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McKean-Vlasov forward-backward doubly stochastic differential equations and applications to stochastic control

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Abstract. This paper investigates first the existence and uniqueness of solutions for McKean-Vlasov forward-backward doubly stochastic differential equations (MV-FBDSDEs) in infinite-dimensional real separable Hilbert spaces. These equations combine the features of forward-backward doubly stochastic differential equations with the mean-field approach, allowing the coefficients to depend on the solution distribution. We establish the existence and uniqueness of solutions for MV-FBDSDEs using the method of continuation and provide an example and a counterexample to illustrate our findings. Moreover, we extend the practical applicability of our results by employing them within the context of the stochastic maximum principle for a control problem governed by MV-FBDSDEs. This study contributes to the field of stochastic control problems and presents the first analysis of MV-FBDSDEs in infinite-dimensional spaces.

1. Introduction

Pardoux and Peng [20] introduced backward doubly stochastic differential equation (BDSDE) in 1994 to give probabilistic interpretation for the solutions of a class of semilinear stochastic PDEs. Since then, the theory of BDSDEs has developed and found applications in various fields, including stochastic control, stochastic PDEs, and finance.

Motivated by BDSDEs, there has been a growing interest in doubly stochastic optimal control problems (see e.g., [7, 24]). Stochastic Hamilton systems, derived from the stochastic maximum principle of stochastic optimal control problems, fall under the category of forward-backward doubly stochastic differential equations (FBDSDEs). The existence and uniqueness of solutions for these equations, which can be fully coupled, have been studied in various works such as [3–5, 22], along with references therein. Peng and Shi [22] established the existence and uniqueness of FBDSDE solutions under certain monotone assumptions using the method of time continuation. Zhu et al. [26] extended the results of [22] to FBDSDEs in different

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dimensional Euclidean spaces, relaxing the imposed monotonicity assumptions. Additionally, Al-Hussein and Gherbal [5] studied FBDSDEs with Poisson jumps, while Al-Hussein [3] explored FBDSDEs in infinite dimensions.

Mean-field stochastic differential equations (SDEs), also known as SDEs of McKean-Vlasov type, represent another type of SDEs where the coefficients can depend on the distribution of the solution, as shown in [14] and the references therein. In accordance to Lasry and Lions [16] and the related references therein, these equations have been widely used in finance, quantum chemistry, and game theory. Mean-field backward stochastic differential equations, called also BSDEs of McKean-Vlasov type (MV-BSDEs), were introduced by Buckdahn et al. [10] as the mean square limit of an interacting particle system of BSDEs.

It is worth knowing that the stochastic maximum principle approaches to the solutions of optimal control problems for mean field SDEs naturally reduce to the solutions of mean field FBSDE systems; cf. e.g., [9, 12–14, 23]. The existence of solutions for MV-BSDEs and McKean-Vlasov FBSDEs (MV-FBSDEs) has been investigated in various works, including [1, 9, 13, 14, 18, 19], along with relevant references therein. Additionally, the works [2] and [17] provide insights into McKean-Vlasov equations in Hilbert spaces and their applications.

In this paper, we have two main objectives. Firstly, we aim to establish the existence and uniqueness of the solution for the following McKean-Vlasov forward-backward doubly stochastic differential equations (MV-FBDSDEs):

$$\begin{cases} dy_t = f\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) dt + g\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) dW_t - z_t \overleftarrow{dB}_t, \\ dY_t = F\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) dt + G\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) \overleftarrow{dB}_t + Z_t \, dW_t, \\ y_0 = x, \ Y_T = h\left(y_T, \mathbb{P}_{y_T}\right). \end{cases} \tag{1}$$

We consider these equations in infinite dimensional real separable Hilbert spaces. The system (1) incorporates mutually independent cylindrical Wiener processes $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ on real separable Hilbert spaces E_1 and E_2, respectively. The mappings f, g, F, G are allowed to depend on all random variables (y, Y, z, Z) in addition to their distribution $\mathbb{P}_{(y,Y,z,Z)}$, thereby enhancing the generality of the system, besides being fully-coupled.

Secondly, we demonstrate that this work contributes to laying a solid foundation for studying stochastic control problems governed by MV-FBDSDEs. Specifically, in Section 4.1, we apply the results here to the stochastic maximum principle for MV-FBDSDEs. As is well-known, dynamic programming requires the solution to satisfy the Markov property, which does not hold in general due to the presence of distributions in the system (1). Therefore, the maximum principle remains the suitable tool for studying such control problems. To the best of our knowledge, our present work is the first to address MV-FBDSDEs in infinite-dimensional spaces and their applications to stochastic optimal control.

The paper is organized as follows: Section 2 introduces the problem formulation by presenting the MV-FBDSDEs and stating the assumptions on the coefficients. In Section 3, we rigorously establish the existence and uniqueness of the solution for MV-FBDSDEs (1), providing detailed proofs. At the end of Section 3, an illustrative example and a counterexample are given to highlight the implications of our results. Finally, Section 4 demonstrates practical applications of MV-FBDSDEs to stochastic optimal control.

2. Notation and Formulation of the Problem

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed time duration $T > 0$. The class of $\mathbb{P}$-null sets of $\mathcal{F}$ is denoted as $\mathcal{N}$. Let E_1 and E_2 be real and separable Hilbert spaces. We suppose that $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are two mutually independent cylindrical Wiener processes on E_1 and E_2, respectively. For each $t \in [0, T]$, we define the σ-algebra $\mathcal{F}_t := \mathcal{F}^B_{t,T} \vee \mathcal{F}^W_t$, which is generated by $\mathcal{F}^B_{t,T} \cup \mathcal{F}^W_t$. Here, $\mathcal{F}^\theta_{s,t} = \sigma\left(\theta_r - \theta_s, s \leq r \leq t\right) \vee \mathcal{N}$ and $\mathcal{F}^\theta_t = \mathcal{F}^\theta_{0,t}$, for any process θ_t. The collection $(\mathcal{F}_t)_{0\leq t\leq T}$ is neither increasing, nor decreasing, and thus does not form a filtration on $(\Omega, \mathcal{F})$.

We shall investigate systems governed by nonlinear MV-FBDSDEs. These systems are described by the equations, presented in (1). In these equations, the integral with respect to $\overleftarrow{dB}_t$ represents a backward Itô integral, while the integral with respect to dW_s is a standard forward Itô integral. These two types of integrals are particular cases of Itô-Sokorohod integral. Here, for a random variable X in a separable Hilbert space, $\mathbb{P}_X$ denotes the probability measure induced by X. The term "nonlinear" used to describe the system (1) refers not only to the fact to the fact that the coefficients f, g, F, and G could be nonlinear functions of the vector process (y_t, Y_t, z_t, Z_t) at time t, but also to the fact that they depend on its distribution $\mathbb{P}_{(y_t,Y_t,z_t,Z_t)}$.

If S is a separable real Hilbert space with norm $\|\cdot\|$, we denote by $\mathcal{P}(S)$ to the space of all probability measures on $(S, \mathcal{B}(S))$, and by $\mathcal{P}_2(S)$ to the subspace of $\mathcal{P}(S)$ of all probability measures having finite second order moments on S. We endow $\mathcal{P}_2(S)$ with the 2-*Wasserstein distance* as follows:

$$\begin{aligned}\varpi_2(\mu_1,\mu_2) &= \inf\left\{ \left(\int_{S\times S} \|x-y\|^2 \lambda(dx,dy)\right)^{\frac{1}{2}} \,\middle|\, \lambda \in \mathcal{P}(S\times S) \text{ with marginals } \mu_1 \text{ and } \mu_2 \right\} \\ &= \inf\left\{ \left(\mathbb{E}\left[\|X-Y\|^2\right]\right)^{\frac{1}{2}} \,\middle|\, X, Y : \Omega \to S \text{ with } \mathbb{P}_{X_1} = \mu_1 \text{ and } \mathbb{P}_{X_2} = \mu_2 \right\}. \end{aligned} \tag{2}$$

This definition makes $\mathcal{P}_2(S)$ a complete separable metric space. We observe that if X_1 and X_2 are two square integrable random variables taking their values in S, then the following inequality holds:

$$\|\mathbb{E}[X_1] - \mathbb{E}[X_2]\| \leq \varpi_2(\mathbb{P}_{X_1}, \mathbb{P}_{X_2}) \leq \left(\mathbb{E}\left[\|X_1 - X_2\|^2\right]\right)^{\frac{1}{2}}. \tag{3}$$

Let H be a separable real Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and norm $|\cdot|_H$. We denote by $L_2(E_i, H)$ to the space of all Hilbert-Schmidt operators from E_i into H, where $i = 1, 2$. The inner product on $L_2(E_i, H)$ is denoted by $\langle\cdot,\cdot\rangle_{L_2(E_i,H)}$, and the norm induced by this inner product is denoted by $\|\cdot\|_{L_2(E_i,H)}$. For any $v^1 = \left(y^1, Y^1, z^1, Z^1\right)$, $v^2 = \left(y^2, Y^2, z^2, Z^2\right) \in \mathbb{H}^2 := H \times H \times L_2(E_2, H) \times L_2(E_1, H)$, we define

$$\left(v^1, v^2\right) = \left\langle y^1, y^2\right\rangle_H + \left\langle Y^1, Y^2\right\rangle_H + \left\langle z^1, z^2\right\rangle_{L_2(E_2,H)} + \left\langle Z^1, Z^2\right\rangle_{L_2(E_1,H)},$$

and let $\left|v^1\right| = \left[\left(v^1, v^1\right)\right]^{\frac{1}{2}}$ be its norm. Finally, for a separable Hilbert space E, we denote by $\mathfrak{M}^2([0,T], E)$ to the space of all E-valued stochastic processes $(X_t)_{0\leq t\leq T}$ such that for each $t \in [0,T]$, X_t is $\mathcal{F}_t$-measurable, and $\mathbb{E}\left[\int_0^T |X_t|_E^2\, dt\right] < +\infty$. Then it is evident that $\mathfrak{M}^2([0,T], E)$ is a Hilbert space endowed with the canonical norm

$$\|X\| = \left(\mathbb{E}\left[\int_0^T |X_t|_E^2\, dt\right]\right)^{1/2}.$$

Definition 2.1. *A quadruple* $(y, Y, z, Z) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$ *is called a* solution *of MV-FBDSDEs (1) if it satisfies (*$\mathbb{P}$*-almost surely) the following integral systems for each* $t \in [0,T]$*:*

$$\begin{cases} y_t = x + \displaystyle\int_0^t f\left(s, y_s, Y_s, z_s, Z_s, \mathbb{P}_{(y_s,Y_s,z_s,Z_s)}\right) ds + \int_0^t g\left(s, y_s, Y_s, z_s, Z_s, \mathbb{P}_{(y_s,Y_s,z_s,Z_s)}\right) dW_s - \int_0^t z_s\, \overleftarrow{dB}_s, \\ Y_t = h\left(y_T, \mathbb{P}_{y_T}\right) - \displaystyle\int_t^T F\left(s, y_s, Y_s, z_s, Z_s, \mathbb{P}_{(y_s,Y_s,z_s,Z_s)}\right) ds - \int_t^T G\left(s, y_s, Y_s, z_s, Z_s, \mathbb{P}_{(y_s,Y_s,z_s,Z_s)}\right) \overleftarrow{dB}_s - \int_t^T Z_s\, dW_s. \end{cases}$$

For convenience, we introduce the notation:

$$v = (y, Y, z, Z),\ \ A(t, v, \mu) = (F, f, G, g)(t, v, \mu),\ \ (A, v) = \langle F, y\rangle_H + \langle f, Y\rangle_H + \langle G, z\rangle_{L_2(E_2,H)} + \langle g, Z\rangle_{L_2(E_1,H)},$$

where μ is a probability measure on $\mathbb{H}^2$.

We now state our main assumptions on the mappings:

$$(f, F) : \Omega \times [0, T] \times \mathbb{H}^2 \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to H,$$

$$g : \Omega \times [0, T] \times \mathbb{H}^2 \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to L_2(E_1, H),$$

$$G : \Omega \times [0, T] \times \mathbb{H}^2 \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to L_2(E_2, H),$$

$$h : \Omega \times H \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to H.$$

($\mathbf{A}_1$) (**Lipschitz conditions**) There exist $C > 0$ and $\gamma \in (0, \frac{1}{2})$ such that for every $\left(v^i, \mu^i\right) := \left(y^i, Y^i, z^i, Z^i, \mu^i\right)$ in $\mathbb{H}^2 \times \mathcal{P}_2\left(\mathbb{H}^2\right)$, if we denote $v^{i,z} = \left(y^i, Y^i, Z^i\right)$ and $v^{i,Z} = \left(y^i, Y^i, z^i\right)$ for $i = 1, 2$, then the following inequalities hold for each $t \in [0, T]$:

(i) $\left|(f, F)\left(t, v^1, \mu^1\right) - (f, F)\left(t, v^2, \mu^2\right)\right|_H \leq C\left(\left|v^1 - v^2\right| + \varpi_2\left(\mu^1, \mu^2\right)\right)$

(ii) $\left\|G\left(t, v^1, \mu^1\right) - G\left(t, v^2, \mu^2\right)\right\|^2_{L_2(E_2,H)} \leq C\left|v^{1,Z} - v^{2,Z}\right|^2 + \gamma\left(\left\|Z^1 - Z^2\right\|^2_{L_2(E_1,H)} + \varpi_2^2\left(\mu^1, \mu^2\right)\right)$

(iii) $\left\|g\left(t, v^1, \mu^1\right) - g\left(t, v^2, \mu^2\right)\right\|^2_{L_2(E_1,H)} \leq C\left|v^{1,z} - v^{2,z}\right|^2 + \gamma\left(\left\|z^1 - z^2\right\|^2_{L_2(E_2,H)} + \varpi_2^2\left(\mu^1, \mu^2\right)\right)$

(iv) $\left|h\left(y^1, \mu^1\right) - h\left(y^2, \mu^2\right)\right|_H \leq C\left(\left|y^1 - y^2\right|_H + \varpi_2\left(\mu^1, \mu^2\right)\right)$.

($\mathbf{A}_2$) (**Monotonicity conditions**) Assume that there exist non-negative constants θ_1, θ_2, and α_1 with $\theta_1 + \theta_2 > 0$, $\alpha_1 + \theta_2 > 0$ such that for any random variables $v^1 := \left(y^1, Y^1, z^1, Z^1\right)$ and $v^2 := \left(y^2, Y^2, z^2, Z^2\right)$ taking values in $\mathbb{H}^2$ and for any $t \in [0, T]$, we have

$$\text{(i) } \mathbb{E}\left[\left(A\left(t, v^1, \mathbb{P}_{(y^1,Y^1,z^1,Z^1)}\right) - A\left(t, v^2, \mathbb{P}_{(y^2,Y^2,z^2,Z^2)}\right), v^1 - v^2\right)\right]$$
$$\leq -\theta_1\, \mathbb{E}\left[\left|y^1 - y^2\right|^2_H + \left\|z^1 - z^2\right\|^2_{L_2(E_2,H)}\right] - \theta_2\, \mathbb{E}\left[\left|Y^1 - Y^2\right|^2_H + \left\|Z^1 - Z^2\right\|^2_{L_2(E_1,H)}\right],$$

$$\text{(ii) } \mathbb{E}\left[\left\langle h\left(y^1, \mathbb{P}_{y^1}\right) - h\left(y^2, \mathbb{P}_{y^2}\right), y^1 - y^2\right\rangle_H\right] \geq \alpha_1\, \mathbb{E}\left[\left|y^1 - y^2\right|^2_H\right].$$

($\mathbf{A}_3$) For each element $v = (y, Y, z, Z)$ of $\mathbb{H}^2$ and for each $\mu \in \mathcal{P}_2\left(\mathbb{H}^2\right)$, we have $A(\cdot, v, \mu) \in \mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$ and $h(y, \mu) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$.

Condition ($\mathbf{A}_2$) is sometimes referred to as the Lasry–Lions monotonicity condition.

The observation that Wasserstein's distance of two probability measures is bounded below by the Euclidean norm of the difference of their respective expectations, as demonstrated in (3), motivates considering the same research problem under different influences of various Lipschitz constraints.

Remark 2.2. (i) *As a special case, when h does not depend on (y, μ), i.e., $h(y, \mu) = \xi$ for a given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, the two monotonicity conditions imposed in ($\mathbf{A}_2$) collapse to the following condition:*

$$\mathbb{E}\left[\left(A\left(t, v^1, \mathbb{P}_{(y^1,Y^1,z^1,Z^1)}\right) - A\left(t, v^2, \mathbb{P}_{(y^2,Y^2,z^2,Z^2)}\right), v^1 - v^2\right)\right]$$
$$\leq -\theta_1\, \mathbb{E}\left[\left|y^1 - y^2\right|^2_H + \left\|z^1 - z^2\right\|^2_{L_2(E_2,H)}\right] - \theta_2\, \mathbb{E}\left[\left|Y^1 - Y^2\right|^2_H + \left\|Z^1 - Z^2\right\|^2_{L_2(E_1,H)}\right]$$

for some constants $\theta_1 \geq 0$ and $\theta_2 > 0$.

(ii) *The assumption ($\mathbf{A}_2$) can be replaced by the following conditions, while preserving the essential structure of the proofs for the theorems and their corresponding lemmas in the next section.*

$(\mathbf{A}_2)'$: *For all* $v^1 := \left(y^1, Y^1, z^1, Z^1\right), v^2 := \left(y^2, Y^2, z^2, Z^2\right) \in \mathbb{H}^2$, *and for all* $t \in [0,T]$,

$$\mathbb{E}\left[\left(A\left(t, v^1, \mathbb{P}_{(y^1,Y^1,z^1,Z^1)}\right) - A\left(t, v^2, \mathbb{P}_{(y^2,Y^2,z^2,Z^2)}\right), v^1 - v^2\right)\right] \geq \theta_1\, \mathbb{E}\left[\left|y^1 - y^2\right|_H^2 + \left\|z^1 - z^2\right\|_{L_2(E_2,H)}^2\right] + \theta_2\, \mathbb{E}\left[\left|Y^1 - Y^2\right|_H^2 + \left\|Z^1 - Z^2\right\|_{L_2(E_1,H)}^2\right]$$

and

$$\mathbb{E}\left[\left\langle h\left(y^1, \mathbb{P}_{y^1}\right) - h\left(y^2, \mathbb{P}_{y^2}\right), y^1 - y^2\right\rangle_H\right] \leq -\alpha_1\, \mathbb{E}\left[\left|y^1 - y^2\right|_H^2\right].$$

3. Existence and Uniqueness Theorems

In this section, we establish our main result of the existence and uniqueness of the solution to MV-FBDSDEs, which is a system of nonlinear fully coupled FBDSDEs of McKean-Vlasov type.

3.1. Uniqueness of the Solutions of MV-FBDSDEs (1)

The following theorem gives conditions that guarantee the uniqueness of the solution of MV-FBDSDEs (1).

Theorem 3.1. *Under* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$, *MV-FBDSDEs (1) has at most one solution* (y, Y, z, Z) *in* $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$.

Let us begin by introducing the integration by parts formula, commonly referred to as Itô's formula. This formula is derived from the classical Itô's formula, as it can be gleaned, for instance, from [20].

Proposition 3.2. *Let* $(\alpha, \beta, \gamma, \delta)$ *and* $\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\right)$ *be elements of* $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$ *and* $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$, *respectively. Assume that*

$$\begin{cases} \alpha_t = \alpha_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, \overleftarrow{dB}_s + \int_0^t \gamma_s\, dW_s, \\ \tilde{\alpha}_t = \tilde{\alpha}_0 + \int_0^t \tilde{\beta}_s\, ds + \int_0^t \tilde{\delta}_s\, \overleftarrow{dB}_s + \int_0^t \tilde{\gamma}_s\, dW_s, \end{cases}$$

for all $t \in [0,T]$. *Then, for each* $t \in [0,T]$,

$$\langle \alpha_t, \tilde{\alpha}_t \rangle_H = \langle \alpha_0, \tilde{\alpha}_0 \rangle_H + \int_0^t \langle \alpha_s, d\tilde{\alpha}_s \rangle_H + \int_0^t \langle \tilde{\alpha}_s, d\alpha_s \rangle_H + \int_0^t d\,\langle \alpha_s, \tilde{\alpha}_s \rangle_H \quad \mathbb{P} - a.s.,$$

and

$$\mathbb{E}\left[\langle \alpha_t, \tilde{\alpha}_t \rangle_H\right] = \mathbb{E}\left[\langle \alpha_0, \tilde{\alpha}_0 \rangle_H\right] + \mathbb{E}\left[\int_0^t \langle \alpha_s, d\tilde{\alpha}_s \rangle_H\right] + \mathbb{E}\left[\int_0^t \langle \tilde{\alpha}_s, d\alpha_s \rangle_H\right] - \mathbb{E}\left[\int_0^t \langle\, \delta_s, \tilde{\delta}_s\, \rangle_{L_2(E_2,H)}\, ds\right] + \mathbb{E}\left[\int_0^t \langle \gamma_s, \tilde{\gamma}_s \rangle_{L_2(E_1,H)}\, ds\right].$$

Proof of Theorem 3.1. Let $v^i = \left(y^i, Y^i, z^i, Z^i\right)$, for $i = 1, 2$, be two solutions of system (1). To simplify the notation, we denote

$$\Delta v = (\Delta y, \Delta Y, \Delta z, \Delta Z) = \left(y^1 - y^2, Y^1 - Y^2, z^1 - z^2, Z^1 - Z^2\right),$$
$$\Delta f_t = f\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1,Y_t^1,z_t^1,Z_t^1)}\right) - f\left(t, y_t^2, Y_t^2, z_t^2, Z_t^2, \mathbb{P}_{(y_t^2,Y_t^2,z_t^2,Z_t^2)}\right),$$
$$\Delta g_t = g\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1,Y_t^1,z_t^1,Z_t^1)}\right) - g\left(t, y_t^2, Y_t^2, z_t^2, Z_t^2, \mathbb{P}_{(y_t^2,Y_t^2,z_t^2,Z_t^2)}\right),$$
$$\Delta F_t = F\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1,Y_t^1,z_t^1,Z_t^1)}\right) - F\left(t, y_t^2, Y_t^2, z_t^2, Z_t^2, \mathbb{P}_{(y_t^2,Y_t^2,z_t^2,Z_t^2)}\right),$$
$$\Delta G_t = G\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1,Y_t^1,z_t^1,Z_t^1)}\right) - G\left(t, y_t^2, Y_t^2, z_t^2, Z_t^2, \mathbb{P}_{(y_t^2,Y_t^2,z_t^2,Z_t^2)}\right),$$
$$\Delta h_T = h\left(y_T^1, \mathbb{P}_{y_T^1}\right) - h\left(y_T^2, \mathbb{P}_{y_T^2}\right),$$

where $0 \leq t \leq T$.

By applying Itô's formula (see Proposition 3.2) and $(\mathbf{A}_2)$ (i) to $\langle \Delta y_t, \Delta Y_t \rangle_H$, we obtain

$$\begin{aligned}\mathbb{E}\left[\langle \Delta y_T, \Delta h_T \rangle_H\right] &= \mathbb{E}\left[\int_0^T \left(A\left(t, v_t^1, \mathbb{P}_{(y_t^1, Y_t^1, z_t^1, Z_t^1)}\right) - A\left(t, v_t^2, \mathbb{P}_{(y_t^2, Y_t^2, z_t^2, Z_t^2)}\right), \Delta v_t\right) dt\right] \\ &\leq -\theta_1 \, \mathbb{E}\left[\int_0^T \left(\left|y_t^1 - y_t^2\right|_H^2 + \left\|z_t^1 - z_t^2\right\|_{L_2(E_2,H)}^2 \right) dt\right] - \theta_2 \, \mathbb{E}\left[\int_0^T \left(\left|Y_t^1 - Y_t^2\right|_H^2 + \left\|Z_t^1 - Z_t^2\right\|_{L_2(E_1,H)}^2 \right) dt\right].\end{aligned}$$

Hence, according to $(\mathbf{A}_2)$ (ii), it follows that

$$\begin{aligned}0 &\leq \alpha_1 \, \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] \\ &\leq -\theta_1 \, \mathbb{E}\left[\int_0^T \left(\left|y_t^1 - y_t^2\right|_H^2 + \left\|z_t^1 - z_t^2\right\|_{L_2(E_2,H)}^2\right) dt\right] - \theta_2 \, \mathbb{E}\left[\int_0^T \left(\left|Y_t^1 - Y_t^2\right|_H^2 + \left\|Z_t^1 - Z_t^2\right\|_{L_2(E_1,H)}^2\right) dt\right].\end{aligned}$$

If both $\theta_1 > 0$ and $\theta_2 > 0$ (e.g., when $\theta_1 = \theta_2$), this inequality directly proves the uniqueness of (y, Y, z, Z), so that we would not need to assume that $0 < \gamma < 1/2$ for the purpose of establishing the uniqueness of the solutions of MV-FBDSDE (1). Therefore, let us consider the general case in the remaining part of the proof.

If $\theta_2 > 0$, we obtain

$$\mathbb{E}\left[\left|Y_t^1 - Y_t^2\right|_H^2\right] = 0 \quad \text{and} \quad \mathbb{E}\left[\left\|Z_t^1 - Z_t^2\right\|_{L_2(E_1,H)}^2\right] = 0,$$

which imply that $Y_t^1 = Y_t^2$ and $Z_t^1 = Z_t^2$ *a.s.* for all $0 \leq t \leq T$. Hence, according to (1), we have

$$\Delta y_t = \int_0^t \Delta\widehat{f_s} \, ds + \int_0^t \Delta\widehat{g_s} \, dW_s - \int_0^t \Delta z_s \, \overleftarrow{dB}_s, \quad t \in [0, T],$$

where $\Delta\widehat{f_t} = \Delta f_t$ and $\Delta\widehat{g_t} = \Delta g_t$ in this case. Specifically,

$$\Delta\widehat{f_t} = f\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1, Y_t^1, z_t^1, Z_t^1)}\right) - f\left(t, y_t^2, Y_t^1, z_t^2, Z_t^1, \mathbb{P}_{(y_t^2, Y_t^1, z_t^2, Z_t^1)}\right)$$

and

$$\Delta\widehat{g_t} = g\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1, Y_t^1, z_t^1, Z_t^1)}\right) - g\left(t, y_t^2, Y_t^1, z_t^2, Z_t^1, \mathbb{P}_{(y_t^2, Y_t^1, z_t^2, Z_t^1)}\right).$$

Applying Itô's formula to $\left|\Delta y_t\right|_H^2$ yields

$$\begin{aligned}\mathbb{E}\left[\left|\Delta y_t\right|_H^2\right] + \mathbb{E}\left[\int_0^t \|\Delta z_s\|_{L_2(E_2,H)}^2 \, ds\right] &= 2 \, \mathbb{E}\left[\int_0^t \langle \Delta\widehat{f_s}, \Delta y_s \rangle_H \, ds\right] + \mathbb{E}\left[\int_0^t \left\|\Delta\widehat{g_s}\right\|_{L_2(E_1,H)}^2 ds\right] \\ &\leq 2 \, \mathbb{E}\left[\int_0^t |\Delta\widehat{f_s}|_H \left|\Delta y_s\right|_H ds\right] + \mathbb{E}\left[\int_0^t \left\|\Delta\widehat{g_s}\right\|_{L_2(E_1,H)}^2 ds\right].\end{aligned}$$

Hence, based on ($\mathbf{A}_1$) and the inequality $ab \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$, for any $\varepsilon > 0$, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left|\Delta y_t\right|_H^2\right] + \mathbb{E}\left[\int_0^t \|\Delta z_s\|^2_{L_2(E_2,H)}\,ds\right] &\leq 2C\int_0^t \mathbb{E}\Bigg[\Bigg(\left|\Delta y_s\right|_H^2 + \|\Delta z_s\|_{L_2(E_2,H)}\left|\Delta y_s\right|_H \\
&\quad + \left|\Delta y_s\right|_H\left(\mathbb{E}\left[\left|\Delta y_s\right|_H^2\right]\right)^{\frac{1}{2}} + \left|\Delta y_s\right|_H\left(\mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]\right)^{\frac{1}{2}}\Bigg)ds\Bigg] \\
&\quad + \int_0^t\left((C+\gamma)\,\mathbb{E}\left[\left|\Delta y_s\right|_H^2\right] + 2\,\gamma\,\mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]\right)ds \\
&\leq 2C\int_0^t\left(\left(2+\frac{1}{2\,\varepsilon}+\frac{1}{2\,\varepsilon}\right)\mathbb{E}\left[\left|\Delta y_s\right|_H^2\right] + \left(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\right)\mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]\right)ds \\
&\quad + \int_0^t\left((C+\gamma)\,\mathbb{E}\left[\left|\Delta y_s\right|_H^2\right] + 2\gamma\,\mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]\right)ds \\
&\leq \int_0^t\left(\left(5C+\gamma+\frac{2C}{\varepsilon}\right)\mathbb{E}\left[\left|\Delta y_s\right|_H^2\right] + (2\gamma + 2C\,\varepsilon)\,\mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]\right)ds.
\end{aligned}$$

Since $0 < \gamma < 1/2$, if we choose in this case $\varepsilon = \frac{1-2\gamma}{4C}$, we get

$$\mathbb{E}\left[\left|\Delta y_t\right|_H^2\right] + \left(\frac{1-2\gamma}{2}\right)\int_0^t \mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]ds \leq \left(5C+\gamma+\frac{8C^2}{1-2\gamma}\right)\int_0^t \mathbb{E}\left[\left|\Delta y_s\right|_H^2\right]ds.$$

Now, Gronwall's inequality implies $y_t^1 = y_t^2$ *a.s.* for all $0 \leq t \leq T$. This leads to $\int_0^t \mathbb{E}\left[\|\Delta z_s\|^2_{L_2(E_2,H)}\right]ds = 0$, which yields $z_t^1 = z_t^2$ *a.s.* for all $0 \leq t \leq T$.

If $\alpha_1 > 0$ and $\theta_1 > 0$, we have $\mathbb{E}\left[\left|y_t^1 - y_t^2\right|_H^2\right] = 0$, $\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] = 0$, and $\mathbb{E}\left[\left\|z_t^1 - z_t^2\right\|^2_{L_2(E_2,H)}\right] = 0$. Consequently, $y_t^1 = y_t^2$ and $z_t^1 = z_t^2$ *a.s.* for all $0 \leq t \leq T$. Therefore, $h\left(y_T^1, P_{y_T^1}\right) = h\left(y_T^2, P_{y_T^2}\right)$, and so

$$\Delta Y_t = -\int_t^T \Delta\widehat{F}_s\,ds - \int_t^T \Delta\widehat{G}_s\,\overleftarrow{dB}_s - \int_t^T \Delta Z_s\,dW_s,$$

where here

$$\begin{aligned}
\Delta\widehat{F}_t &= F\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1, Y_t^1, z_t^1, Z_t^1)}\right) - F\left(t, y_t^1, Y_t^2, z_t^1, Z_t^2, \mathbb{P}_{(y_t^1, Y_t^2, z_t^1, Z_t^2)}\right), \\
\Delta\widehat{G}_t &= G\left(t, y_t^1, Y_t^1, z_t^1, Z_t^1, \mathbb{P}_{(y_t^1, Y_t^1, z_t^1, Z_t^1)}\right) - G\left(t, y_t^1, Y_t^2, z_t^1, Z_t^2, \mathbb{P}_{(y_t^1, Y_t^2, z_t^1, Z_t^2)}\right).
\end{aligned}$$

Next, apply Itô's formula to $|\Delta Y_t|_H^2$ and utilize ($\mathbf{A}_1$) to find that

$$\begin{aligned}
\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\,ds\right] &\leq 2\,\mathbb{E}\left[\int_t^T |\Delta\widehat{F}_s|_H\,|\Delta Y_s|_H\,ds\right] + \mathbb{E}\left[\int_t^T \|\Delta\widehat{G}_s\|^2_{L_2(E_2,H)}\,ds\right] \\
&\leq 2\,C\,\mathbb{E}\left[\int_t^T\left(|\Delta Y_s|_H + \|\Delta Z_s\|_{L_2(E_1,H)}\right)|\Delta Y_s|_H\,ds\right] \\
&\quad + 2\,C\,\mathbb{E}\left[\int_t^T\left(\left(\mathbb{E}\left[|\Delta Y_s|_H^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\|\Delta Z_s\|^2_{L_2(E_1,H)}\right]\right)^{\frac{1}{2}}\right)|\Delta Y_s|_H\,ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T\left(C\,|\Delta Y_s|_H^2 + \gamma\,\mathbb{E}\left[|\Delta Y_s|_H^2\right]\right)ds\right] + \mathbb{E}\left[\int_t^T \gamma\left(\|\Delta Z_s\|^2_{L_2(E_1,H)} + \mathbb{E}\left[\|\Delta Z_s\|^2_{L_2(E_1,H)}\right]\right)ds\right].
\end{aligned} \tag{4}$$

For any $\varepsilon > 0$, we then observe

$$\begin{aligned}
&\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] \\
&\leq 2C\int_t^T \mathbb{E}\left[\left(|\Delta Y_s|_H^2 + \|\Delta Z_s\|_{L_2(E_1,H)}\, |\Delta Y_s|_H \;+\; |\Delta Y_s|_H \left(\mathbb{E}\left[|\Delta Y_s|_H^2\right]\right)^{\frac{1}{2}} + |\Delta Y_s|_H \left(\mathbb{E}\left[\, \|\Delta Z_s\|^2_{L_2(E_1,H)}\,\right]\right)^{\frac{1}{2}}\right)\right] ds \\
&\qquad\qquad + \int_t^T \left((C+\gamma)\; \mathbb{E}\left[|\Delta Y_s|_H^2\right] + 2\gamma\, \mathbb{E}\left[\, \|\Delta Z_s\|^2_{L_2(E_1,H)}\,\right]\right) ds \\
&\leq 2C\int_t^T \left(\left(2 + \frac{1}{2\,\varepsilon} + \frac{1}{2\,\varepsilon}\right)\mathbb{E}\left[|\Delta Y_s|_H^2\right] + \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right)\mathbb{E}\left[\, \|\Delta Z_s\|^2_{L_2(E_1,H)}\,\right]\right) ds \\
&\qquad\qquad + \int_t^T \left((C+\gamma)\; \mathbb{E}\left[|\Delta Y_s|_H^2\right] + 2\gamma\, \mathbb{E}\left[\, \|\Delta Z_s\|^2_{L_2(E_1,H)}\,\right]\right) ds \\
&\leq \int_t^T \left(\left(5C + \gamma + \frac{2C}{\varepsilon}\right)\mathbb{E}\left[|\Delta Y_s|_H^2\right] + (2\gamma + 2C\,\varepsilon)\, \mathbb{E}\left[\, \|\Delta Z_s\|^2_{L_2(E_1,H)}\,\right]\right) ds.
\end{aligned}$$

Thus, choosing $\varepsilon = \frac{1-2\gamma}{4C}$ (recalling $0 < \gamma < 1/2$) yields the following inequality:

$$\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \left(\frac{1-2\gamma}{2}\right)\mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] \leq \left(5C + \gamma + \frac{8C^2}{1-2\gamma}\right)\int_t^T \mathbb{E}\left[|\Delta Y_s|_H^2\right] ds.$$

Consequently, by Gronwall's inequality, we deduce that $Y_t^1 = Y_t^2$ and $Z_t^1 = Z_t^2$ *a.s.* for all $t \in [0,T]$. □

3.2. Existence of Solutions of MV-FBDSDEs

In this section, we establish the existence of solution for the MV-FBDSDEs (1) under assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$. We will follow the method of continuation, a method which is explained in [21] for the purpose of solving BSDEs with an arbitrary terminal time and also [25] for FBSDEs.

Theorem 3.3. *Under* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$*, MV-FBDSDEs (1) has a solution* (y, Y, z, Z) *in* $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$.

We shall employ the method of continuation and divide the proof of this theorem into two separate cases.

Case 1: Let $\theta_1 > 0, \theta_2 \geq 0$, and $\alpha_1 > 0$. We shall need first Lemma 3.5 below which involves a priori estimates of solutions of the following family of MV-FBDSDEs parameterized by $\alpha \in [0,1]$:

$$\begin{cases}
dy_t = \left(f^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) + \varphi_t\right) dt + \left(g^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) + \phi_t\right) dW_t - z_t\, \overleftarrow{dB}_t, \\
dY_t = \left(F^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) + \psi_t\right) dt + \left(G^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) + \kappa_t\right) \overleftarrow{dB}_t + Z_t\, dW_t, \\
y_0 = x, \qquad Y_T = h^\alpha\left(y_T, \mathbb{P}_{y_T}\right) + \xi,
\end{cases} \tag{5}$$

where $v_t = (y_t, Y_t, z_t, Z_t)\,, \mathbb{P}_{v_t} = \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}, (\varphi, \psi, \kappa, \phi) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$ and $\xi \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{P}, H\right)$, and for any given $\alpha \in [0,1]$:

$$\begin{cases}
f^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) = \alpha\, f\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{v_t}\right), \\
g^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) = \alpha\, g\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{v_t}\right), \\
F^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) = \alpha\, F\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{v_t}\right) + (1-\alpha)\,\theta_1\left(-y_t\right), \\
G^\alpha\left(t, v_t, \mathbb{P}_{v_t}\right) = \alpha\, G\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{v_t}\right) + (1-\alpha)\,\theta_1\left(-z_t\right), \\
h^\alpha\left(y_T, P_{y_T}\right) = \alpha\, h\left(y_T, \mathbb{P}_{y_T}\right) + (1-\alpha)\, y_T.
\end{cases}$$

When $\alpha = 1$, the existence of the solution of (5) implies clearly that of (1) by letting $\left(\varphi, \psi, \phi, \kappa\right) = (0, 0, 0, 0)$. On the other hand, if $\alpha = 0$ then (4) reduces to the following linear FBDSDEs:

$$\begin{cases} dy_t = \varphi_t\, dt + \phi_t\, dW_t - z_t\, \overleftarrow{dB}_t, \\ dY_t = \left(\theta_1 \left(-y_t\right) + \psi_t\right) dt + \left(\theta_1 \left(-z_t\right) + \kappa_t\right) \overleftarrow{dB}_t + Z_t\, dW_t, \\ y_0 = x, \quad Y_T = y_T + \xi. \end{cases} \tag{6}$$

Lemma 3.4. *The system (6), which is that of (5) when $\alpha = 0$, has a unique solution $(y, Y, z, Z) \in \mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$.*

Proof. It is easy to verify that MV-FBDSDEs (6) satisfies $(\mathbf{A}_1)$–$(\mathbf{A}_3)$. From Theorems (5.3, 5.4) in [3], we know that (5) has a unique solution (y, Y, z, Z) in $\mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$. For more details, we refer the readers to the arguments presented in [3, 4]. □

The following lemma is a key step in the proof of the method of continuation.

Lemma 3.5. *Assume that $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ holds with $\theta_1 > 0$, $\theta_2 \geq 0$, and $\alpha_1 > 0$. Suppose that there exists a constant $\alpha_0 \in [0, 1)$ such that, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $\left(\varphi, \psi, \kappa, \phi\right) \in \mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$, MV-FBDSDEs (5) has a unique solution.*

Then there exists $\delta_0 \in (0, 1)$, which only depends on $C, \gamma, \alpha_1, \theta_1, \theta_2$, and T, such that for any $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, MV-FBDSDEs (5) has a unique solution.

Proof. Assume that for each $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $\left(\varphi, \psi, \kappa, \phi\right) \in \mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$, MV-FBDSDEs (5) has a unique solution for a constant $\alpha = \alpha_0 \in [0, 1)$. Then, for each element $\bar{v} = (\bar{y}, \bar{Y}, \bar{z}, \bar{Z})$ of $\mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$, there exists a unique quadruple $v = (y, Y, z, Z) \in \mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$ satisfying the following MV-FBDSDEs:

$$\begin{cases} dy_t = \left(f^{\alpha_0}\left(t, v_t, \mathbb{P}_{v_t}\right) + \delta\, f\left(t, \bar{v}_t, \mathbb{P}_{\bar{v}_t}\right) + \varphi_t\right) dt + \left(g^{\alpha_0}\left(t, v_t, \mathbb{P}_{v_t}\right) + \delta\, g\left(t, \bar{v}_t, \mathbb{P}_{\bar{v}_t}\right) + \phi_t\right) dW_t - z_t\, \overleftarrow{dB}_t, \\ dY_t = \left(F^{\alpha_0}\left(t, v_t, \mathbb{P}_{v_t}\right) + \delta\left(\theta_1\, \bar{y}_t + F\left(t, \bar{v}_t, \mathbb{P}_{\bar{v}_t}\right)\right) + \psi_t\right) dt \\ \qquad\qquad + \left(G^{\alpha_0}\left(t, v_t, \mathbb{P}_{v_t}\right) + \delta\left(\theta_1\, \bar{z}_t + G\left(t, \bar{v}_t, \mathbb{P}_{\bar{v}_t}\right)\right) + \kappa_t\right) \overleftarrow{dB}_t + Z_t\, dW_t, \\ y_0 = x, \quad Y_T = h^{\alpha_0}\left(y_T, \mathbb{P}_{y_T}\right) + \delta\left(h\left(\bar{y}_T, \mathbb{P}_{\bar{y}_T}\right) - \bar{y}_T\right) + \xi. \end{cases} \tag{7}$$

We will show that the mapping $I_{\alpha_0+\delta}\left(\bar{v}, \bar{y}_T\right) := \left(v, y_T\right)$ from the space $\mathfrak{M}^2\left([0, T], \mathbb{H}^2\right) \times L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ into itself is a contraction, provided that $\delta > 0$ is sufficiently small. To this end, let $I_{\alpha_0+\delta}\left(\bar{v}^i, \bar{y}^i_T\right) := \left(v^i, y^i_T\right)$ for elements $\bar{v}^i = \left(\bar{y}^i, \bar{Y}^i, \bar{z}^i, \bar{Z}^i\right)$ of $\mathfrak{M}^2\left([0, T], \mathbb{H}^2\right)$, and define $v^{i,\bar{Z}} = \left(\bar{y}^i, \bar{Y}^i, \bar{z}^i\right)$ and $v^{i,\bar{z}} = \left(\bar{y}^i, \bar{Y}^i, \bar{Z}^i\right)$ for $i = 1, 2$. Next, introduce the notations: $\Delta\bar{v} = (\Delta\bar{y}, \Delta\bar{Y}, \Delta\bar{z}, \Delta\bar{Z}) = \left(\bar{y}^1 - \bar{y}^2, \bar{Y}^1 - \bar{Y}^2, \bar{z}^1 - \bar{z}^2, \bar{Z}^1 - \bar{Z}^2\right)$, and $\Delta v = (\Delta y, \Delta Y, \Delta z, \Delta Z) = \left(y^1 - y^2, Y^1 - Y^2, z^1 - z^2, Z^1 - Z^2\right)$. Also, set $\Delta h_T = h\left(y^1_T, \mathbb{P}_{y^1_T}\right) - h\left(y^2_T, \mathbb{P}_{y^2_T}\right)$ and $\Delta\bar{h}_T = h\left(\bar{y}^1_T, \mathbb{P}_{\bar{y}^1_T}\right) - h\left(\bar{y}^2_T, \mathbb{P}_{\bar{y}^2_T}\right)$.

By applying Itô's formula to $\langle \Delta y_t, \Delta Y_t\rangle_H$, it follows that

$$\begin{aligned} &\alpha_0\, \mathbb{E}\left[\langle \Delta y_T, \Delta h_T\rangle_H\right] + (1 - \alpha_0)\, \mathbb{E}\left[\left|\Delta y_t\right|^2_H\right] + \delta\, \mathbb{E}\left[\left\langle \Delta y_T, \Delta\bar{h}_T - \Delta\bar{y}_T\right\rangle_H\right] \\ &\quad = \alpha_0\, \mathbb{E}\left[\int_0^T \left(A\left(t, v^1_t, \mathbb{P}_{v^1_t}\right) - A\left(t, v^2_t, \mathbb{P}_{v^2_t}\right), \Delta v_t\right) dt\right] - (1 - \alpha_0)\, \theta_1\, \mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|^2_H + \|\Delta z_t\|^2_{L_2(E_2, H)}\right) dt\right] \\ &\quad + \delta\, \mathbb{E}\left[\int_0^T \left(A\left(t, \bar{v}^1_t, \mathbb{P}_{\bar{v}^1_t}\right) - A\left(t, \bar{v}^2_t, \mathbb{P}_{\bar{v}^2_t}\right), \Delta v_t\right) dt\right] + \delta\, \theta_1\, \mathbb{E}\left[\int_0^T \left(\langle \Delta y_t, \Delta\bar{y}_t\rangle_H + \langle \Delta z_t, \Delta\bar{z}_t\rangle_{L_2(E_2, H)}\right) dt\right]. \end{aligned}$$

Based on the conditions imposed in $(\mathbf{A}_2)$, we can derive the following inequality:

$$
\begin{aligned}
&(\alpha_0\,\alpha_1 + (1-\alpha_0))\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] \\
&\quad\le -\alpha_0\,\theta_2\,\mathbb{E}\left[\int_0^T \left(|\Delta Y_t|_H^2 + \|\Delta Z_t\|^2_{L_2(E_1,H)}\right)dt\right] - \alpha_0\,\theta_1\,\mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|_H^2 + \|\Delta z_t\|^2_{L_2(E_2,H)}\right)dt\right] \\
&\qquad + \delta\,\mathbb{E}\left[\int_0^T \left|A\left(t,\bar{v}_t^1,\mathbb{P}_{\bar{v}_t^1}\right) - A\left(t,\bar{v}_t^2,\mathbb{P}_{\bar{v}_t^2}\right)\right| |\Delta v_t|\,dt\right] \\
&\qquad + \delta\,\theta_1\,\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\left|\Delta y_t\right|_H^2 + \frac{1}{2}\left|\Delta \bar{y}_t\right|_H^2 + \frac{1}{2}\|\Delta z_t\|^2_{L_2(E_2,H)} + \frac{1}{2}\|\Delta \bar{z}_t\|^2_{L_2(E_2,H)}\right)dt\right] \\
&\qquad + \delta\,\mathbb{E}\left[\frac{1}{2}\left|\Delta y_T\right|_H^2 + \frac{1}{2}\left|\Delta \bar{y}_T\right|_H^2 + \left|\Delta y_T\right|_H\left|\Delta \bar{h}_T\right|_H\right].
\end{aligned}
$$

Therefore, we can rewrite the inequality as follows:

$$
\begin{aligned}
&(\alpha_0\,\alpha_1 + (1-\alpha_0))\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \alpha_0\,\theta_2\,\mathbb{E}\left[\int_0^T \left(|\Delta Y_t|_H^2 + \|\Delta Z_t\|^2_{L_2(E_1,H)}\right)dt\right] \\
&\qquad\qquad + \alpha_0\,\theta_1\,\mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|_H^2 + \|\Delta z_t\|^2_{L_2(E_2,H)}\right)dt\right] \\
&\le \delta\,\mathbb{E}\Bigg[\int_0^T \Bigg(\frac{1}{2}|\Delta v_t|^2 + C\left(|\Delta \bar{v}_t|^2 + \bar{w}_2^2\big(\mathbb{P}_{v_t^1},\mathbb{P}_{v_t^2}\big)\right) + \frac{C}{2}\left|v_t^{1,\bar{Z}} - v_t^{2,\bar{Z}}\right|^2 \\
&\qquad + \frac{\gamma}{2}\left(\left\|\Delta \bar{Z}_t\right\|^2_{L_2(E_1,H)} + \bar{w}_2^2\big(\mathbb{P}_{\bar{v}_t^1},\mathbb{P}_{\bar{v}_t^2}\big)\right) + \frac{C}{2}\left|v_t^{1,\bar{z}} - v_t^{2,\bar{z}}\right|^2 + \frac{\gamma}{2}\left(\|\Delta \bar{z}_t\|^2_{L_2(E_2,H)} + \bar{w}_2^2\big(\mathbb{P}_{\bar{v}_t^1},\mathbb{P}_{\bar{v}_t^2}\big)\right)\Bigg)dt\Bigg] \\
&\qquad + \delta\,\theta_1\,\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\left(\left|\Delta y_t\right|_H^2 + \left|\Delta \bar{y}_t\right|_H^2\right) + \frac{1}{2}\left(\|\Delta z_t\|^2_{L_2(E_2,H)} + \|\Delta \bar{z}_t\|^2_{L_2(E_2,H)}\right)\right)dt\right] \\
&\qquad + \delta\,\mathbb{E}\left[\frac{1}{2}\left|\Delta y_T\right|_H^2 + \frac{1}{2}\left|\Delta \bar{y}_T\right|_H^2 + \frac{1}{2}\left|\Delta y_T\right|_H^2 + \frac{C}{2}\left(\left|\Delta \bar{y}_T\right|_H^2 + \bar{w}_2^2\big(\mathbb{P}_{\bar{y}_T^1},\mathbb{P}_{\bar{y}_T^2}\big)\right)\right].
\end{aligned}
$$

We also know from inequality (3) that $\bar{w}_2^2\left(\mathbb{P}_{v_t^1},\mathbb{P}_{v_t^2}\right) \le \mathbb{E}\left[|\Delta v_t|^2\right]$. As a result,

$$
\begin{aligned}
&(\alpha_0\,\alpha_1 + (1-\alpha_0))\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \alpha_0\,\theta_2\,\mathbb{E}\left[\int_0^T \left(|\Delta Y_t|_H^2 + \|\Delta Z_t\|^2_{L_2(E_1,H)}\right)dt\right] \\
&\qquad\qquad + \theta_1\,\mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|_H^2 + \|\Delta z_t\|^2_{L_2(E_2,H)}\right)dt\right] \\
&\qquad \le \delta\,L\,\mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right)dt\right] + \delta\,L\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \left|\Delta \bar{y}_T\right|_H^2\right],
\end{aligned}
$$

for some generic constant $L > 0$, which from here on may vary from place to place and depends at most on the constants C, γ, α_1, θ_1, θ_2, and T. Next, since

$$\alpha_0\,\alpha_1 + (1-\alpha_0) > \min\{1,\alpha_1\} =: \tilde{\alpha}_1$$

and $\tilde{\alpha}_1 > 0$, then by letting $\beta = \min\{\tilde{\alpha}_1,\theta_1\}$, we deduce that $0 < \beta < 1$ and

$$\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|_H^2 + \|\Delta z_t\|^2_{L_2(E_2,H)}\right)dt\right] \le \frac{\delta\,L}{\beta}\left(\mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right)dt\right] + \mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \left|\Delta \bar{y}_T\right|_H^2\right]\right), \quad (8)$$

after neglecting the term $\alpha_0\,\theta_2\,\mathbb{E}\left[\int_0^T \left(|\Delta Y_t|_H^2 + \|\Delta Z_t\|^2_{L_2(E_1,H)}\right)dt\right]$ that contains α_0, as we aim to find a value of δ that is independent of α_0.

We therefore need to find estimates for $\mathbb{E}\left[\int_0^T \left(|\Delta Y_t|_H^2 + \|\Delta Z_t\|^2_{L_2(E_1,H)} \right) dt\right]$. To this end, we apply Itô's formula to $|\Delta Y_t|_H^2$ and take the expectation. Eventually, we find that

$$\begin{aligned}\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] &= \mathbb{E}\left[\left|\Delta h^{\alpha_0}\left(y_T, \mathbb{P}_{y_T}\right) + \delta\left(\Delta \bar{h}_T - \Delta \bar{y}_T\right)\right|_H^2\right] \\ &- 2\,\mathbb{E}\left[\int_t^T \left(\langle \Delta F^{\alpha_0}(s, v_s, \mathbb{P}_{v_s}), \Delta Y_s\rangle_H + \langle \delta\left(\theta_1(\Delta \bar{y}_s) + \Delta F(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s})\right), \Delta Y_s\rangle_H \right) ds\right] \\ &+ \mathbb{E}\left[\int_t^T \left\|\Delta G^{\alpha_0}(s, v_s, \mathbb{P}_{v_s}) + \delta\left(\theta_1(\Delta \bar{z}_s) + \Delta G(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s})\right)\right\|^2_{L_2(E_2,H)}\, ds\right],\end{aligned}$$

where

$$\begin{cases} \Delta F^{\alpha_0}(s, v_s, \mathbb{P}_{v_s}) = \alpha_0\, \Delta F\left(s, v_s, \mathbb{P}_{(y_s, Y_s, z_s, Z_s)}\right) + (1-\alpha_0)\,\theta_1\left(-(\Delta y_s)\right), \\ \Delta G^{\alpha_0}(s, v_s, \mathbb{P}_{v_s}) = \alpha_0\, \Delta G\left(s, v_s, \mathbb{P}_{(y_s, Y_s, z_s, Z_s)}\right) + (1-\alpha_0)\,\theta_1\left(-(\Delta z_s)\right), \\ \Delta h^{\alpha_0}\left(y_T, \mathbb{P}_{y_T}\right) = \alpha_0\, \Delta h_T + (1-\alpha_0)(\Delta y_T), \\ \Delta F(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s}) = F\left(t, \bar{v}_s^1, \mathbb{P}_{(\bar{y}_s^1, \bar{Y}_s^1, \bar{z}_s^1, \bar{Z}_s^1)}\right) - F\left(t, \bar{v}_s^2, \mathbb{P}_{(\bar{y}_s^2, \bar{Y}_s^2, \bar{z}_s^2, \bar{Z}_s^2)}\right), \\ \Delta G(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s}) = G\left(t, \bar{v}_s^1, \mathbb{P}_{(\bar{y}_s^1, \bar{Y}_s^1, \bar{z}_s^1, \bar{Z}_s^1)}\right) - G\left(t, \bar{v}_s^2, \mathbb{P}_{(\bar{y}_s^2, \bar{Y}_s^2, \bar{z}_s^2, \bar{Z}_s^2)}\right). \end{cases}$$

It follows that

$$\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] \leq I_1 + I_2(t) + I_3(t) + I_4(t), \tag{9}$$

where

$$\begin{aligned} I_1 &= 4\,\mathbb{E}\left[\alpha_0^2\, |\Delta h_T|_H^2 + (1-\alpha_0)^2 \left|\Delta y_T\right|_H^2 + \delta^2 \left|\Delta \bar{h}_T\right|_H^2 + \delta^2 \left|\Delta \bar{y}_T\right|_H^2\right], \\ I_2(t) &:= 2\,\mathbb{E}\left[\int_t^T \left(\left|\Delta F^{\alpha_0}(s, v_s, \mathbb{P}_{v_s})\right|_H |\Delta Y_s|_H + \left(\delta\,\theta_1 \left|\Delta \bar{y}_s\right|_H + \left|\delta\, \Delta F(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s})\right|_H\right) |\Delta Y_s|_H \right) ds\right], \\ I_3(t) &:= \mathbb{E}\left[\int_t^T \left(\frac{1+\gamma}{2\gamma}\right) \alpha_0^2 \left\|\Delta G\left(s, v_s, \mathbb{P}_{(y_s, Y_s, z_s, Z_s)}\right)\right\|^2_{L_2(E_2,H)}\, ds\right], \\ I_4(t) &:= 3\left(\frac{1+\gamma}{1-\gamma}\right) \mathbb{E}\left[\int_t^T \left((1-\alpha_0)^2\,\theta_1^2\, \|\Delta z_s\|^2_{L_2(E_2,H)} + \delta^2\,\theta_1^2\, \|\Delta \bar{z}_s\|^2_{L_2(E_2,H)} + \delta^2 \left\|\Delta G(s, \bar{v}_s, \mathbb{P}_{\bar{v}_s})\right\|^2_{L_2(E_2,H)} \right) ds\right]. \end{aligned}$$

On the other hand, from the fact that h is a Lipschitz mapping, we obtain

$$I_1 \leq C\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \delta\left|\Delta \bar{y}_T\right|_H^2\right]. \tag{10}$$

Since F is Lipschitz, we also have

$$\begin{aligned} I_2(t) \leq \mathbb{E}\Bigg[\int_t^T \Bigg\{ &\left(\frac{16\,C\,\alpha_0^2}{1-\gamma}\right) |\Delta Y_s|_H^2 + \left(\frac{1-\gamma}{16\,C}\right) C\left(|\Delta v_s|^2 + \varpi_2^2\left(\mathbb{P}_{v_t^1}, \mathbb{P}_{v_t^2}\right)\right) + (1-\alpha_0)\,\theta_1\left(|\Delta Y_s|_H^2 + \left|\Delta y_s\right|_H^2\right) \\ &+ \delta\,\theta_1\left(\left|\Delta \bar{y}_s\right|_H^2 + |\Delta Y_s|_H^2\right) + \delta\left(|\Delta Y_s|_H^2 + C\left(|\Delta \bar{v}_s|^2 + \varpi_2^2\left(\mathbb{P}_{\bar{v}_t^1}, \mathbb{P}_{\bar{v}_t^2}\right)\right)\right) \Bigg\} ds\Bigg]. \end{aligned}$$

Thus, by using inequality (3), it follows that

$$I_2(t) \le C\,\mathbb{E}\left[\int_t^T \left(|\Delta Y_s|_H^2 + \left|\Delta y_s\right|_H^2 + \|\Delta z_s\|^2_{L_2(E_2,H)}\right) ds\right] + C\,\delta\,\mathbb{E}\left[\int_t^T \left|\Delta \bar{y}_s\right|_H^2 ds\right] + \left(\frac{1-\gamma}{8}\right)\mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] + C\,\delta\,\mathbb{E}\left[\int_t^T |\Delta \bar{v}_s|^2\, ds\right]. \tag{11}$$

For $I_3(t)$ and $I_4(t)$, we apply $(\mathbf{A}_1)$ to see that

$$\begin{aligned} I_3(t) &\le \mathbb{E}\left[\int_t^T \left(\frac{1+\gamma}{4\gamma}\right)\alpha_0\left(C\left|v^{1,Z} - v^{2,Z}\right|^2 + \gamma\left(\|\Delta Z_s\|^2_{L_2(E_1,H)} + \varpi_2^2\left(\mathbb{P}_{v_t^1}, \mathbb{P}_{v_t^2}\right)\right)\right) ds\right] \\ &\le C\,\mathbb{E}\left[\int_t^T \left(|\Delta Y_s|_H^2 + \left|\Delta y_s\right|_H^2 + \|\Delta z_s\|^2_{L_2(E_2,H)}\right) ds\right] + \left(\frac{1+\gamma}{2}\right)\alpha_0\,\mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} I_4(t) &\le 3\left(\frac{4\gamma}{1-\gamma}\right)\mathbb{E}\left[\int_t^T \left((1-\alpha_0)^2\,\theta_1^2\,\|\Delta z_s\|^2_{L_2(E_2,H)} + \delta^2\,\theta_1^2\,\|\Delta \bar{z}_s\|^2_{L_2(E_2,H)}\right.\right. \\ &\qquad \left.\left. + \,\delta^2\left(C\left\|v^{1,\bar{z}} - v^{2,\bar{z}}\right\|^2_{L_2(E_2,H)} + \gamma\left(|\Delta \bar{z}_s|_H^2 + \varpi_2^2\left(\mathbb{P}_{\bar{v}_t^1}, \mathbb{P}_{\bar{v}_t^2}\right)\right)\right)\right) ds\right] \\ &\le C\,\mathbb{E}\left[\int_t^T \left(\|\Delta z_s\|^2_{L_2(E_2,H)} + \delta\left|\Delta \bar{y}_s\right|_H^2 + \delta\left|\Delta \bar{Y}_s\right|_H^2 + \delta\,\|\Delta \bar{z}_s\|^2_{L_2(E_2,H)} + \delta\,C\left(\frac{8\gamma^2}{1-\gamma}\right)\left\|\Delta \bar{Z}_s\right\|^2_{L_2(E_1,H)}\right) ds\right]. \end{aligned} \tag{13}$$

Now substitute (10)–(13) into (9), and use $\alpha_0 < 1$ to conclude that there exists a universal constant $L > 0$, independent of α_0, such that

$$\begin{aligned} &\mathbb{E}\left[|\Delta Y_t|_H^2\right] + \left(1 - \frac{5+3\gamma}{8}\right)\mathbb{E}\left[\int_t^T \|\Delta Z_s\|^2_{L_2(E_1,H)}\, ds\right] \\ &\le C\,\mathbb{E}\left[\int_t^T |\Delta Y_s|_H^2\, ds\right] + L\,\mathbb{E}\left[\left(\left|\Delta y_T\right|_H^2 + \delta\left|\Delta \bar{y}_T\right|_H^2\right)\right] + L\,\mathbb{E}\left[\int_t^T \left(\left|\Delta y_s\right|_H^2 + \|\Delta z_s\|^2_{L_2(E_2,H)} + \delta\,|\Delta \bar{v}_s|^2\right) ds\right]. \end{aligned}$$

Recall that $0 < \gamma < \frac{1}{2}$, and apply Gronwall's inequality to deduce that, for each $t \in [0, T]$,

$$\mathbb{E}\left[|\Delta Y_t|_H^2\right] \le e^{C(T-t)}\left(L\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \delta\left|\Delta \bar{y}_T\right|_H^2\right] + L\,\mathbb{E}\left[\int_t^T \left(\left|\Delta y_s\right|_H^2 + \|\Delta z_s\|^2_{L_2(E_2,H)} + \delta\,|\Delta \bar{v}_s|^2\right) ds\right]\right),$$

noting that $0 < \gamma < 1$ is only needed here. As a result, we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left(|\Delta Y_s|_H^2 + \|\Delta Z_s\|^2_{L_2(E_1,H)}\right) ds\right] &\le L\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \delta\left|\Delta \bar{y}_T\right|_H^2\right] \\ &\quad + L\,\mathbb{E}\left[\int_0^T \left(\left|\Delta y_s\right|_H^2 + \|\Delta z_s\|^2_{L_2(E_2,H)} + \delta\,|\Delta \bar{v}_s|^2\right) ds\right], \end{aligned} \tag{14}$$

where the constant L may vary from one line to line.

Next, we combine the crucial results (8) and (14) to find that

$$\begin{aligned} \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \mathbb{E}\left[\int_0^T |\Delta v_t|^2\, dt\right] &\le \frac{\delta L}{\beta}\left(\mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right) dt\right] + \mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \left|\Delta \bar{y}_T\right|_H^2\right]\right) \\ &\quad + L\,\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + L\,\mathbb{E}\left[\int_0^T \left(\left|\Delta y_t\right|_H^2 + \|\Delta z_t\|^2_{L_2(E_2,H)}\right) dt\right]. \end{aligned}$$

We then apply (8) once more to the last two terms of this latter inequality to derive

$$\mathbb{E}\left[\int_0^T |\Delta v_t|^2\,dt\right] + \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] \le \frac{\delta L}{\beta}\left(\mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right)dt\right] + \mathbb{E}\left[\left|\Delta y_T\right|_H^2 + \left|\Delta \bar{y}_T\right|_H^2\right]\right).$$

By taking $\delta \le \delta_0 := \frac{\beta}{3L}$, it follows that

$$\mathbb{E}\left[\int_0^T |\Delta v_t|^2\,dt + \left|\Delta y_T\right|_H^2\right] \le \frac{1}{2}\,\mathbb{E}\left[\int_0^T |\Delta \bar{v}_t|^2\,dt + \left|\Delta \bar{y}_T\right|_H^2\right].$$

Therefore, the mapping $I_{\alpha_0+\delta}$ is a contraction for all fixed δ in $[0,\delta_0]$. Consequently, $I_{\alpha_0+\delta}$ admits a unique fixed point (y,Y,z,Z) in $\mathfrak{M}^2\left([0,T],\mathbb{H}^2\right)$, which is the solution of MV-FBDSDE (5) for $\alpha = \alpha_0+\delta$, $\delta \in [0,\delta_0]$. □

Case 2: Let $\theta_1 \ge 0$, $\theta_2 > 0$, and $\alpha_1 \ge 0$. Consider the following family of MV-FBDSDEs, parameterized by $\alpha \in [0,1]$:

$$\begin{cases} dy_t = \left(\check{f}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) + \varphi_t\right)dt + \left(\check{g}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) + \phi_t\right)dW_t - z_t\,\overleftarrow{dB}_t, \\ dY_t = \left(\check{F}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) + \psi_t\right)dt + \left(\check{G}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) + \kappa_t\right)\overleftarrow{dB}_t + Z_t\,dW_t, \\ y_0 = x, \qquad Y_T = \check{h}^\alpha\left(y_T,\mathbb{P}_{y_T}\right) + \xi, \end{cases} \tag{15}$$

where

$$\begin{cases} \check{f}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) = \alpha\, f\left(t,y_t,Y_t,z_t,Z_t,\mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) + (1-\alpha)\,\theta_2\left(-Y_t\right), \\ \check{g}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) = \alpha\, g\left(t,y_t,Y_t,z_t,Z_t,\mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) + (1-\alpha)\,\theta_2\left(-Z_t\right), \\ F^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) = \alpha\, F\left(t,y_t,Y_t,z_t,Z_t,\mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right), \\ \check{G}^\alpha\left(t,v_t,\mathbb{P}_{v_t}\right) = \alpha\, G\left(t,y_t,Y_t,z_t,Z_t,\mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right), \\ \check{h}^\alpha\left(y_T,P_{y_T}\right) = \alpha\, h\left(y_T,\mathbb{P}_{y_T}\right). \end{cases}$$

When $\alpha = 1$, the existence of a solution to (15) immediately implies the existence of a solution to system (1) by letting $(\varphi,\psi,\phi,\kappa) = (0,0,0,0)$. On the other hand, if $\alpha = 0$, (15) is uniquely solvable as explained in Case 1. We now state a crucial lemma that will help us complete the proof of Theorem 3.3.

Lemma 3.6. *Assume that* $(\mathbf{A}_1)$*–*$(\mathbf{A}_3)$ *hold with* $\theta_1 \ge 0$*,* $\theta_2 > 0$*, and* $\alpha_1 \ge 0$*. Suppose there exists a constant* $\alpha_0 \in [0,1)$ *such that, for any* $\xi \in L^2\left(\Omega,\mathcal{F}_T,\mathbb{P},H\right)$ *and* $\left(\varphi,\psi,\kappa,\phi\right) \in \mathfrak{M}^2\left([0,T],\mathbb{H}^2\right)$*, MV-FBDSDEs (15) has a unique solution.*

Then there exists $\delta_0 \in (0,1)$ *which only depends on* $C,\gamma,\alpha_1,\theta_1,\theta_2$*, and* T*, such that for any* $\alpha \in [\alpha_0,\alpha_0+\delta_0]$*, MV-FBDSDEs (15) has a unique solution.*

Proof. Assume that, for all $\xi \in L^2\left(\Omega,\mathcal{F}_T,\mathbb{P},H\right)$ and $\left(\varphi,\psi,\kappa,\phi\right) \in \mathfrak{M}^2\left([0,T],\mathbb{H}^2\right)$, MV-FBDSDEs (15) has a unique solution for a constant $\alpha = \alpha_0 \in [0,1)$. Then, for each element $\bar{v} = (\bar{y},\bar{Y},\bar{z},\bar{Z})$ of $\mathfrak{M}^2\left([0,T],\mathbb{H}^2\right)$, there exists a unique quadruple $v = (y,Y,z,Z) \in \mathfrak{M}^2\left([0,T],\mathbb{H}^2\right)$ satisfying the following MV-FBDSDEs:

$$\begin{cases} dy_t = \left(\check{f}^{\alpha_0}\left(t,v_t,\mathbb{P}_{v_t}\right) + \delta\,\left(\theta_2\,\bar{Y}_t + f\left(t,\bar{v}_t,\mathbb{P}_{\bar{v}_t}\right)\right) + \varphi_t\right)dt \\ \qquad\qquad + \left(g^{\alpha_0}\left(t,v_t,\mathbb{P}_{v_t}\right) + \delta\left(\theta_2\,\bar{Z}_t + g\left(t,\bar{v}_t,\mathbb{P}_{\bar{v}_t}\right)\right) + \phi_t\right)dW_t - z_t\,\overleftarrow{dB}_t, \\ dY_t = \left(\check{F}^{\alpha_0}\left(t,v_t,\mathbb{P}_{v_t}\right) + \delta\, F\left(t,\bar{v}_t,\mathbb{P}_{\bar{v}_t}\right) + \psi_t\right)dt \\ \qquad\qquad + \left(\check{G}^{\alpha_0}\left(t,v_t,\mathbb{P}_{v_t}\right) + \delta\, G\left(t,\bar{v}_t,\mathbb{P}_{\bar{v}_t}\right) + \kappa_t\right)\overleftarrow{dB}_t + Z_t\,dW_t, \\ y_0 = x, \qquad Y_T = \check{h}^{\alpha_0}\left(y_T,\mathbb{P}_{y_T}\right) + \delta\, h\left(\bar{y}_T,\mathbb{P}_{\bar{y}_T}\right) + \xi. \end{cases}$$

We argue as in Case 1. Let us consider the mapping $I_{\alpha_0+\delta}$ defined in the proof of Lemma 3.4 and retain the same notations as set there after system (7). By applying Itô's formula to $\langle \Delta y_t, \Delta Y_t \rangle_H$ and disregarding the terms involving α_0, we obtain

$$\theta_2\, \mathbb{E}\left[\int_0^T |\Delta v_t|_H^2\, dt\right] \leq \delta\, L\, \mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right) dt\right] + \delta\, \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \delta\, L\, \mathbb{E}\left[\left|\Delta \bar{y}_T\right|_H^2\right]. \tag{16}$$

On the other hand, we can follow a similar approach as in (9) by applying Itô's formula to $\left|\Delta y_s\right|_H^2$ to obtain

$$\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \mathbb{E}\left[\int_0^T \|\Delta z_t\|_{L_2(E_2,H)}^2\, dt\right] \leq L\, \mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + \delta\, |\Delta \bar{v}_t|^2\right) dt\right]. \tag{17}$$

These two inequalities play a crucial role here.

Now, let $\beta' := \min\{\theta_2, 1\}$ to observe that $0 < \beta' < 1$. By combining (16) and (17), we can derive the following inequality:

$$\begin{aligned} &\beta' \left(\mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \mathbb{E}\left[\int_0^T |\Delta v_t|^2\, dt\right]\right) \\ &\quad \leq \delta\, L\, \mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right) dt\right] + \delta\, L\, \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \delta\, L\, \mathbb{E}\left[\left|\Delta \bar{y}_T\right|_H^2\right] + L\, \mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + \delta\, |\Delta \bar{v}_t|^2\right) dt\right]. \end{aligned}$$

Furthermore, applying (16) again to the term $\mathbb{E}\left[\int_0^T |\Delta v_t|^2\, dt\right]$ and utilizing the preceding inequality, we obtain

$$\mathbb{E}\left[\int_0^T |\Delta v_t|^2\, dt + \left|\Delta y_T\right|_H^2\right] \leq \frac{\delta\, L}{\beta'} \left(\mathbb{E}\left[\int_0^T \left(|\Delta v_t|^2 + |\Delta \bar{v}_t|^2\right) dt\right] + \mathbb{E}\left[\left|\Delta y_T\right|_H^2\right] + \mathbb{E}\left[\left|\Delta \bar{y}_T\right|_H^2\right]\right).$$

Thus, if we choose $\delta \leq \delta_0 := \frac{\beta'}{3L}$, we conclude that

$$\mathbb{E}\left[\int_0^T |\Delta v_t|^2\, dt + \left|\Delta y_T\right|_H^2\right] \leq \frac{1}{2}\, \mathbb{E}\left[\int_0^T |\Delta \bar{v}_t|^2\, dt + \left|\Delta \bar{y}_T\right|_H^2\right].$$

The remainder of the proof follows a similar approach as in Lemma 3.5. $\square$

We emphasize that the condition $0 < \gamma < 1/2$ in assumption $(\mathbf{A}_1)$ is needed frankly in Case 2 to establish the proof of Theorem 3.1 and so the proof of the preceding lemma. The reader can find similar details in [5, Lemma 3.8].

We are now ready to conclude the proof of Theorem 3.3.

Proof completion of Theorem 3.3. In Case 1 (when $\theta_2 > 0$), we already know that for each element ξ of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ and $\left(\varphi, \psi, \phi, \kappa\right) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$, the MV-FBDSDEs (5) has a unique solution when $\alpha = 0$. It then follows from Lemma 3.5 that there exists a positive constant $\delta_0 = \delta_0(C, \gamma, \alpha_1, \theta_1, \theta_2, T)$ such that for any $\delta \in [0, \delta_0]$, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$, and $\left(\varphi, \psi, \phi, \kappa\right) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$, (5) has a unique solution for $\alpha = \delta$. Moreover, since δ_0 depends only on $C, \gamma, \alpha_1, \theta_1, \theta_2, T$, we can repeat this process N times with $1 \leq N\delta_0 < 1 + \delta_0$. In particular, for $\alpha = 1$ with $\left(\varphi, \psi, \phi, \kappa\right) \equiv 0, \phi \equiv 0$, we deduce that MV-FBDSDEs (1) has a unique solution in $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$.

For Case 2 (when $\alpha_1 > 0$ and $\theta_1 > 0$), given any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ and $\left(\varphi, \psi, \phi, \kappa\right) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$, MV-FBDSDEs (15) has a unique solution when $\alpha = 0$. Consequently, Lemma 3.5 implies that there exists a constant $\delta_0 > 0$ that depends only on $C, \gamma, \alpha_1, \theta_1, \theta_2$, and T, such that, for any element $\delta \in [0, \delta_0]$,

$\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$, and $\left(\varphi, \psi, \phi, \kappa\right) \in \mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$, system (5) has a unique solution for $\alpha = \delta$. Therefore, similar to the preceding case, we conclude that the MV-FBDSDEs (1) attains a unique solution in $\mathfrak{M}^2\left([0,T], \mathbb{H}^2\right)$. □

We conclude this section by providing two examples to illustrate the results of Theorems (3.1, 3.3) and to demonstrate how to handle our conditions.

Example 3.7. *Let E and H be two real separable Hilbert spaces. Suppose B and W are cylindrical Wiener processes on E. Consider the following system on H:*

$$\begin{cases} dy_t = \left(\frac{1}{2}\,\mathbb{E}\left[Y_t\right] - Y_t\right)dt + \left(\frac{1}{4}\,\mathbb{E}\left[Z_t\right] - \frac{1}{2}\,Z_t\right)dW_t - z_t\,\overleftarrow{dB}_t, \\ dY_t = \left(\frac{1}{2}\,\mathbb{E}\left[y_t\right] - y_t\right)dt + \left(\frac{1}{4}\,\mathbb{E}\left[z_t\right] - \frac{1}{2}\,z_t\right)\overleftarrow{dB}_t + Z_t\,dW_t, \\ y_0 = x\ (\in H), \quad Y_T = -\frac{1}{2}\,\mathbb{E}\left[y_T\right] + y_T. \end{cases} \tag{18}$$

In order to relate this system to MV-FBDSDEs (1), we define for $t \in [0,T]$,

$$\begin{aligned} &f\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) = \tfrac{1}{2}\,\mathbb{E}\left[Y_t\right] - Y_t, \\ &g\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) = \tfrac{1}{4}\,\mathbb{E}\left[Z_t\right] - \tfrac{1}{2}\,Z_t, \\ &F\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) = \tfrac{1}{2}\,\mathbb{E}\left[y_t\right] - y_t, \\ &G\left(t, y_t, Y_t, z_t, Z_t, \mathbb{P}_{(y_t,Y_t,z_t,Z_t)}\right) = \tfrac{1}{4}\,\mathbb{E}\left[z_t\right] - \tfrac{1}{2}\,z_t, \\ &h\left(y_T, \mathbb{P}_{y_T}\right) = -\tfrac{1}{2}\,\mathbb{E}\left[y_T\right] + y_T. \end{aligned}$$

In particular, we have used here

$$\mathbb{E}\left[Y_t\right] = \int_H x_1\,d\mathbb{P}_{Y_t}(x_1) = \int_{\mathbb{H}^2} \Psi(x_1,x_2,x_3,x_4)\,d\mathbb{P}_{(y_t,Y_t,z_t,Z_t)},$$

through Fubini's theorem, where $\Psi(x_1,x_2,x_3,x_4) = x_1 \cdot 1 \cdot 1 \cdot 1 = x_1$. Similar expressions hold for $\mathbb{E}\left[Z_t\right]$, $\mathbb{E}\left[y_t\right]$, and $\mathbb{E}\left[z_t\right]$. So the dependence of these mappings f, g, F, G, h on a measure $\mu \in \mathcal{P}_2(\mathbb{H}^2)$ is only through its first moment $\int u\,d\mu$.

Now, with the help of (3) and the Cauchy-Schwarz inequality, we observe

$$\begin{aligned} &\left|f\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - f\left(t, v_t^2, \mathbb{P}_{v_t^2}\right)\right| \le \left|Y_t^1 - Y_t^2\right| + \tfrac{1}{2}\,\varpi_2\left(\mathbb{P}_{Y_t^1}, \mathbb{P}_{Y_t^2}\right), \\ &\left|F\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - F\left(t, v_t^2, \mathbb{P}_{v_t^2}\right)\right| \le \left|y_t^1 - y_t^2\right| + \tfrac{1}{2}\,\varpi_2\left(\mathbb{P}_{y_t^1}, \mathbb{P}_{y_t^2}\right), \\ &\left|g\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - g\left(t, v_t^2, \mathbb{P}_{v_t^2}\right)\right|^2 \le \tfrac{1}{2}\left\|Z_t^1 - Z_t^2\right\|^2 + \tfrac{1}{8}\,\varpi_2^2\left(\mathbb{P}_{Z_t^1}, \mathbb{P}_{Z_t^2}\right), \\ &\left|G\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - G\left(t, v_t^2, \mathbb{P}_{v_t^2}\right)\right|^2 \le \tfrac{1}{2}\left\|z_t^1 - z_t^2\right\|^2 + \tfrac{1}{8}\,\varpi_2^2\left(\mathbb{P}_{z_t^1}, \mathbb{P}_{z_t^2}\right). \end{aligned}$$

We also have

$$\mathbb{E}\left[\left(A\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - A\left(t, v_t^2, \mathbb{P}_{v_t^2}\right), v_t^1 - v_t^2\right)\right] \le -\frac{1}{4}\,\mathbb{E}\left[\left|\Delta y_t\right|^2 + |\Delta Y_t|^2 + ||\Delta z_t||^2 + ||\Delta Z_t||^2\right]$$

and

$$\mathbb{E}\left[\left\langle h\left(y_T^1, \mathbb{P}_{y_T^1}\right) - h\left(y_T^2, \mathbb{P}_{y_T^2}\right), y_T^1 - y_T^2\right\rangle\right] \ge \frac{1}{2}\,\mathbb{E}\left[\left|\Delta y_T\right|^2\right].$$

Therefore, by setting $C = 1, \gamma = \frac{1}{8}$, $\theta_1 = \theta_2 = \frac{1}{4}$, and $\alpha_1 = \frac{1}{2}$, it follows that assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ are satisfied. As a result, based on Theorems (3.1, 3.3), we deduce that system (18) has a unique solution.

We will now provide a counter example to show that the assumption ($\mathbf{A}_2$) in Theorems (3.1, 3.3) is necessary and cannot be dropped.

Example 3.8. *Let us consider the following MV-FBDSDEs on $H = \mathbb{R}$, with spaces $E_1 = E_2 = \mathbb{R}$:*

$$\begin{cases} dy_t = \mathbb{E}\left[Y_t\right] dt - z_t\, \overleftarrow{dB}_t, \\ dY_t = -\,\mathbb{E}\left[y_t\right] dt - z_t\, \overleftarrow{dB}_t + Z_t\, dW_t, \\ y_0 = 0, \qquad Y_T = -\,\mathbb{E}\left[y_T\right], \end{cases} \tag{19}$$

for $T = \frac{3\pi}{4}$. Here, B and W are 1-dimensional Brownian motions.

Using the notation set in assumption ($\mathbf{A}_2$), we have, for $v = (y, Y, z, Z)$,

$$A\left(t, v_t, \mathbb{P}_{v_t}\right) = \left(-\,\mathbb{E}\left[y_t\right], \mathbb{E}\left[Y_t\right], -z_t, 0\right).$$

Moreover, noting that

$$\mathbb{E}\left[\left(A\left(t, v_t^1, \mathbb{P}_{v_t^1}\right) - A\left(t, v_t^2, \mathbb{P}_{v_t^2}\right), v_t^1 - v_t^2\right)\right] = -\left(\mathbb{E}\left[\Delta y_t\right]\right)^2 + \left(\mathbb{E}\left[\Delta Y_t\right]\right)^2 - \mathbb{E}\left[\|(\Delta z_t)\|^2\right],$$

we realize that assumption ($\mathbf{A}_2$) does not hold. As a result, (19) might not have a unique solution. Indeed, $(\sin t, \cos t, 0, 0)$ is a solution of (19) in addition to the trivial solution $(y_t, Y_t, z_t, Z_t) = (0, 0, 0, 0)$.

4. Application to Stochastic Optimal Control

Let E be a separable real Hilbert space, and let U be a nonempty convex subset of K. A process $u_\cdot : [0, T] \times \Omega \to K$ is called *admissible* if $u_\cdot \in \mathfrak{M}^2\left([0, T], K\right)$ and $u_t \in U$ for all $t \in [0, T]$. The dot in $u_\cdot$ distinguishes the control process from the elements of U. The set of all such admissible controls is denoted by $\mathcal{U}_{ad}$.

In this section, we derive sufficient optimality conditions for a stochastic control problem governed by MV-FBDSDEs in infinite-dimensional separable real Hilbert spaces. Specifically, we consider the problem of minimizing the *cost functional* (or *objective functional*):

$$J\left(u_\cdot\right) = \mathbb{E}\left[\varphi\left(y_T^{u_\cdot}, \mathbb{P}_{y_T^{u_\cdot}}\right) + \psi\left(Y_0^{u_\cdot}, \mathbb{P}_{Y_0^{u_\cdot}}\right) + \int_0^T \ell\left(t, y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}, u_t, \mathbb{P}_{\left(y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}\right)}\right) dt\right], \tag{20}$$

over all $\mathcal{U}_{ad}$, subject to the state dynamic:

$$\begin{cases} dy_t^{u_\cdot} = f\left(t, y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}, u_t, \mathbb{P}_{\left(y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}\right)}\right) dt + g\left(t, y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}, u_t, \mathbb{P}_{\left(y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}\right)}\right) dW_t - z_t^{u_\cdot}\, \overleftarrow{dB_t}, \\ dY_t^{u_\cdot} = -F\left(t, y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}, u_t, \mathbb{P}_{\left(y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}\right)}\right) dt - G\left(t, y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}, u_t, \mathbb{P}_{\left(y_t^{u_\cdot}, Y_t^{u_\cdot}, z_t^{u_\cdot}, Z_t^{u_\cdot}\right)}\right) \overleftarrow{dB_t} + Z_t^{u_\cdot}\, dW_t, \\ y_0^{u_\cdot} = x, \; Y_T^{u_\cdot} = c\, y_T^{u_\cdot} + \xi, \end{cases} \tag{21}$$

where the coefficients are given by measurable mappings:

$$(f, F) : [0, T] \times \mathbb{H}^2 \times K \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to H, \quad g : [0, T] \times \mathbb{H}^2 \times K \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to L_2\left(E_1, H\right),$$
$$G : [0, T] \times \mathbb{H}^2 \times K \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to L_2\left(E_2, H\right),$$

ensuring that the cost functional is defined. Here, ξ is an $\mathcal{F}_T$-measurable random variable, and c is a constant.

We say that $u_\cdot^* \in \mathcal{U}_{ad}$ is an *optimal control* if it satisfies

$$J(u_\cdot^*) = \inf_{u_\cdot \in \mathcal{U}_{ad}} J(u_\cdot). \tag{22}$$

To address this control problem (20)–(22), we need to introduce the concept of *L-differentiability* with respect to probability measure. This is necessary due to the dependence of distribution appearing in both (20) and (21). We can then obtain the adjoint equations of (21), which resemble the MV-FBDSDEs studied in Section 3.

In our control problem (20)–(22), both the state process and the cost functional depend on the distribution $\mathbb{P}_{(y_t^{u.},Y_t^{u.},z_t^{u.},Z_t^{u.})}$ of the state process, providing more generality to cover cases such as those considered in Examples (3.7, 3.8).

4.1. The L-Differentiability and Convexity of Functions of Measures

In this subsection, we recall the definition of the L-derivative of functions of measures. The L-derivative was introduced by P. Lions, and in this regard, we refer to [14, Chapter 5] for more details on such a notion. Bensoussan et al., [8], gave an alternative equivalent definition. We shall be working over Hilbert spaces. The idea is to view the probability measures in $\mathcal{P}_2(E)$ over a separable real Hilbert space E as laws of random variables $X \in L^2(\Omega,\mathcal{F},\mathbb{P},E)$ so that $\mu = \mathbb{P}_X$. To be more precise, we assume that probability space $(\Omega,\mathcal{F},\mathbb{P})$ is rich enough in the sense that for every $\mu \in \mathcal{P}_2(E)$, there is a random variable $X \in L^2(\Omega,\mathcal{F},\mathbb{P},E)$ such that $\mu = \mathbb{P}_X$. A function $\Phi : \mathcal{P}_2(E) \to \mathbb{R}$ is said to be *L-differentiable* at μ_0 if there exists $X \in L^2(\Omega,\mathcal{F},\mathbb{P},E)$ with $\mu_0 = \mathbb{P}_{X_0}$ such that the *lifted function* $\hat{\Phi} : L^2(\Omega,\mathcal{F},\mathbb{P},E) \to \mathbb{R}$, given by $\hat{\Phi}(X) := \Phi(\mathbb{P}_X)$ for $X \in L^2(\Omega,\mathcal{F},\mathbb{P},E)$, is Fréchet differentiable at X_0, i.e., there exists a continuous linear functional

$$D\hat{\Phi}(X_0) : L^2(\Omega,\mathcal{F},\mathbb{P},E) \to \mathbb{R}$$

satisfying

$$\hat{\Phi}(X_0 + \Delta X) - \hat{\Phi}(X_0) = D\hat{\Phi}(X_0)(\Delta X) + o(\|\Delta X\|),$$

where ΔX represents a perturbation.

By the Riesz representation theorem, there exists a unique random variable ζ_0 in $L^2(\Omega,\mathcal{F},\mathbb{P},E)$ such that $D\hat{\Phi}(X_0)(X) = \langle \zeta_0, X\rangle$, for each $X \in L^2(\Omega,\mathcal{F},\mathbb{P},E)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega,\mathcal{F},\mathbb{P},E)$.

It is known (see [11] and [14]) that there exists a measurable function $\rho : H \to H$ depending only on μ_0 such that $\zeta_0 = \rho(Y)$ *a.s.* for all Y with $\mathbb{P}_Y = \mu_0$. We define the *L-derivative* $\partial_\mu \Phi(\mu_0)(Y)$ of $\hat{\Phi}$ at μ_0 along the random variable Y by $\rho(Y)$. Therefore, we have *a.s.*

$$\partial_\mu \Phi(\mathbb{P}_Y)(Y) = \rho(Y) = \nabla\hat{\Phi}(X_0),$$

where $\nabla\hat{\Phi}(X_0)$ is the gradient of $\hat{\Phi}$ at the point X_0.

The continuity of $\partial_\mu \Phi(x,\mu)$ is understood as the continuity of the mapping $X \mapsto \partial_\mu \Phi(\mathbb{P}_X)(X)$ from $L^2(\Omega,\mathcal{F},\mathbb{P},E)$ to $L^2(\Omega,\mathcal{F},\mathbb{P},E)$.

Similarly, for each fixed $t \in [0,T]$, a function $\Phi : \mathcal{P}_2(\mathbb{H}^2) \to \mathbb{R}$ is *differentiable* at μ if there exists a quadruple of random variables (y,Y,z,Z) in $\mathbb{H}^2$ with $\mu = \mathbb{P}_{(y,Y,z,Z)}$ so that the lifted function $\hat{\Phi}$, given by $\hat{\Phi}(y,Y,z,Z) = \Phi\left(\mathbb{P}_{(y,Y,z,Z)}\right)$, is Fréchet differentiable at (y,Y,z,Z). The partial L-derivatives $\partial_{\mu_y}\Phi$, $\partial_{\mu_Y}\Phi$, $\partial_{\mu_z}\Phi$, and $\partial_{\mu_Z}\Phi$ at μ along (y,Y,z,Z) can be viewed uniquely as an element $\nabla\hat{\Phi}(V)$ of $L^2\left(\Omega,\mathcal{F},\mathbb{P},\mathbb{H}^2\right)$, which can be represented as

$$\left(\partial_{\mu_y}\Phi(\mathbb{P}_V), \partial_{\mu_Y}\Phi(\mathbb{P}_V), \partial_{\mu_z}\Phi(\mathbb{P}_V), \partial_{\mu_Z}\Phi(\mathbb{P}_V)\right)(V),$$

where $V = (y,Y,z,Z)$.

Finally, let us introduce the following notation. Consider $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ as a copy of the probability space $(\Omega,\mathcal{F},\mathbb{P})$. For any pair of random variables (X,X') in $L^2(\Omega,\mathcal{F},\mathbb{P},E) \times L^2(\Omega,\mathcal{F},\mathbb{P},E)$, we denote their independent copies on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ as $(\tilde{X},\tilde{X}')$. Furthermore, we denote the expectation under the probability measure $\tilde{\mathbb{P}}$ as $\tilde{\mathbb{E}}$.

We say that Φ is *L-convex* (or merely *convex*) if for every $\mu,\mu' \in \mathcal{P}_2(E)$, we have

$$\Phi(\mu') - \Phi(\mu) - \tilde{\mathbb{E}}\left[\left\langle \partial_\mu \Phi(\mu)(X), X' - X\right\rangle\right] \geq 0, \tag{23}$$

whenever $X, X' \in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ with distributions μ and μ', respectively.

4.2. The Maxmum Principle

To establish the maximum principle for optimality, we need the following assumptions.

($\mathbf{A}_4$): Assume that

$$\begin{cases} \text{(i)} & F, f, G, g, \ell \text{ are continuous and continuously Fréchet differentiable with respect to } (y, Y, z, Z, u) \in \mathbb{H}^2 \times K, \text{ and } \varphi, \psi \text{ are continuously differentiable with respect to } y \in H \text{ and } Y \in H, \text{ respectively.} \\ \text{(ii)} & \text{The Fréchet derivatives of } F, f, G, g \text{ with respect to the above arguments are continuous and bounded, uniformly in } (t, \mu). \text{ Moreover, the Fréchet derivatives of } \phi = g, G \text{ satisfy } \left|\frac{\partial}{\partial z}\phi(t, y, Y, z, Z, v, \mu)\right|^2 < \gamma \text{ and } \left|\frac{\partial}{\partial Z}\phi(t, y, Y, z, Z, v, \mu)\right|^2 < \gamma, \text{ with } 0 < \gamma < \frac{1}{6}. \\ \text{(iii)} & \text{The derivatives of } \ell \text{ are bounded by } C(1 + |y|_H + |Y|_H + \|z\|_{L_2(E_2,H)} + \|Z\|_{L_2(E_1,H)} + \bar{w}_2(\mu, \delta_0)). \\ \text{(iv)} & \text{The derivatives of } \varphi \text{ and } \psi \text{ are bounded by } C(1 + |y|_H + \bar{w}_2(\mu, \delta_0)) \text{ and } C(1 + |Y|_H + \bar{w}_2(\mu, \delta_0)), \text{ respectively,} \end{cases}$$

for some constant $C > 0$, where δ_0 denotes the Dirac measure at 0.

($\mathbf{A}_5$): Suppose that the following conditions hold:

$$\begin{cases} \text{(i)} & F, f, G, g, \ell \text{ are } L\text{-differentiable with respect to } \mu \in \mathcal{P}_2(\mathbb{H}^2), \text{ and } \varphi, \psi \text{ are continuously } L\text{-differentiable with respect to } \mu. \\ \text{(ii)} & \text{The } L\text{-derivatives of } F, f, G, g \text{ are continuous and bounded, uniformly in } (t, y, Y, z, Z, v, \mu)\text{; in particular, we require} \\ & \displaystyle\int_{\mathbb{H}^2} \left|\partial_{\mu_z}\phi(t, y, Y, z, Z, v, \mu)(y', Y', z', Z')\right|^2 d\mu(y', Y', z', Z') < \frac{1}{3}\gamma, \text{ and} \\ & \displaystyle\int_{\mathbb{H}^2} \left|\partial_{\mu_Z}\phi(t, y, Y, z, Z, v, \mu)(y', Y', z', Z')\right|^2 d\mu(y', Y', z', Z') < \frac{1}{3}\gamma. \\ \text{(iii)} & \text{The } L\text{-derivatives of } \ell \text{ are bounded by } C(1 + |y|_H + |Y|_H + \|z\|_{L_2(E_2,H)} + \|Z\|_{L_2(E_1,H)} + \bar{w}_2(\mu, \delta_0)). \\ \text{(iv)} & \text{The } L\text{-derivatives of } \varphi \text{ and } \psi \text{ are bounded by } C(1 + |y|_H + \bar{w}_2(v, \delta_0)) \text{ and } C(1 + |Y|_H + \bar{w}_2(v, \delta_0)), \text{ respectively.} \end{cases}$$

($\mathbf{A}_6$): Denoting $A(t, v, \mu) = (-F, f, -G, g)(t, v, \mu)$ as in hypothesis ($\mathbf{A}_2$), we assume that either $c > 0$ and A satisfies ($\mathbf{A}_2$), or $c < 0$ with A satisfying $(\mathbf{A}_2)'$.

As we saw in Section 3, the condition $0 < \gamma < \frac{1}{2}$ crucial to guarantee the existence of solutions to the adjoint equations of MV-FBDSDE (21) in Theorem 4.2 below, which is one of the main theorems of this subsection. For additional clarification and similar discussions, refer to Remark 4.2 (i) in [6].

The following theorem addresses the existence and uniqueness of the solution of MV-FBDSDEs (21).

Theorem 4.1. *For any given admissible control u., if assumptions* ($\mathbf{A}_4$)–($\mathbf{A}_6$) *hold, then system (21) possesses a unique solution.*

Proof. Considering that C^1 mappings with bounded derivatives are globally Lipschitz, it is evident and straightforward to verify that assumptions ($\mathbf{A}_4$)–($\mathbf{A}_6$) imply ($\mathbf{A}_1$)–($\mathbf{A}_3$). For instance, ($\mathbf{A}_4$)(i, iii) and ($\mathbf{A}_5$)(ii), along with the definitions of $\bar{w}$ in (2) and the lifted functions, imply ($\mathbf{A}_1$)(ii, iii). As a result, the proof of the theorem can be derived from Theorem 3.1 and Theorem 3.3. □

Let $u.$ be an arbitrary element of $\mathcal{U}_{ad}$, and let $(y_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.})$ be the corresponding solution of system (21). Suppose that $(\mathbf{A}_4)$–$(\mathbf{A}_6)$ hold. First, we want to introduce the adjoint equations of the MV-FBDSDEs (21), and then we present our main result regarding the maximum principle for the optimal control of system (21). To this end, let us define the *Hamiltonian* $\mathcal{H} : [0,T] \times \Omega \times \mathbb{H}^2 \times K \times \mathbb{H}^2 \times \mathcal{P}_2\left(\mathbb{H}^2\right) \to \mathbb{R}$ by the formula:

$$\begin{aligned}\mathcal{H}(t,y,Y,z,Z,v,p,P,q,Q,\mu) := \langle p, F(t,y,Y,z,Z,v,\mu)\rangle - \langle P, f(t,y,Y,z,Z,v,\mu)\rangle + \langle q, G(t,y,Y,z,Z,v,\mu)\rangle \\ - \langle Q, g(t,y,Y,z,Z,v,\mu)\rangle - \ell(t,y,Y,z,Z,v,\mu).\end{aligned}$$

Using the notation in Section 4.1, the adjoint equations of MV-FBDSDEs (21) are the following MV-FBDSDEs:

$$\begin{cases} dp_t^{u.} = \nabla_Y \mathcal{H}\Big(t, V_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big) + \tilde{\mathbb{E}}\left[\partial_{\mu_Y}\mathcal{H}\Big(t, \tilde{V}_t^{u.}, u_t, \tilde{\chi}_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big)\Big(V_t^{u.}\Big)\right]\Big)\, dt \\ \qquad + \Big(\nabla_Z \mathcal{H}\Big(t, V_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big) + \tilde{\mathbb{E}}\left[\partial_{\mu_Z}\mathcal{H}\Big(t, \tilde{V}_t^{u.}, u_t, \tilde{\chi}_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big)\Big(V_t^{u.}\Big)\right]\Big)\, dW_t - q_t^{u.}\, \overleftarrow{dB}_t, \\ dP_t^{u.} = \Big(\nabla_y \mathcal{H}\Big(t, V_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big) + \tilde{\mathbb{E}}\left[\partial_{\mu_y}\mathcal{H}\Big(t, \tilde{V}_t^{u.}, u_t, \tilde{\chi}_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big)\Big(V_t^{u.}\Big)\right]\Big)\, dt \\ \qquad + \Big(\nabla_z \mathcal{H}\Big(t, V_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big) + \tilde{\mathbb{E}}\left[\partial_{\mu_z}\mathcal{H}\Big(t, \tilde{V}_t^{u.}, u_t, \tilde{\chi}_t^{u.}, \mathbb{P}_{V_t^{u.}}\Big)\Big(V_t^{u.}\Big)\right]\Big)\, \overleftarrow{dB}_t + Q_t^{u.}\, dW_t, \\ p_0^{u.} = -\nabla_Y \psi\left(Y_0^{u.}, \mathbb{P}_{Y_0^{u.}}\right) - \tilde{\mathbb{E}}\left[\partial_{\mu_Y}\psi\left(\tilde{Y}_0^{u.}, \mathbb{P}_{Y_0^{u.}}\right)\left(Y_0^{u.}\right)\right], \\ P_T^{u.} = \nabla_y \varphi\left(y_T^{u.}, \mathbb{P}_{y_T^{u.}}\right) + \tilde{\mathbb{E}}\left[\partial_{\mu_y}\varphi\left(\tilde{y}_T^{u.}, \mathbb{P}_{y_T^{u.}}\right)\left(y_T^{u.}\right)\right] - c\, p_T^{u.}, \end{cases} \tag{24}$$

where $V_t^{u.} \triangleq (y_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.})$, $\tilde{V}_t^{u.} \triangleq (\tilde{y}_t^{u.}, \tilde{Y}_t^{u.}, \tilde{z}_t^{u.}, \tilde{Z}_t^{u.})$, $\chi_t^{u.} \triangleq (p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.})$, and $\tilde{\chi}_t^{u.} \triangleq (\tilde{p}_t^{u.}, \tilde{P}_t^{u.}, \tilde{q}_t^{u.}, \tilde{Q}_t^{u.})$. Here, $\nabla_Y \mathcal{H}(t, V_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{V_t^{u.}})$ is the gradient, defined via the Gâteaux differential $D\mathcal{H}(t, Y_t^{u.})(h) = \left\langle \nabla_Y \mathcal{H}(t, Y_t^{u.}), h\right\rangle_H$ at the point $Y_t^{u.}$ in the direction $h \in H$, where $\mathcal{H}(t, Y_t^{u.}) := \mathcal{H}(t, y_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, u_t, \chi_t^{u.}, \mathbb{P}_{(y_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.})})$, etc.

In view of our results in Section 3, we observe the following theorem.

Theorem 4.2. *Under $(\mathbf{A}_4)$–$(\mathbf{A}_6)$, there exists a unique solution $(p^{u.}, P^{u.}, q^{u.}, Q^{u.})$ of the adjoint equations (24).*

Proof. This system (24) can be expressed as a linear system of MV-FBDSDEs on the arguments $(p^{u.}, P^{u.}, q^{u.}, Q^{u.})$. With the assumptions $(\mathbf{A}_4)$–$(\mathbf{A}_6)$, it is evident that this linear system satisfies $(\mathbf{A}1)$, $(\mathbf{A}2)'$, and $(\mathbf{A}_3)$. For more detailed information on a similar approach for FBDSDEs, one can refer to the methodology employed in our previous work [6]. Therefore, the desired result follows from Theorem 3.1, Theorem 3.3, and Remark 2.2 (ii). $\square$

Now, we present the main theorem of this section.

Theorem 4.3 (Sufficient conditions for optimality). *Assume that conditions $(\mathbf{A}_4)$–$(\mathbf{A}_6)$ hold. Given $\widehat{u.} \in \mathcal{U}_{ad}$, let $V_t^{\widehat{u.}} \equiv (y_t^{\widehat{u.}}, Y_t^{\widehat{u.}}, z_t^{\widehat{u.}}, Z_t^{\widehat{u.}})$ and $\left(p^{\widehat{u}}, P^{\widehat{u}}, q^{\widehat{u}}, Q^{\widehat{u}}\right)$ be the corresponding solutions of MV-FBDSDEs (21) and (24), respectively. Assume the following:*

(i) *φ and ψ are convex.*

(ii) *For all $t \in [0,T]$, $\mathbb{P}-a.s.$, the function $\mathcal{H}(t,\cdot,\cdot,\cdot,\cdot,\cdot,p^{\widehat{u}}, P^{\widehat{u}}, q^{\widehat{u}}, Q^{\widehat{u}})$ is concave.*

(iii) $\mathcal{H}\Big(t, V_t^{\widehat{u.}}, \widehat{u}_t, \chi_t^{\widehat{u.}}, \mathbb{P}_{V_t^{\widehat{u.}}}\Big) = \max_{v\in K} \mathcal{H}\Big(t, V_t^{\widehat{u.}}, v, \chi_t^{\widehat{u.}}, \mathbb{P}_{V_t^{\widehat{u.}}}\Big)$, *a.e. t, $\mathbb{P}-a.s.$*

Then $(y^{\widehat{u.}}, Y^{\widehat{u.}}, z^{\widehat{u.}}, Z^{\widehat{u.}}, \widehat{u.})$ is an optimal solution of the control problem (20)-(22).

Proof. Let $\widehat{u.} \in \mathcal{U}_{ad}$ be an arbitrary candidate for an optimal control, and denote its associated trajectory by $V_t^{\widehat{u.}} \triangleq (y_t^{\widehat{u.}}, Y_t^{\widehat{u.}}, z_t^{\widehat{u.}}, Z_t^{\widehat{u.}})$. For any $u. \in \mathcal{U}_{ad}$ with corresponding trajectory $V_t^{u.} \triangleq (y_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.})$, we have

$$\begin{aligned} J(u.) - J(\hat{u}.) = \mathbb{E}\left[\varphi(y_T^{u.}, \mathbb{P}_{y_T^{u}}) - \varphi(y_T^{\hat{u}.}, \mathbb{P}_{y_T^{\widehat{u.}}})\right] + \mathbb{E}\left[\psi(Y_0^{u.}, \mathbb{P}_{Y_0^{u.}}) - \psi(Y_0^{\widehat{u.}}, \mathbb{P}_{Y_0^{\widehat{u.}}})\right] \\ + \ \mathbb{E}\left[\int_0^T \Big(\ell(t, V_t^{u.}, u_t, \mathbb{P}_{V_t^{u.}}) - \ell(t, V_t^{\widehat{u.}}, \widehat{u}_t, \mathbb{P}_{V_t^{\widehat{u.}}})\Big) dt\right]. \end{aligned}$$

By the convexity of φ and ψ (see (23)), we deduce

$$\begin{aligned}\mathbb{E}\Big[\varphi(y_T^{u.},\mathbb{P}_{y_T^{u}})-\varphi(y_T^{\hat{u}.},\mathbb{P}_{y_T^{\widehat{u}.}})\Big] &\geq \mathbb{E}\Big[\Big\langle \nabla_y\,\varphi\Big(y_T^{\widehat{u}.},\mathbb{P}_{y_T^{\widehat{u}.}}\Big),y_T^{u.}-y_T^{\widehat{u}.}\Big\rangle_H\Big]+\mathbb{E}\Big[\tilde{\mathbb{E}}\Big[\Big\langle\partial_{\mu_y}\varphi\Big(y_T^{\widehat{u}.},\mathbb{P}_{y_T^{\widehat{u}.}}\Big)\Big(\tilde{y}_T^{\widehat{u}.}\Big),\tilde{y}_T^{u.}-\tilde{y}_T^{\widehat{u}.}\Big\rangle_H\Big]\Big]\\ &=\mathbb{E}\Big[\Big\langle\nabla_y\,\varphi\Big(y_T^{\widehat{u}.},\mathbb{P}_{y_T^{\widehat{u}.}}\Big)+\tilde{\mathbb{E}}\Big[\partial_{\mu_y}\varphi\Big(\tilde{y}_T^{\widehat{u}.},\mathbb{P}_{y_T^{\widehat{u}.}}\Big)\Big(y_T^{\widehat{u}.}\Big)\Big],y_T^{u.}-y_T^{\widehat{u}.}\Big\rangle_H\Big].\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}\Big[\psi(Y_0^{u.},\mathbb{P}_{Y_0^{u.}})-\psi(Y_0^{\widehat{u}.},\mathbb{P}_{Y_0^{\widehat{u}.}})\Big] &\geq \mathbb{E}\Big[\Big\langle\nabla_Y\psi\Big(Y_0^{\widehat{u}.},\mathbb{P}_{Y_0^{\widehat{u}.}}\Big),Y_0^{u.}-Y_0^{\widehat{u}.}\Big\rangle_H\Big]+\mathbb{E}\Big[\tilde{\mathbb{E}}\Big[\Big\langle\partial_{\mu_Y}\psi\Big(Y_0^{\widehat{u}.},\mathbb{P}_{Y_0^{\widehat{u}.}}\Big)\Big(\tilde{Y}_0^{v.}\Big),\tilde{Y}_0^{u.}-\tilde{Y}_0^{\widehat{u}.}\Big\rangle_H\Big]\Big]\\ &=\mathbb{E}\Big[\Big\langle\nabla_Y\psi\Big(Y_0^{\widehat{u}.},\mathbb{P}_{Y_0^{\widehat{u}.}}\Big)+\tilde{\mathbb{E}}\Big[\partial_{\mu_Y}\psi\Big(\tilde{Y}_0^{\widehat{u}.},\mathbb{P}_{Y_0^{\widehat{u}.}}\Big)\Big(Y_0^{v.}\Big)\Big],Y_0^{u.}-Y_0^{\widehat{u}.}\Big\rangle_H\Big].\end{aligned}$$

Therefore, by implementing (24), it follows that

$$\begin{aligned}J(u.)-J(\widehat{u.}) \geq{}& \mathbb{E}\Big[\langle P_T^{\widehat{u}.}+c\,p_T^{\widehat{u}.},y_T^{u.}-y_T^{\widehat{u}.}\rangle_H\Big]-\mathbb{E}\Big[\langle p_0^{\widehat{u}.},Y_0^{u.}-Y_0^{\widehat{u}.}\rangle_H\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big(\ell(t,V_t^{u.},u_t,\mathbb{P}_{V_t^{u.}})-\ell(t,V_t^{\widehat{u}.},\widehat{u}_t,\mathbb{P}_{V_t^{\widehat{u}.}})\Big)\,dt\Big].\end{aligned}$$

Next, by computing $\mathbb{E}\Big[\langle P_T^{\widehat{u}.},y_T^{u.}-y_T^{\widehat{u}.}\rangle_H\Big]$ and $\mathbb{E}\Big[\langle p_0^{\widehat{u}.},Y_0^{u.}-Y_0^{\widehat{u}.}\rangle_H\Big]$ via Itô's formula (e.g., as in [14] or [15]) applied to $\langle P_t^{\widehat{u}.},y_t^{u.}-y_t^{\widehat{u}.}\rangle_H$ and $\langle p_t^{\widehat{u}.},Y_t^{u.}-Y_t^{\widehat{u}.}\rangle_H$, respectively (cf. e.g., [6]), and then using the equality

$$\mathbb{E}\Big[\langle c\,p_T^{\widehat{u}.},y_T^{u.}-y_T^{\widehat{u}.}\rangle_H\Big]=\mathbb{E}\Big[\langle p_T^{\widehat{u}.},Y_T^{u.}-Y_T^{\widehat{u}.}\rangle_H\Big],$$

which follows directly from the definition of $Y_T^{u.}$ in (21), we ultimately obtain

$$\begin{aligned}J(u.)-J(\widehat{u.}) \geq{}& \mathbb{E}\Big[\int_0^T\Big\langle\nabla_y\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big),y_t^{u.}-y_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\tilde{\mathbb{E}}\Big[\partial_{\mu_y}\mathcal{H}\Big(t,\tilde{V}_t^{\widehat{u}.},\widehat{u}_t,\tilde{\chi}_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big)\Big(V^{\widehat{u}.}\Big)\Big],y_t^{u.}-y_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\nabla_Y\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big),Y_t^{u.}-Y_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\tilde{\mathbb{E}}\Big[\partial_{\mu_Y}\mathcal{H}\Big(t,\tilde{V}_t^{\widehat{u}.},\widehat{u}_t,\tilde{\chi}_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big)\Big(V^{\widehat{u}.}\Big)\Big],Y_t^{u.}-Y_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\nabla_z\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big),z_t^{u.}-z_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\tilde{\mathbb{E}}\Big[\partial_{\mu_z}\mathcal{H}\Big(t,\tilde{V}_t^{\widehat{u}.},\widehat{u}_t,\tilde{\chi}_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big)\Big(V^{\widehat{u}.}\Big)\Big],z_t^{u.}-z_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\nabla_Z\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big),Z_t^{u.}-Z_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &+\mathbb{E}\Big[\int_0^T\Big\langle\tilde{\mathbb{E}}\Big[\partial_{\mu_Z}\mathcal{H}\Big(t,\tilde{V}_t^{\widehat{u}.},\widehat{u}_t,\tilde{\chi}_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big)\Big(V^{\widehat{u}.}\Big)\Big],Z_t^{u.}-Z_t^{\widehat{u}.}\Big\rangle_H dt\Big]\\ &-\mathbb{E}\Big[\int_0^T\Big(\mathcal{H}\Big(t,V_t^{u.},u_t,\chi_t^{u.},\mathbb{P}_{V_t^{u.}}\Big)-\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big)\Big)dt\Big].\end{aligned}$$

Using the concavity of $\mathcal{H}$ (assumption (ii)), it follows that

$$J(u.)-J(\widehat{u.}) \geq -\,\mathbb{E}\Big[\int_0^T\Big\langle\nabla_u\mathcal{H}\Big(t,V_t^{\widehat{u}.},\widehat{u}_t,\chi_t^{\widehat{u}.},\mathbb{P}_{V_t^{\widehat{u}.}}\Big),u_t-\widehat{u}_t\Big\rangle_K dt\Big]. \tag{25}$$

However, condition (iii) implies that the function $v \mapsto \mathcal{H}\left(t, V_t^{\widehat{u}.}, v, \chi_t^{\widehat{u}.}, \mathbb{P}_{V_t^{\widehat{u}.}}\right)$ is maximal for $v = \widehat{u}_t$, so we have

$$\left\langle \nabla_u \mathcal{H}\left(t, V_t^{\widehat{u}.}, \widehat{u}_t, \chi_t^{\widehat{u}.}, \mathbb{P}_{V_t^{\widehat{u}.}}\right), u_t - \widehat{u}_t \right\rangle_K \leq 0, \quad a.e.\ t,\ \mathbb{P} - a.s. \tag{26}$$

Consequently, (25) simplifies to

$$J(u.) - J(\widehat{u.}) \geq 0.$$

Since $u.$ is an arbitrary admissible control, we conclude that $\widehat{u.}$ is an optimal control to the control problem (20)–(22). □

Remark 4.4. *1) As seen in the proof of Theorem 4.3, condition* (iii) *presented in its global form can be replaced by the local counterpart (26).*

2) The general case of optimal control problems governed by MV-FBDSDEs with a non-convex control set U remains an interesting research problem which we aim to address in future work.

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