying generalized Itô formula on function $\ln I_1(t)$ it can be obtained

$$\begin{aligned}
d\ln I_1(t) = &\left[\frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)} - \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2I_1^2(t)} - (\mu+\varepsilon_1+\gamma_1)\right]dt\\
&+ \int_Y \left[\ln\left(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\right) - \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\right]\lambda(du)dt\\
&+ \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)}dB_1(t) + \int_Y \ln\left(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\right)\tilde{N}(dt,du).
\end{aligned}$$

Using the fact that function $f(x) = \beta_1 x - \frac{\sigma_1^2}{2}x^2$, for $x = \frac{S(t)I_1(t-\delta_1(t))}{I_1(t)}$, is increasing on $\left[0, \frac{\beta_1}{\sigma_1^2}\right]$ and therefore has maximum in $\frac{\beta_1}{\sigma_1^2}$ which is $\frac{\beta_1^2}{2\sigma_1^2}$ and $\ln(1+z) \leq z$, for any $z > 0$ we get

$$\begin{aligned}
d\ln I_1(t) \leq &\left[\frac{\beta_1^2}{2\sigma_1^2} - (\mu+\varepsilon_1+\gamma_1)\right]dt + \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)}dB_1(t)\\
&+ \int_Y \ln\left(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\right)\tilde{N}(dt,du).
\end{aligned}$$

Integrating both sides of previous inequality from 0 to t and dividing with t leads to

$$\frac{\ln I_1(t) - \ln I_1(0)}{t} \leq \frac{1}{t}\int_0^t \Big[\frac{\beta_1^2}{2\sigma_1^2} - (\mu + \varepsilon_1 + \gamma_1)\Big]ds + \frac{1}{t}M(t) + \frac{1}{t}\bar{M}(t), \tag{1}$$

where

$$M(t) = \int_0^t \frac{\sigma_1 S(s) I_1(s-\delta_1(s))}{I_1(s)} dB_1(s), \quad \bar{M}(t) = \int_0^t \int_Y \ln\Big(1 + \frac{\eta_1(u)S(s^-)I_1\big((s-\delta_1(s))^-\big)}{I_1(s)}\Big)\tilde{N}(ds,du),$$

are continuous martingale and local martingale, respectively, such that $M(0) = 0$ and $\bar{M}(0) = 0$. Due to the fact that $0 < I_1(t) \leq N(t) \leq \frac{\Lambda}{\mu}$, for all $t \geq -\tau$ (which is proven in Theorem 3.1) there exist constant $C_1 > 0$ such that $\frac{I_1(t-\delta_1(t))}{I_1(t)} < C_1$, so

$$[M_1, M_1](t) = \int_0^t \frac{\sigma_1^2 S^2(s) I_1^2(s-\delta_1(s))}{I_1^2(s)} ds \leq \sigma_1^2 C_1^2 \frac{\Lambda^2}{\mu^2} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_1, M_1](t)}{t} \leq \sigma_1^2 C_1^2 \frac{\Lambda^2}{\mu^2} < \infty,$$

$$\langle \bar{M}, \bar{M}\rangle(t) = \int_0^t \int_Y \ln^2\Big(1 + \frac{\eta_1(u)S(s^-)I_1\big((s-\delta_1(s))^-\big)}{I_1(s)}\Big)\lambda(du)ds \leq \ln^2(1 + hC_1)\lambda(Y)t$$
$$\Rightarrow \quad \lim_{t\to\infty} \int_0^t \frac{d\langle \bar{M}, \bar{M}\rangle(s)}{(1+s)^2} \leq \ln^2(1 + hC_1)\lambda(Y) < \infty.$$

Hence, using Lemma 2.4 and Lemma 2.5 leads to

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\bar{M}(t)}{t} = 0 \quad a.s.$$

Taking limes superior of both sides of (1) we can conclude that

$$\limsup_{t\to+\infty} \frac{\ln I_1(t)}{t} \leq \frac{\beta_1^2}{2\sigma_1^2} - (\mu + \varepsilon_1 + \gamma_1).$$

∘ On page 2543, first line should be replaced with "Using condition (24) we obtain", while beginning of second line should be replaced with " $\limsup_{t\to\infty} \frac{I_1(t)}{t} \leq$ ". The rest of proof stays the same.
∘ On page 2543 in line 7, expression "$V(t) := \ln(I_2(t) + 1)$" should be replaced with "$V(t) := \ln I_2(t)$".
∘ On page 2543-2545 Case (3.1) should be erased.
∘ On page 2545 lines 14-27 should be replaced with:

3. Applying generalized Itô formula on function $\ln(I_1(t) + I_2(t))$ it is obtained

$$\begin{aligned} d\ln(I_1(t) + I_2(t)) = \Big[&\frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+I_2(t)} - \frac{(\mu+\varepsilon_1+\gamma_1)I_1(t)}{I_1(t)+I_2(t)} + \frac{\beta_2 S(t) I_2(t-\delta_2(t))}{I_1(t)+I_2(t)} - \frac{(\mu+\varepsilon_2+\gamma_2)I_2(t)}{I_1(t)+I_2(t)} \\ &- \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+I_2(t))^2} - \frac{\sigma_2^2 S^2(t) I_2^2(t-\delta_2(t))}{2(I_1(t)+I_2(t))^2}\Big]dt \\ &+ \int_Y \Big[\ln\Big(1 + \frac{\big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big) + \eta_2(u)I_2\big((t-\delta_2(t))^-\big)\big)S(t^-)}{I_1(t)+I_2(t)}\Big) \\ &\qquad - \frac{\big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big) + \eta_2(u)I_2\big((t-\delta_2(t))^-\big)\big)S(t^-)}{I_1(t)+I_2(t)}\Big]\lambda(du)dt \\ &+ \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+I_2(t)}dB_1(t) + \frac{\sigma_2 S(t) I_2(t-\delta_2(t))}{I_1(t)+I_2(t)}dB_2(t) \\ &+ \int_Y \ln\Big(1 + \frac{\big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big) + \eta_2(u)I_2\big((t-\delta_2(t))^-\big)\big)S(t^-)}{I_1(t)+I_2(t)}\Big)\tilde{N}(dt,du). \end{aligned}$$

Hence,

$$\begin{aligned}
d\ln(I_1(t)+I_2(t)) \le \Big[&\frac{\beta_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)} - \frac{\sigma_1^2 S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+I_2(t))^2} - \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}\\
&+ \frac{\beta_2 S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)} - \frac{\sigma_2^2 S^2(t)I_2^2(t-\delta_2(t))}{2(I_1(t)+I_2(t))^2}\Big]dt\\
&+ \int_Y \Big[\ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)}\Big)\\
&\quad - \frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)}\Big]\lambda(du)dt\\
&+ \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)}dB_1(t) + \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)}dB_2(t)\\
&+ \int_Y \ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)}\Big)\tilde{N}(dt,du).
\end{aligned}$$

Using the same consideration as in the previous part of proof for maximizing functions we obtain

$$\begin{aligned}
d\ln(I_1(t)+I_2(t)) \le \Big[&\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} - \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}\Big]dt\\
&+ \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)}dB_1(t) + \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)}dB_2(t)\\
&+ \int_Y \ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)}\Big)\tilde{N}(dt,du).
\end{aligned}$$

Integrating both sides of previous inequality from 0 to t and dividing with t leads to

$$\begin{aligned}
\frac{\ln(I_1(t)+I_2(t)) - \ln(I_1(0)+I_2(0))}{t} \le &\frac{1}{t}\int_0^t \Big[\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} - \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}\Big]ds\\
&+ \frac{1}{t}\bar{M}_1(t) + \frac{1}{t}\bar{M}_2(t) + \frac{1}{t}\bar{M}_3(t),
\end{aligned} \tag{2}$$

where

$$\bar{M}_i(t) = \int_0^t \frac{\sigma_i S(s)I_i(s-\delta_i(s))}{I_1(s)+I_2(t)}dB_i(s),\ i=1,2$$

are continuous martingales such that $\bar{M}_i(0)=0$, $i=1,2$ and

$$\bar{M}_3(t) = \int_0^t \int_Y \ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((s-\delta_1(s))^-\Big)+\eta_2(u)I_2\Big((s-\delta_2(s))^-\Big)\Big)S(s^-)}{I_1(s)+I_2(s)}\Big)\tilde{N}(ds,du)$$

is local martingale such that $\bar{M}_3(0)=0$. Furthermore, applying Lemma 2.4 and Lemma 2.5 (due to the similar consideration as in the previous part of proof) yields to

$$\lim_{t\to\infty}\frac{\bar{M}_i(t)}{t} = 0,\ i=1,2 \quad \text{and} \quad \lim_{t\to\infty}\frac{\bar{M}_3(t)}{t} = 0 \quad a.s.$$

Taking limes superior of both sides of (2) leads to

$$\limsup_{t\to\infty}\frac{\ln(I_1(t)+I_2(t))}{t} \le \frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} - \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}.$$

◦ On page 2456, first two lines should be replaced with: "Using (28) leads to

$$\limsup_{t\to+\infty} \frac{\ln\Big(I_1(t)+I_2(t)\Big)}{t} \leq \min\{\mu+\varepsilon_1+\gamma_1, \mu+\varepsilon_2+\gamma_2\}(\mathcal{R}_0^{1,2*}-1) < 0 \quad a.s."$$

The rest of the proof stays the same.

- In the proof of Theorem 5.2. on pages 2546-2552 we made following changes:

◦ On page 2546, in line 13 after text "Assumption 2.3" it should be added "for $k_1 \geq 1$".
◦ On page 2547, in line 1 and on page 2549, in line 4 expression "$V := V_1 + V_2 : \mathbb{R}_+^4 \to [0,\infty)$" should be replaced with "$V := V_1 + V_2$".
◦ On page 2547, left side of (55) should be replaced with "$dV_2(t) = \frac{\beta_1\Lambda}{\mu}\Big(I_1(t) - (1-\delta_1'(t))I_1(t-\delta_1(t))\Big)$", while on the page 2550, line 14 should be replaced with
"$dV_2(t) = \frac{\beta_1\Lambda}{\mu}\Big(I_1(t) - (1-\delta_1'(t))I_1(t-\delta_1(t))\Big) + \frac{\beta_2\Lambda}{\mu}\Big(I_2(t) - (1-\delta_2'(t))I_2(t-\delta_2(t))\Big)$".
◦ On page 2548, line 13, instead of expression "$M_1(t)$ and $M_2(t)$ are defined in (32) and (33), respectively" it should be

$$"M_1(t) = \int_0^t \frac{\sigma_1 S(s) I_1(s-\delta_1(s))}{I_1(s)+1} dB_1(s)$$

and

$$M_2(t) = \int_0^t \int_Y \ln\Big(1 + \frac{\eta_1(u)S(s^-)I_1\big((s-\delta_1(s))^-\big)}{I_1(s)+1}\Big)\tilde{N}(ds,du)$$

are continuous martingale and local martingale, respectively, such that $M_1(0) = 0$ and $M_2(0) = 0$. Also,

$$[M_1, M_1](t) = \int_0^t \frac{\sigma_1^2 S^2(s) I_1^2(s-\delta_1(s))}{(I_1(s)+1)^2} ds \leq \frac{\sigma_1^2\Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_1,M_1](t)}{t} \leq \frac{\sigma_1^2\Lambda^4}{\mu^4} < \infty,$$

$$\langle M_2, M_2\rangle(t) = \int_0^t \int_Y \ln^2\Big(1 + \frac{\eta_1(u)S(s^-)I_1\big((s-\delta_1(s))^-\big)}{I_1(s)+1}\Big)\lambda(du)ds \leq \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)\lambda(Y)t$$

$$\Rightarrow \quad \lim_{t\to\infty} \int_0^t \frac{d\langle M_2, M_2\rangle(s)}{(1+s)^2} \leq \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)\lambda(Y) < \infty.$$

Hence, using Lemma 2.4 and Lemma 2.5 leads to

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \quad a.s."$$

◦ On page 2548, in line 18, expression "$-\ln(I_1(t)+1) \geq -\ln\frac{\Lambda}{\mu}$" should be replaced with "$-\ln(I_1(t)+1) \geq -\ln\Big(\frac{\Lambda}{\mu}+1\Big)$", while on page 2552, in line 14, expression "$-\ln(I_1(t)+1), -\ln(I_2(t)+1) \geq -\ln\frac{\Lambda}{\mu}$" should be replaced with "$-\ln(I_1(t)+1), -\ln(I_2(t)+1) \geq -\ln\Big(\frac{\Lambda}{\mu}+1\Big)$".
◦ On page 2552, in line 15 in numerator of fraction, expression "$\varepsilon - 2$" should be replaced with "ε_2".
The rest of the proof stays the same.

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References

[1] M. Đorđević, J. Đorđević, *SIRS model for computer viruses propagation with time dependent delay and jumps*, Filomat **39**(8) (2025), 2525–2556.