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S-Weyl's theorem for bounded linear operators on Banach spaces

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Abstract. In this paper we consider a new variant of the classical Weyl type theorems for operators defined on Banach spaces. This variant, called *S-Weyl's theorem* entails two other variants, a-Weyl's theorem and property (w), studied by different authors in the last two decades. The theory is also exemplified by considering many classes of operators that satisfy it. In particular, the theory is exemplified for Toeplitz operators defined on the Hardy space $H^2(\mathbf{T})$, where $\mathbf{T}$ is the unit circle.

1. Introduction

A classical result by H. Weyl [20] states that for self-adjoint operators on Hilbert spaces the spectral points λ of T for which $\lambda I - T$ is Weyl (i.e., $\lambda I - T$ is Fredholm operator having index 0) are exactly all isolated points of the spectrum that are eigenvalues having finite multiplicity. A similar result has been observed for operators that are non-normal, see Chapter 6 of the monograph [1]. A bounded linear operator $T \in L(X)$ on an infinite dimensional Banach space X, whose spectrum has the structure above mentioned for self-adjoint operators, is said, in the modern literature, to satisfy Weyl's theorem. Weyl's theorem admits several stronger variants, among these a-Weyl theorem and property (w) for many reason seem to be the more important, and have been studied in several papers by different authors. Property (w) and a-Weyl's theorem are independent one each other, but each one of them entails Weyl's theorem.

In order to give a certain order to the intricate variety of variants of Weyl's theorem, we introduce another variant of Weyl's theorem, that we call S-Weyl's theorem, (abbreviation of Strong Weyl's theorem), that implies both a-Weyl theorem and property (w). S-Weyl's theorem is a rather strong property, the same self-adjoint operators satisfy a-Weyl's theorem, as well as property (w), but may fail S-Weyl's theorem. Another example of operator which satisfies both a-Weyl theorem and property (w), but not S-Weyl's theorem, is given by every finite-dimensional operator. Nevertheless, the classes of operators that satisfy S-Weyl' s theorem is considerably large, and deserve to be studied. For instance, Riesz operators having infinite spectrum, and a less trivial examples are provided by Toeplitz operators T_ϕ, where $\phi \in H^\infty(\mathbf{T})$, or with a continuous symbol ϕ on $\mathbf{T}$ for which the orientation of the curve $\phi(\mathbf{T})$ with respect to each hole is counterclockwise. If the orientation of the curve $\phi(\mathbf{T})$ is clockwise then the adjoint of T_ϕ satisfies S-Weyl's. In particular, if ϕ is a trigonometric polynomial on $\mathbf{T}$ then S-Weyl's theorem holds for T_ϕ. Other important examples that satisfy S-Weyl's theorem are given by the dual T^* of a non-invertible symmetry T which is not quasi-nilpotent.

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2. Definitions and preliminary results

Let X be an infinite-dimensional complex Banach space and let $T \in L(X)$ be a bounded linear operator defined on X. By $\alpha(T)$ and $\beta(T)$, the dimension of the kernel $\ker T$ and the codimension of the range $R(T) := T(X)$, respectively. Recall that $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$ if $\beta(T) < \infty$. The class of *Fredholm* operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, while the class of *semi-Fredholm* operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$. If $T \in \Phi_\pm(X)$ then the index is defined by $\text{ind}\,(T) := \alpha(T) - \beta(T)$. The set of *Weyl operators* is defined by $W(X) := \{T \in \Phi(X) : \text{ind}\, T = 0\}$, the class of *upper semi-Weyl operators* is defined by $W_+(X) := \{T \in \Phi_+(X) : \text{ind}\, T \leq 0\}$, and the class of *lower semi-Weyl operators* is defined by $W_-(X) := \{T \in \Phi_-(X) : \text{ind}\, T \geq 0\}$. Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the *Weyl spectrum*, defined by $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$, the *upper semi-Weyl spectrum* $\sigma_{\mathrm{uw}}(T)$ and the *lower semi-Weyl spectrum* $\sigma_{\mathrm{lw}}(T)$, defined similarly.

The *ascent* $p := p(T)$ of an operator T is the smallest non-negative integer p, if it does exist, such that $\ker T^p = \ker T^{p+1}$. Analogously, the *descent* $q := q(T)$ of T is the smallest non-negative integer q, if it does exist, such that $T^q(X) = T^{q+1}(X)$. If $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [19, Proposition 50.2].

The class of all *Browder operators* is defined as the set $B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}$; the class of all *upper semi-Browder operators* is defined $B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}$, and the class of all *lower semi-Browder operators* is defined $B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$ and $B_-(X) \subseteq W_-(X)$.

In the sequel we denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum*, and $\sigma_{\mathrm{s}}(T)$ *surjectivity spectrum*. Evidently, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{s}}(T)$.

An operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, , where the operator $T \in L(X)$ is said to satisfy *a- Brower's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, or equivalently $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}^a(T)$, where $p_{00}^a(T) := \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. It is known that a-Browder's theorem entails Browder's theorem.

Recall that given a compact set $\sigma \subset \mathbb{C}$, a *hole* of σ is a bounded component of the complement $\mathbb{C} \setminus \sigma$. Since $\mathbb{C} \setminus \sigma$ has always an unbounded component, $\mathbb{C} \setminus \sigma$ is connected precisely when σ has no hole.

For a proof of the following result see [6]

Lemma 2.1. *Let $T \in L(X)$. Then*

(i) $\sigma_{\mathrm{uw}}(T)$ *has no hole if and only if* $\sigma_{\mathrm{ap}}(T)$ *has no hole and T satisfies a-Browder's theorem.*

(ii) $\sigma_{\mathrm{lw}}(T)$ *has no hole if and only if* $\sigma_s(T)$ *has no hole and T satisfies a-Browder's theorem.*

Recall that $T \in L(X)$ has the *single valued extension property* at λ_0, abbreviated the SVEP at λ_0, if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of $\sigma(T)$.

Remark 2.2. We have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda \tag{2}$$

If T is semi-Fredholm then $p(\lambda I - T) < \infty \Leftrightarrow T$ has SVEP at λ, and dually, $q(\lambda I - T) < \infty \Leftrightarrow T^*$ has SVEP at λ, see [1, Chapter 2]. Furthermore, if T has SVEP then $\sigma(T) = \sigma_s(T)$, if T^* has SVEP then $\sigma(T) = \sigma_{ap}(T)$, see [1, Theorem 2.68].

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if $p(T) = q(T) < \infty$. Clearly, $\lambda I - T \in L(X)$ is Drazin invertible if and only if $\lambda I - T$ is invertible or λ is a pole of the resolvent. Drazin invertibility for bounded operators suggests the following definition:

The concept of pole of the resolvent suggests the following definition:

Definition 2.3. *An operator $T \in L(X)$, is said to be left Drazin invertible, if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$, is said to be right Drazin invertible, if $q := q(T) < \infty$ and $T^q(X)$ is closed. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\rm ap}(T)$ then λ is said to be a* left pole*. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\rm s}(T)$ then λ is said to be a* right pole*.*

It should be noted that there is a perfect duality, i.e., T (respectively, T^*) is left Drazin invertible if and only if T^* (respectively, T) is right Drazin invertible. Furthermore, $T \in L(X)$ is Drazin invertible if and only if T is both left Drazin invertible and right Drazin invertible.

Denote by $\Pi(T)$ and $\Pi_a(T)$ the set of all poles, the set of left poles of T, respectively. Clearly, $\Pi(T) = \sigma(T) \setminus \sigma_{\rm d}(T)$, $\Pi_a(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm ld}(T)$. Let $\mathrm{iso}\, A$ denote the set of all isolated points of a subset $A \subseteq \mathbb{C}$.

Lemma 2.4. *If $T \in L(X)$ then*

$$\Pi_a(T) \subseteq iso\, \sigma_{\rm ap}(T) \quad and \quad \Pi_s(T) \subseteq iso\, \sigma_{\rm s}(T). \tag{3}$$

Proof. In fact, if $\lambda_0 \in \Pi_a(T)$ then $\lambda I - T$ is left Drazin invertible and hence $p(\lambda_0 I - T) < \infty$. Since $\lambda I - T$ has topological uniform descent (see [18], for definition and details), it then follows, from [18, Corollary 4.8], that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0. An analogous reasoning shows that $\Pi_s(T) \subseteq \mathrm{iso}\, \sigma_{\rm s}(T)$ for all $T \in L(X)$. ∎

The *Drazin spectrum* is defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

The *left Drazin spectrum* $\sigma_{\rm ld}(T)$ and the *right Drazin spectrum* $\sigma_{\rm rd}(T)$ are defined similarly. Evidently, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$.

The next lemma has been proved in [3, Lemma 2.4].

Lemma 2.5. *Let $T \in L(X)$. Then we have*

$$\sigma_{\rm ld}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm a}(T) = \sigma(T).$$

Analogously,

$$\sigma_{\rm rd}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm s}(T) = \sigma(T).$$

Let now consider an analytic function defined on an open disc containing the spectrum $\sigma(T)$). Then $f(T_\phi)$ is defined by the classical functional calculus:

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, \mathrm{d}\lambda,$$

where Γ is a contour that surrounds $\sigma(T)$ in U. The set of all analytic functions defined on an open disc containing the spectrum $\sigma(T)$ will be denoted by $\mathcal{H}(\sigma(T))$. In the sequel we shall need the following simple result.

Lemma 2.6. *Let $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$. Then $iso\, \sigma_{\rm ap}(f(T)) \subseteq f(iso\, \sigma_{\rm ap}(T))$, $iso\, \sigma_{\rm s}(f(T)) \subseteq f(iso\, \sigma_{\rm s}(T))$ and $iso\, \sigma(f(T)) \subseteq f(iso\, \sigma(T))$*

Proof. The proof of the first inclusion may be found in [2]. This proof may be adapted in very simple way to prove the second and the third inclusion. ∎

A bounded operator $T \in L(X)$ is said to be *polaroid* (respectively, *a-polaroid*) if every $\lambda \in \operatorname{iso}\sigma(T)$ (respectively, $\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T)$) is a pole of the resolvent. Every a-polaroid operator is polaroid. T is said *finite-polaroid* if every $\lambda \in$ *iso* $\sigma(T)$ is a pole of finite rank, in other world $\lambda I - T \in B(X)$. The next Lemma has been proved in [7].

Lemma 2.7. *If* $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$ *then* T *is finite-polaroid.*

Proof. Assume that $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$ and $\lambda_0 \in \operatorname{iso}\sigma(T)$. Then there exists $\varepsilon > 0$ for which $\lambda I - T$ is invertible for all $0 < |\lambda - \lambda_0| < \varepsilon$. We have either $\lambda_0 \in \sigma_{\mathrm{uw}}(T)$ or $\lambda_0 \notin \sigma_{\mathrm{uw}}(T)$. If $\lambda_0 \in \sigma_{\mathrm{uw}}(T)$ then $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$, and this is impossible. Hence $\lambda_0 \notin \sigma_{\mathrm{uw}}(T)$, so $\lambda I - T \in W_+(X)$. Since both T and T^* have SVEP at λ_0 we have $p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$, and this implies $\alpha(\lambda_0 I - T) = \beta(\lambda_0 I - T) < \infty$. Therefore $\lambda_0 I - T \in B(X)$, so λ_0 is a pole of finite rank. ∎

3. *S*-Weyl's theorem

Semi-Fredholm operators have been generalized by Berkani ([11] and [12]) in the following way: if $T \in L(X)$ and $n \in \mathbb{N}$ by $T_{[n]}$ we denote the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm*, (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([12]) with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl. lower semi-Weyl). The *B-Weyl spectrum* is defined by $\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}$, the *upper semi B-Weyl spectrum* $\sigma_{\mathrm{ubw}}(T)$ and the *lower semi B-Weyl spectrum* $\sigma_{\mathrm{lbw}}(T)$ are defined similarly. We have, see [1, Chapter 3]:

$$\sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T) \quad \text{and } \sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{d}}(T).$$

Lemma 3.1. *If* $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$ *then* $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ubw}}(T)$.

Proof. The equality

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ubw}}(T) \cup \operatorname{iso}\sigma_{\mathrm{uw}}(T) \tag{4}$$

holds for every operator $T \in L(X)$, see [1, Theorem 3.55]. If $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$ then $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ubw}}(T)$. ∎

Define

$$\Delta_a^g(T) := \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ubw}}(T) \quad \text{and} \quad \Delta_1^g(T) := \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T).$$

Since $\sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$, we have

$$\Pi_a(T) \subseteq \Delta_a^g(T) \subseteq \Delta_1^g(T).$$

Let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, and $p_{00}^a(T) := \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. It is easy to check that $p_{00}(T) \subseteq p_{00}^a(T)$ for all $T \in L(X)$ and, obviously, every point of $p_{00}(T)$ is an isolated point of $\sigma(T)$, and hence of $\sigma_{\mathrm{ap}}(T)$, since $\sigma_{\mathrm{ap}}(T)$ contains the boundary of $\sigma(T)$.

The following property has been introduced in [22] and succesively studied in [3].

Definition 3.2. *An operator* $T \in L(X)$ *is said to verify* property (gaz) *if* $\Delta_1^g(T) = \Pi_a(T)$.

Define

$$\pi_{00}(T) := \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

and

$$\pi^a_{00}(T) := \{\lambda \in \text{iso}\,\sigma_{\text{ap}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

It is easily seen that

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T) \quad \text{and} \quad p^a_{00}(T) \subseteq \pi^a_{00}(T). \tag{5}$$

Property (gaz) entails remarkable spectral equalities:

Theorem 3.3. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T has property (gaz);*

(ii) *a-Browder's theorem holds for T and $\sigma_{\text{ap}}(T) = \sigma(T)$, or equivalently $\sigma_{\text{ld}}(T) = \sigma_{\text{d}}(T)$.*

(iii) *$\sigma(T) \setminus \sigma_{\text{uw}}(T) = p^a_{00}(T)$.*

(iv) *T^* has SVEP at every $\lambda \notin \sigma_{\text{ubw}}(T)$;*

(v) *T^* has SVEP at every $\lambda \notin \sigma_{\text{uw}}(T)$.*

Furthermore, if T has property (gaz) then

$$\sigma_{\text{ubw}}(T) = \sigma_{\text{bw}}(T) = \sigma_{\text{ld}}(T) = \sigma_{\text{d}}(T). \tag{6}$$

and

$$\sigma_{\text{uw}}(T) = \sigma_{\text{w}}(T) = \sigma_{\text{ub}}(T) = \sigma_{\text{b}}(T). \tag{7}$$

Consequently, $\Pi(T) = \Pi_a(T)$.

Proof. The equivalences (i) ⇔ (ii) ⇔ (iii)⇔ (iv) have been shown in [3]. Evidently, (iv) ⇒ (v), since $\sigma_{\text{ubw}}(T) = \sigma_{\text{uw}}(T)$.

(v) ⇒ (iii) Suppose that (v) holds, and let $\lambda \in \sigma(T) \setminus \sigma_{\text{uw}}(T)$. Then T^* has SVEP at λ and, since $\lambda I - T$ is upper semi-Weyl, we have $q(\lambda I - T) < \infty$. This implies, ind $(\lambda I - T) \geq 0$, and hence, since $\lambda I - T \in W_+(X)$ we have ind $(\lambda I - T) = 0$, i.e. $\lambda I - T \in W(X)$. From [1, Theorem 1.22] it then follows that $\lambda \in p_{00}(T) \subseteq p^a_{00}(T)$. On the other hand, the inclusion

$$p^a_{00}(T) = \sigma_{\text{ap}}(T) \setminus \sigma_{\text{ub}}(T) \subseteq \sigma(T) \setminus \sigma_{\text{uw}}(T)$$

is immediate, so the equality (iii) holds.

Also the equalities (6) have been shown in [3], so it remains to prove (7). The equalities $\sigma_{\text{uw}}(T) = \sigma_{\text{ub}}(T)$ and $\sigma_{\text{w}}(T) = \sigma_{\text{b}}(T)$ are true, since T satisfies a-Browder's theorem and hence Browder's theorem. It suffices to prove the inclusion $\sigma_{\text{w}}(T) \subseteq \sigma_{\text{uw}}(T)$. Let $\lambda \notin \sigma_{\text{uw}}(T)$. As above, the SVEP for T^* at λ entails that $\lambda I - T$ is Weyl, hence $\lambda \notin \sigma_{\text{w}}(T)$. ∎

In the sequel we set $\rho_{\text{ap}}(T) := \mathbb{C} \setminus \sigma_{\text{ap}}(T)$ and $\rho_{\text{uw}}(T) := \mathbb{C} \setminus \sigma_{\text{uw}}(T)$.

Theorem 3.4. *Let $T \in L(X)$.*

(i) *If $\sigma_{\text{uw}}(T)$ has no hole then T satisfies property (gaz).*

(ii) *If $\sigma_{\text{lw}}(T)$ has no hole then T^* satisfies property (gaz).*

Proof. (i) By Lemma 2.1, T satisfies a-Browder's theorem, so it suffices, by Theorem 3.3, to prove that $\sigma_{\text{ap}}(T) = \sigma(T)$. Since $\rho_{\text{ap}}(T)$ is connected, by Lemma 2.1, then there is no bounded open connected component of $\rho_{\text{ap}}(T)$. Let $\lambda \notin \sigma_{\text{ap}}(T)$. Then $\lambda \in \rho_{\text{ap}}(T)$, and if Ω is the unique unbounded open connected component of $\rho_{\text{ap}}(T)$, we have

$$\rho(T) \subseteq \Omega \subseteq \rho_{\text{ap}}(T).$$

Now, T^* has SVEP at λ, see [9], so $q(\lambda I - T) < \infty$ by [1, Theorem 2.98], and hence $\beta(\lambda I - T) \leq \alpha(\lambda I - T) = 0$, by [1, Theorem 1.22]. Thus $\lambda \notin \sigma(T)$, and hence $\sigma(T) \subseteq \sigma_{\text{ap}}(T)$. The reverse inclusion is clear.

(ii) This follows from (i), since $\sigma_{\text{uw}}(T^*) = \sigma_{\text{lw}}(T)$. ∎

The classical Weyl type theorems are defined as follows, see [1, Chapter 6] for details.

Definition 3.5. *A bounded operator $T \in L(X)$ is said to satisfy* Weyl's theorem *if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$, $T \in L(X)$ is said to satisfy* a-Weyl's theorem *if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$, $T \in L(X)$ is said to satisfy property (w) if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$.*

The following diagram resume the relationships between Weyl's theorems, a-Browder's theorem and property (w).

$$\begin{array}{ccc} \text{Property } (w) & \Rightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Leftarrow & a\text{-Weyl's theorem} \end{array}$$

(see [1] and [5]). Examples of operators satisfying Weyl's theorem but not property (w) may be found in [5]. Property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [5] for examples. Note that property (w) and a-Weyl's theorem are satisfied by a certain number of Hilbert space operators, see [1, Chapter 6].

We now introduce a stronger variant of Weyl's theorem.

Definition 3.6. *A bounded operator $T \in L(X)$ is said to satisfy the* S-Weyl's theorem, *abbreviated (SW), if $\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \pi_{00}(T)$.*

S-Weyl's theorem may be characterized as follows:

Theorem 3.7. *An operator $T \in L(X)$ satisfies S-Weyl's theorem if and only if T has property (gaz) and $\Pi_a(T) = \pi_{00}(T)$.*

Proof. Suppose that $\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \pi_{00}(T)$. We show first that T satisfies property (gaz). Let $\lambda \notin \sigma_{\mathrm{ubw}}(T)$ arbitrary given. Then there are two possibilities: $\lambda \notin \sigma(T)$ or $\lambda \in \sigma(T)$. If $\lambda \notin \sigma(T) = \sigma(T^*)$ then, trivially, T^* has SVEP at λ. Suppose that $\lambda \in \sigma(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = p_{00}(T)$, so λ is a pole of T and hence an isolated point of $\sigma(T) = \sigma(T^*)$, thus T^* has SVEP at λ. By Theorem 3.3 then T satisfies property (gaz). Therefore, $\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \Pi_a(T)$.

We show now that $\Pi_a(T) = \pi_{00}(T)$. From (6) we know that property (gaz) entails $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ and $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$, so

$$\Pi_a(T) = \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ld}}(T) = \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \pi_{00}(T).$$

Conversely, assume that T satisfies property (gaz) and $\Pi_a(T) = \pi_{00}(T)$. Then

$$\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ld}}(T) = \Pi_a(T) = \pi_{00}(T).$$

■

Since the SVEP for T^* ensures property (gaz) for T, we have:

Corollary 3.8. *Suppose that T^* has SVEP. Then T satisfies S-Weyl's theorem if and only if $\pi_{00}(T) = \Pi_a(T)$.*

Theorem 3.9. *Suppose that T^* has SVEP and $\sigma(T)$ is connected. Then $f(T)$ satisfies S-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. If T^* has SVEP then $f(T^*) = (f(T))^*$ has SVEP for every $f \in \mathcal{H}(\sigma(T))$, see [1, Theorem 2.86], hence $\sigma(f((T)) = \sigma_{\mathrm{ap}}(f((T))$. Since $\sigma(T)$ is connected then $f(\sigma(T)) = \sigma(f(T))$ is connected, hence $\mathrm{iso}\,\sigma(f(T)) = \mathrm{iso}\,\sigma_{\mathrm{ap}}(f(T)) = \emptyset$. This show that $\Pi_a(f(T)) = \pi_{00}(f(T)) = \emptyset$. By Corollary 3.8 then $f(T)$ satisfies S-Weyl's theorem. ■

Recall that if T is upper semi B-Weyl and $\alpha(T) < \infty$ then T is upper semi-Weyl.

Theorem 3.10. *If $T \in L(X)$ satisfies S-Weyl's theorem then T satisfies both a-Weyl's theorem and property (w).*

Proof. Suppose that S-Weyl's theorem holds for T. Then T has property (gaz), by Theorem 3.7, hence $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ and $\pi^a_{00}(T) = \pi_{00}(T)$. Moreover, the property (gaz) is equivalent, by Theorem 3.3, to the equality $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = p^a_{00}(T)$, and since $p^a_{00}(T) \subseteq \pi^a_{00}(T) = \pi_{00}(T)$ we have $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T) = \pi^a_{00}(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$, i.e. $\lambda I - T$ is upper semi B-Weyl. But $\alpha(\lambda I - T) < \infty$, since $\lambda \in \pi_{00}(T)$, so $\lambda I - T$ is upper semi-Weyl, hence $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. This shows that

$$\pi_{00}(T) = \pi^a_{00}(T) = \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T).$$

∎

Later we shall give examples which show that the converse of Theorem 3.10 in general is not true.

Corollary 3.11. *If $T \in L(X)$ satisfies S-Weyl's theorem then*

$$\pi^a_{00}(T) = \pi_{00}(T) = \Pi_a(T) = \Pi(T). \tag{8}$$

Proof. By Theorem 3.7 T satisfies property (gaz), hence $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ and by [3, Lemma 2.7] this is equivalent to saying $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$. Therefore,

$$\Pi_a(T) = \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ld}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T) \quad \text{and} \quad \pi^a_{00}(T) = \pi_{00}(T).$$

From Theorem 3.7 then we have $\Pi_a(T) = \pi_{00}(T)$. ∎

Theorem 3.12. *If iso $\sigma_{\mathrm{ap}}(T) = \emptyset$ and T^* has SVEP, then T satisfies S-Weyl's theorem. Analogously, if iso $\sigma_{\mathrm{s}}(T) = \emptyset$ and T has SVEP, then T^* satisfies S-Weyl's theorem.*

Proof. By Remark 2.2 the SVEP for T^* entails $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. Therefore, iso $\sigma(T) =$ iso $\sigma_{\mathrm{ap}}(T) = \emptyset$, so

$$\pi_{00}(T) \subseteq \operatorname{iso} \sigma(T) = \emptyset,$$

and $\Pi_a(T) \subseteq$ iso $\sigma_{\mathrm{ap}}(T) = \emptyset$. Since T^* satisfies property (gaz) then S-Weyl's theorem holds for T, by Theorem 3.7.

Analogously, the SVEP for T entails property (gaz) for T^*, see [3, Corollary 3.7], and $\sigma(T) = \sigma_{\mathrm{s}}(T)$, by Remark 2.2. Therefore, iso $\sigma(T) =$ iso $\sigma_{\mathrm{s}}(T) = \emptyset$, so

$$\pi_{00}(T^*) \subseteq \operatorname{iso} \sigma(T^*) = \operatorname{iso} \sigma(T) = \emptyset,$$

and

$$\Pi_a(T^*) = \Pi_s(T) \subseteq \operatorname{iso} \sigma_{\mathrm{ap}}(T) = \emptyset.$$

By Theorem 3.7 it then follows that T^* satisfies S-Weyl's theorem. ∎

Corollary 3.13. *Let $T \in L(X)$.*

(i) *If $\sigma_{\mathrm{uw}}(T)$ has no hole and iso $\sigma_{\mathrm{ap}}(T) = \emptyset$ then T satisfies S-Weyl's theorem.*
(ii) *If $\sigma_{\mathrm{lw}}(T)$ has no hole and iso $\sigma_{\mathrm{s}}(T) = \emptyset$ then T^* satisfies S-Weyl's theorem.*

Proof. (i) By Theorem 3.4 T satisfies property (gaz), and hence, by Theorem 3.3, $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. Therefore, iso $\sigma_{\mathrm{ap}}(T) =$ iso $\sigma(T) = \emptyset$, so

$$\Pi_a(T) = \pi_{00}(T) = \emptyset.$$

By Theorem 3.7 then S-Weyl's theorem holds for T.

(i) By Theorem 3.4 T^* satisfies property (gaz), and hence $\sigma_{\mathrm{ap}}(T^*) = \sigma_{\mathrm{s}}(T) = \sigma(T)$. Consequently,

$$\operatorname{iso} \sigma_{\mathrm{ap}}(T^*) = \operatorname{iso} \sigma_{\mathrm{s}}(T) = \operatorname{iso} \sigma(T) = \operatorname{iso} \sigma(T^*) = \emptyset,$$

thus $\Pi_a(T^*) = \pi_{00}(T) = \emptyset$. By Theorem 3.7 then S-Weyl's theorem holds for T^*. ∎

The result of Theorem 3.12 may be, considerably, improved:

Theorem 3.14. *If* $\operatorname{iso}\sigma_{\rm ap}(T) = \emptyset$ *and* T^* *has SVEP, then* $f(T+K)$ *satisfies S-Weyl's theorem for every finite dimensional operator* $K \in L(X)$ *for which* $KT = TK$ *and* $f \in \mathcal{H}(\sigma(T+K))$*. Analogously, if* $\operatorname{iso}\sigma_{\rm s}(T) = \emptyset$ *and* T *has SVEP, then* $f(T^* + K^*)$ *satisfies S-Weyl's theorem.*

Proof. Suppose first that $\operatorname{iso}\sigma_{\rm ap}(T) = \emptyset$ and that T^* has SVEP. By [1, Theorem 3.29] then $\operatorname{iso}\sigma_{\rm ap}(T+K) = \operatorname{iso}\sigma_{\rm ap}(T) = \emptyset$, and from Lemma 2.6 we obtain that $\operatorname{iso}\sigma_{\rm ap}(f(T+K)) \subseteq f(\operatorname{iso}\sigma_{\rm ap}(T+K)) = \emptyset$. Since a finite-dimensional operator is algebraic, then $T^* + K^*$ has SVEP, see [4], and consequently, by [1, Theorem 2.86], $f(T^* + K^*) = [f(T+K)]^*$ has SVEP for every $f \in \mathcal{H}(\sigma(T+K))$. By Theorem 3.12 we conclude that $f(T+K)$ satisfies S-Weyl's theorem.

Analogously, suppose that $\operatorname{iso}\sigma_{\rm ap}(T^*) = \operatorname{iso}\sigma_{\rm s}(T) = \emptyset$ and that T has SVEP. Since K^* is finite-dimensional and $T^*K^* = K^*T^*$, we have, always by [1, Theorem 3.29], that

$$\operatorname{iso}\sigma_{\rm s}(T+K) = \operatorname{iso}\sigma_{\rm ap}(T^+K^*) = \operatorname{iso}\sigma_{\rm ap}(T^*) = \emptyset.$$

Moreover, the SVEP for T is transmitted to $T+K$ and hence to $f(T+K)$. Again by Theorem 3.12 we conclude that $f(T^* + K^*)$ satisfies S-Weyl's theorem. ∎

Theorem 3.15. *If* T^* *has SVEP and* T *is finite-polaroid then* T *satisfies S-Weyl's theorem.*

Proof. By Theorem 3.8 it suffices to prove the equality $\pi_{00}(T) = \Pi_a(T)$. Now, the SVEP for T^* entails property (gaz), hence $\sigma_{\rm ap}(T) = \sigma(T)$ and this is equivalent to saying $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$. Therefore, we have

$$\Pi_a(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm ld}(T) = \sigma(T) \setminus \sigma_{\rm d}(T) = \Pi(T).$$

Let $\lambda \in \Pi(T)$ arbitrary given. Then $0 < \alpha(\lambda I - T) < \infty$, since λ is an eigenvalue of T and T is finite-polaroid. Therefore, $\lambda \in \pi_{00}(T)$, so

$$\Pi_a(T) = \Pi(T) \subseteq \pi_{00}(T).$$

On the other hand, if $\lambda \in \pi_{00}(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$ and the finite-polaroid condition entails that λ is a pole of T having finite rank. But $p(\lambda I - T) = q(\lambda I - T)$ implies, by [1, Theorem 1.22], that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda I - T$ is Browder. Hence

$$\lambda \in \sigma(T) \setminus \sigma_{\rm b}(T) = p_{00}(T) \subseteq p_{00}^a(T) \subseteq \Pi_a(T),$$

from which we conclude that $\pi_{00}(T) = \Pi_a(T)$. ∎

Recall that the spectral mapping theorem holds for $\sigma(T)$ and $\sigma_{\rm ap}(T)$.

Theorem 3.16. *Suppose that* $T \in L(X)$ *commutes with a quasi-nilpotent operator* $Q \in L(X)$*. If* $\sigma(T) = \sigma_{\rm ap}(T)$ *and* $\sigma(T)$ *is connected then* $f(T)$ *satisfies S-Weyl's theorem for every* $f \in \mathcal{H}(\sigma(T))$*.*

Proof. Evidently Q commutes with $(\lambda I - T)^{-1}$ for every $\lambda \notin \sigma(T)$, hence $f(T)Q = Qf(T)$, so $\sigma_{\rm ap}(f(T)) = \sigma_{\rm uw}(f(T))$, by [8, Theorem 3.8]. If $\sigma(T) = \sigma_{\rm ap}(T)$ then

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{\rm ap}(T)) = \sigma_{\rm ap}(f(T)).$$

Consequently, $\sigma(f((T)) = \sigma_{\rm uw}(f(T))$.

Let $\lambda \in \pi_{00}(f(T))$. Then $0 < \alpha(\lambda I - f(T)) < \infty$, and this is impossible, by [8, Theorem 3.1]. Hence $\pi_{00}(f(T)) = \emptyset$. Since $\operatorname{iso}\sigma_{\rm uw}(f(T)) = \emptyset$, by Lemma 3.1 we also have $\sigma_{\rm uw}(f(T)) = \sigma_{\rm ubw}(f(T))$. Hence

$$\pi_{00}(f(T)) = \emptyset = \sigma_{\rm ap}(f(T)) \setminus \sigma_{\rm uw}(f(T)) = \sigma(f(T)) \setminus \sigma_{\rm ubw}(f(T)),$$

so S-Weyl's theorem holds for T, by Theorem 3.7. ∎

The *Volterra operator* V on $L_2[0,1]$ is defined by means

$$(Vf)(x) := \int_0^x f(t)\mathrm{d}t \quad \text{for all } f \in X \quad \text{and } x \in [0,1].$$

The Volterra operator is injective and quasi-nilpotent, so Theorem 3.16 applies to the operators $T \in L(L^2([0,1]))$ that commute with the Volterra operator.

In the sequel we give some examples of operators that satisfy, or do not satisfy, S-Weyl's theorem.

a) Let $Q \in L(X)$ denote a quasi-nilpotent operator for which $\ker Q$ is infinite-dimensional, or Q is injective. Then $\pi_{00}(T) = \emptyset$ and $\sigma_{\rm ap}(Q) = \sigma_{\rm ld}(Q) = \{0\}$. Therefore, $\Pi_a(Q) = \emptyset$. Since Q^* has SVEP then, by Corollary 3.8, Q satisfies S-Weyl's theorem.

b) Let $R \in L(X)$ be a Riesz operator. Recall that R is said *Riesz* if $\lambda I - R \in \Phi(X)$ for every $\lambda \neq 0$, or equivalently, $\lambda I - R \in B(X)$ for every $\lambda \neq 0$. Every Riesz operator has a finite spectrum, or $\sigma(R)$ consists of a sequence of eigenvalues (of finite multiplicity) which clusters at 0. Every Riesz operator with an infinite spectrum satisfies S-Weyl's theorem. Indeed, in this case $\pi_{00}(R) = \Pi_a(R) = \mathbb{C} \setminus \{0\}$. Moreover, R^* has SVEP, so Corollary 3.8 applies.

c) An operator $T \in L(X)$ is said to be *algebraic* if there exists a complex polynomial h such that $h(T) = 0$. Examples of algebraic operators are nilpotent operators and operators K for which K^n is a finite-dimensional operator for some $n \in \mathbb{N}$. Every algebraic operator has a finite spectrum, say $\{\lambda_1, \dots, \lambda_n\}$. Moreover, $\sigma_{\rm ld}(K) = \emptyset$, see [1, Theorem 3.93], $\sigma_{\rm ap}(K) = \sigma(K)$, and hence, $\Pi_a(K) = \{\lambda_1, \dots, \lambda_n\}$. In general an algebraic operator may not satisfy S-Weyl's theorem. To see this, we show that every finite dimensional operator does not satisfy S-Weyl's theorem. Indeed, let K be a finite-dimensional operator on a infinite-dimensional Banach space X. Evidently, $0 \in \sigma(K)$, since $K(X) \neq X$. Hence $0 < p := p(K) = q(K) < \infty$, since 0 is a pole. If were $\alpha(K) < \infty$, from the decomposition $X = \ker K^p \oplus K^p(X)$, see [1, Theorem 1.35], where $\ker K^p$ and $K^p(X)$ are both finite-dimensional, we would have that X is finite-dimensional. Thus, $\alpha(K) = \infty$, and hence $0 \notin \pi_{00}(K)$ while $0 \in \Pi_a(K)$, from which we conclude that S-Weyl's theorem fails for K.

d) A self-adjoint operator on a Hilbert space, satisfy both property (w) and a-Weyl's theorem, but may fail S-Weyl's theorem. For instance, let K be the integral operator defined on $L^2([a,b])$ by

$$(Kx)(s) := \int_a^b p(s,t)x(t)\mathrm{d}t,$$

where the kernel $p(s,t)$ is a polynomial defined on the square $[a,b]\times[a,b]$ such that the Hermiticity condition $p(s,t) = \overline{p(t,s)}$ holds. Then K is self-adjoint and finite-dimensional. From above we know that K does not satisfies S-Weyl's theorem. Note that if kernel is a continuous function which is not a polynomial then S-Weyl's theorem holds for K, since K is a compact operator (and hence a Riesz operator) whose spectrum is infinite.

e) Let $T \in L(X)$, X an infinite-dimensional Banach space, be non-invertible isometry. Denotes by $r(T)$ the spectral radius of T, and set

$$i(T) := \lim_{n\to\infty} k(T^n)^{1/n},$$

where the *lower bound* $k(T)$ is defined by

$$k(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

If $i(T) = r(T)$ then, see [1, Chapter 3], we have $\sigma(T) = \sigma_{\rm w}(T) = \mathbf{D}(0, r(T))$, while $\sigma_{\rm ap}(T) = \partial\mathbf{D}(0, r(T))$. An example of operator for which the equality $i(T) = r(T)$ holds is given by a non-invertible operator, see [1, Chapter 4]. A non-invertible symmetry for which $r(T) > 0$ (i.e., T not quasi-nilpotent) provides another example of operator T which satisfies Weyl's theorem, see [6, Theorem 3.14], while T does not satisfy S-Weyl's theorem, since $\sigma_{\rm ap}(T) \neq \sigma(T)$. However, we have:

Theorem 3.17. *If T is a non-invertible isometry which is not quasi-nilpotent, then T^* satisfies S-Weyl's theorem.*

Proof. Since $\sigma_{\rm ap}(T)$ is the boundary of the spectrum then T has SVEP, by [1, Lemma 2.94], and hence $\sigma_{\rm s}(T) = \sigma(T) = \mathbf{D}(0, r(T))$. Therefore, $\operatorname{iso}\sigma_{\rm s}(T) = \emptyset$. By Theorem 3.12 then T^* satisfies S-Weyl's theorem. ∎

Examples of non-invertible isometries are the *semishifts* , i.e., those isometries for which the *hyperrange*

$$T^{\infty}(X) := \bigcap_{n=1}^{\infty} T^n(X) = \{0\}.$$

Every *right translation operator* on $L^p([0,\infty])$, with $1 \le p < \infty$, as well as every *multiplication operator* T_f on the disc algebra $\mathcal{A}(\mathbb{D})$, is a semishift.

4. Toeplitz operators on Hardy spaces

In this section we shall apply the results of the previous section to Toeplitz operators. Let μ denote the normalized Lebesque measure on $\mathbf{T}$, and denote by $L^2(\mathbf{T})$, $\mathbf{T}$ the unit circle, the classical Hilbert space defined with respect to μ. Let χ_n be the function on $\mathbf{T}$ defined by

$$\chi_n(e^{it}) := e^{int} \quad \text{for all } n \in \mathbb{N}.$$

The set $\{\chi_n\}_{n\in\mathbb{Z}}$ is a orthogonal basis of $L^2(\mathbf{T})$. The *Hardy space* $H^2(\mathbf{T})$ is defined as the closed subspace of all $f \in L^2(\mathbf{T})$ for which

$$\frac{1}{2\pi}\int_0^{2\pi} f\chi_n \mathrm{dt} = 0 \quad \text{for } n = 1, 2, \ldots .$$

The Hilbert space $H^2(\mathbf{T})$ is the closed linear span of the set $\{\chi_n\}_{n\ge 0}$. Moreover, $H^2(\mathbf{T})$ is a closed subspace of $L^{\infty}(\mathbf{T})$. Let $H^{\infty}(\mathbf{T})$ denote the Banach space of all $\phi \in L^{\infty}(\mathbf{T})$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\chi_n \mathrm{d}t = 0 \quad \text{for all } n = 1, 2, \ldots .$$

$H^{\infty}(\mathbf{T})$ is a closed subalgebra of $L^{\infty}(\mathbf{T})$, as well as the closed subalgebra of all continuous functions on $\mathbf{T}$, and $H^{\infty}(\mathbf{T}) = L^{\infty}(\mathbf{T}) \cap H^2(\mathbf{T})$.

Let P denote the projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$. The *Toeplitz operator T_ϕ on $H^2(\mathbf{T})$, with symbol ϕ*, is defined by

$$T_\phi f := P(\phi f) \quad \text{for } f \in H^2(\mathbf{T}).$$

A classical result due to Coburn [13], for a proof see also [1, Chapter 4], shows that if $\phi \in L^{\infty}(\mathbf{T})$ is not almost everywhere 0, then either $\alpha(T_\phi) = 0$ or $\beta(T_\phi) = \alpha(T'_\phi) = 0$. If $\phi \in L^{\infty}(\mathbf{T})$ the spectrum $\sigma(T_\phi)$ and the essential spectrum $\sigma_{\mathrm{e}}(T_\phi)$ are connected, see [15, Chapter 7]. Moreover,

$$\sigma(T_\phi) = \sigma_{\mathrm{w}}(T) \quad \text{and} \quad \sigma_{\mathrm{e}}(T_\phi) = \phi(\mathbf{T}).$$

Furthermore, if $\phi \in L^{\infty}(\mathbf{T})$ is not almost everywhere 0, we have

$$\sigma_{\mathrm{ap}}(T_\phi) = \sigma_{\mathrm{uw}}(T_\phi) \quad \text{and} \quad \sigma_{\mathrm{s}}(T_\phi) = \sigma_{\mathrm{lw}}(T_\phi),$$

see [2, Theorem 3.3]. Recall that the winding number $wn(\phi,\lambda)$ of a closed curve in the plane around a given point λ is an integer representing the total number of times that curve travels counterclockwise around the point. It is known that if the symbol ϕ is continuous on $\mathbf{T}$ and T_ϕ is Fredholm, then

$$\operatorname{ind} T_\phi = -wn(\phi, 0),$$

where $wn(\phi,0)$ denotes the winding number of the curve $\phi(\mathbf{T})$ traced by ϕ with respect to the origin, see Widom [21]. In particular, T_ϕ is Weyl (or equivalently, invertible) if and only if $wn(\phi,0) = 0$ and $\sigma(T_\phi)$ consists of $\phi(\mathbf{T})$ and the points λ of the holes for which $wn(\phi,\lambda) \neq 0$. In [2] it has been proved that if ϕ is continuous then

$$\phi \text{ is nonconstant } \Leftrightarrow \operatorname{iso}\sigma_{\mathrm{w}}(T_\phi) = \emptyset \Leftrightarrow \operatorname{iso}\sigma_{\mathrm{uw}}(T_\phi) = \emptyset. \tag{9}$$

In [2] it is shown that *a*-Weyl's theorem and property (w) hold for Toeplitz operator T_ϕ with continuous symbol. The following counterexample shows that Toeplitz operator may fail *S*-Weyl's theorem.

Example 4.1. Let ϕ be the continuous function on $\mathbf{T}$, defined as

$$\phi(e^{i\theta}) := \begin{cases} -e^{2i\theta} + 1 & \text{if } 0 \le \theta \le \pi, \\ e^{-2i\theta} - 1 & \text{if } \pi \le \theta \le 2\pi. \end{cases}$$

The spectrum $\sigma(T_\phi)$ has two holes Ω_1 and Ω_2, see [1, Example 4.101], where the orientation of $\phi(\mathbf{T})$ traced out by ϕ in Ω_1 is counterclockwise, while the orientation of $\phi(\mathbf{T})$ traced out by ϕ in Ω_2 is clockwise. Moreover,

$$\sigma(T_\phi) = \Omega_1 \cup \Omega_2 \cup \phi(\mathbf{T}) \quad \text{and} \quad \sigma_{\rm ap}(T_\phi) = \Omega_2 \cup \phi(\mathbf{T}).$$

so, $\sigma(T_\phi) \neq \sigma_{\rm ap}(T_\phi)$ and hence S-Weyl's theorem does not hold for T_ϕ, since property (gaz) fails for T_ϕ.

It is known that for a Hilbert space operator T with adjoint T' we have:

$$\sigma_{\rm ap}(T') = \overline{\sigma_{\rm s}(T)} \quad \text{and} \quad \sigma_{\rm s}(T') = \overline{\sigma_{\rm ap}(T)}.$$

Theorem 4.2. *Suppose that $\phi \in H^\infty(\mathbf{T})$ is not almost everywhere 0. Then $f(T'_\phi)$ satisfies S-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T_\phi))$.*

Proof. The operator T_ϕ, $\phi \in H^\infty(\mathbf{T})$, is subnormal and hence hyponormal, see Conway [14, Proposition 2.4.2]. Every hyponormal operator has SVEP, see [1, §4.3], hence $f(T_\phi)$ has SVEP for every $f \in \mathcal{H}(\sigma(T_\phi))$. According Remark 2.2 this implies that $\sigma_{\rm s}(f(T_\phi)) = \sigma(f(T_\phi))$ and hence

$$\sigma_{\rm ap}(f(T'_\phi)) = \sigma_{\rm ap}((f(T_\phi))') = \overline{\sigma_{\rm s}(f(T_\phi))} = \overline{\sigma(f(T_\phi))} = \overline{f(\sigma(T_\phi))}.$$

Since $\sigma(T_\phi)$ is connected also $\overline{f(\sigma(T_\phi))}$ is connected, hence $\sigma_{\rm ap}(f(T'_\phi))$ has no isolated points, from which we deduce that $\Pi_a(f(T'_\phi)) = \emptyset$. On the other hand, also $\sigma(f(T'_\phi)) = \overline{\sigma(f(T_\phi))}$ is connected, so $\pi_{00}(f(T'_\phi)) = \emptyset$. By Theorem 3.7 we conclude that $f(T'_\phi)$ satisfies S-Weyl's theorem. ∎

In general, for symbols $\phi \in L^\infty(\mathbf{T})$, the operator T_ϕ is not hyponormal, also if the symbol ϕ is continuous. For instance, the operator T_ϕ defined in Example 4.1 fails the SVEP, see [7], so it cannot be hyponormal. It is easy to see that for a Hilbert space operator T^* has SVEP if and only if T' has SVEP.

Theorem 4.3. *Let ϕ be a nonconstant continuous on $\mathbf{T}$, and suppose that $\sigma(T_\phi)$ has no holes. Then both T_ϕ and T'_ϕ satisfy S-Weyl's theorem.*

Proof. Since $\sigma(T_\phi)$ consists of $\phi(\mathbf{T})$ and some of its holes, then $\sigma(T_\phi) = \phi(\mathbf{T})$ is the boundary of $\sigma(T_\phi)$, so, both T_ϕ and T'_ϕ (or equivalently, the dual of T_ϕ) have SVEP and hence $\sigma(T_\phi) = \sigma_{\rm ap}(T_\phi) = \sigma_{\rm s}(T_\phi)$, according Remark 2.2. Since $\operatorname{iso}\sigma(T_\phi) = \operatorname{iso}\sigma_{\rm ap}(T_\phi) = \operatorname{iso}\sigma_{\rm s}(T_\phi) = \emptyset$, being $\sigma(T_\phi)$ connected, then S-Weyl's theorem holds for T_ϕ and T'_ϕ by Corollary 3.12. ∎

Write $\rho_{\rm ap}(T) := \mathbb{C} \setminus \sigma_{\rm ap}(T)$ and $\rho_{\rm s}(T) := \mathbb{C} \setminus \sigma_{\rm s}(T)$.

Theorem 4.4. *Suppose that symbol ϕ is continuous and nonconstant.*

(i) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is counterclockwise then T_ϕ satisfies S-Weyl's theorem.*

(ii) *If the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is clockwise then T'_ϕ satisfies S-Weyl's theorem.*

Proof. (i) Suppose that the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is counterclockwise. We prove that T_ϕ satisfies property (gaz). We show first that $\rho(T_\phi)$ is connected. Suppose for this, that $\rho_{\rm uw}(T_\phi) = \rho_{\rm ap}(T_\phi)$ is not connected, i.e. that $\sigma_{\rm uw}(T_\phi)$ admits some holes Ω. If $\lambda \in \Omega$ then $\lambda \notin \phi(\mathbf{T}) = \sigma_{\rm e}(T_\phi)$, so $\lambda I - T_\phi$ is Fredholm and hence $\operatorname{ind}(\lambda I - T_\phi) = -wn(\phi, \lambda) < 0$. This implies that $\lambda I - T_\phi$ is upper semi-Weyl, and hence $\lambda \notin \sigma_{\rm uw}(T_\phi)$, a contradiction. Therefore, $\rho_{\rm uw}(T_\phi)$ is connected. From Theorem 3.4 it then follows that T_ϕ satisfies property (gaz).

To show that S-Weyl's theorem holds for T_ϕ it suffices, by Theorem 3.7, to prove that $\Pi_a(T_\phi) = \pi_{00}(T_\phi)$. We have $\operatorname{iso}\sigma(T_\phi) = \operatorname{iso}\sigma_{\mathrm{w}}(T_\phi) = \emptyset$, by (9), so $\pi_{00}(T) \subseteq \operatorname{iso}(\sigma(T_\phi)) = \emptyset$, and analogously we have $\Pi_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{ap}}(T_\phi) = \operatorname{iso}\sigma_{\mathrm{uw}}(T_\phi) = \emptyset$. Therefore, $\pi_{00}(T) = \Pi_a(T_\phi)$ and hence, by Theorem 3.7, T_ϕ satisfies S-Weyl's theorem.

(ii) Suppose that the orientation of the curve $\phi(\mathbf{T})$ traced out by ϕ is clockwise. We show first that T'_ϕ satisfies property (gaz). Suppose for this, that $\rho_{\mathrm{lw}}(T_\phi) = \rho_{\mathrm{s}}(T_\phi)$ is not connected, i.e., $\sigma_{\mathrm{uw}}(T_\phi)$ admits some holes Ω. If $\lambda \in \Omega$ then $\lambda I - T_\phi$ is Fredholm and hence $\operatorname{ind}(\lambda I - T_\phi) = -wn(\phi, \lambda) > 0$. This implies that $\lambda I - T_\phi$ is lower semi-Weyl, and hence $\lambda \notin \sigma_{\mathrm{lw}}(T_\phi) = \sigma_{\mathrm{s}}(T_\phi)$, a contradiction. Therefore, $\rho_{\mathrm{lw}}(T_\phi)$ is connected and by Theorem 3.4 it then follows that T'_ϕ satisfies property (gaz). From (9) we have $\operatorname{iso}(\sigma(T'_\phi)) = \operatorname{iso}\overline{(\sigma(T_\phi))} = \emptyset$, so $\pi_{00}(T'_\phi) = \emptyset$. Always from (9) we have

$$\operatorname{iso}\sigma_{\mathrm{ap}}(T'_\phi) = \operatorname{iso}\overline{\sigma_{\mathrm{s}}(T_\phi)} = \operatorname{iso}\overline{\sigma_{\mathrm{lw}}(T_\phi)} = \emptyset,$$

and consequently, $\Pi_a(T'_\phi) \subseteq \operatorname{iso}\sigma_{\mathrm{ap}}(T'_\phi) = \emptyset$. By Theorem 3.7 it then follows that S-Weyl's theorem holds for T'_ϕ. ∎

Part (i) of Theorem 4.4 applies in particular to the case where ϕ is a trigonometric polynomial

$$\phi(\mathrm{e}^{i\theta}) =: \Sigma_{k=n}^{-n} a_k \mathrm{e}^{ik\theta},$$

or also in the case that T_ϕ is hyponormal, since these operators have SVEP, and hence the index $\operatorname{ind}(\lambda I - T_\phi)$ on a hole is less or equal to 0. Note that if ϕ is a trigonometric polynomial then T_ϕ may be not hyponormal, see [16].

Corollary 4.5. *If ϕ is a trigonometric polynomial on $\mathbf{T}$ then S-Weyl's theorem holds for T_ϕ.*

A *conjugation operator* on a Hilbert space H is an antilinear operator C for which $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in H$. According Garcia and Putnar [17], an operator $T \in L(H)$ is said to be *complex symmetric* if there exists a conjugation C on H such that $CT = T'C$.

Theorem 4.6. *Let $\phi \in C(\mathbf{T})$ and suppose that T_ϕ is complex symmetric. Then both $f(T_\phi)$ and $f(T'_\phi)$ satisfy S-Weyl's theorem for all $f \in \mathcal{H}(\sigma(T_\phi))$,*

Proof. As observed in [2], both T_ϕ and T'_ϕ have SVEP and hence $f(T_\phi)$ and $f(T'_\phi)$ have SVEP for all $f \in \mathcal{H}(\sigma(T_\phi))$. Furthermore, $\operatorname{iso}\sigma_{\mathrm{ap}}(T_\phi) = \emptyset$ and $\operatorname{iso}\sigma_{\mathrm{s}}(T_\phi) = \emptyset$, since $\sigma_{\mathrm{ap}}(T_\phi) = \sigma_{\mathrm{s}}(T_\phi)$ coincides with the boundary $\partial\sigma(T_\phi) = \phi(\mathbf{T})$, see [2]. By Lemma 2.6 $\operatorname{iso}\sigma_{\mathrm{ap}}(f(T_\phi)) = \operatorname{iso}\sigma_{\mathrm{s}}(f(T_\phi)) = \emptyset$, thus Theorem 3.12 applies. ∎

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