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Killing vectors and magnetic curves associated to canonical connection and Kobayashi-Nomizu connection in Heisenberg group

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Abstract. This study establishes the canonical connection and Kobayashi-Nomizu connection within the Heisenberg group $(\mathbb{H}_3, g)$ through a systematic derivation based on the Levi-Civita connection framework. Following this, explicit expressions for Killing vector fields are determined, and analytical formulae for Killing magnetic curves associated with both the canonical connection and the Kobayashi-Nomizu connection are derived.

1. Introduction

In electromagnetism, the trajectory analysis of charged particles under the action of magnetic fields constitutes a fundamental research area. Within the framework of differential geometry, these trajectories are mathematically modeled as magnetic curves. A particularly significant case arises when the magnetic field corresponds to a Killing vector field, in which case the magnetic curves are termed Killing magnetic curves.

The investigation of magnetic curves corresponding to various manifolds has emerged as a pivotal direction in both differential geometry and theoretical physics. Extensive studies have been conducted in various geometric settings, including Euclidean spaces [21], Sasakian manifolds [9], cosymplectic manifolds [10], Walker manifolds [6], and the 3-torus [23]. Notable contributions also include characterizations of these curves in Thurston geometries by Erjavec and Inoguchi (see [14, 15, 17]).

For Killing magnetic curves specifically, significant results have been established in Euclidean 3-space [11], Minkowski 3-space [12], $\mathbb{S}^2 \times \mathbb{R}$ [22], and $SL(2, \mathbb{R})$ [13]. Recent advancements include explicit formulas for Heisenberg group configurations under Riemannian and Lorentzian metrics (see [2, 8]), as well as comprehensive characterizations in 3-dimensional almost paracontact manifolds [7]. The geometric analysis of Killing magnetic curves has further expanded to include local properties and specialized connections (see [18, 25, 26]). With recent works [20], Liu, Hua, and Chen addressed some explicit formulas for Killing magnetic curves in Bott connection formulations.

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The canonical connection and Kobayashi-Nomizu connection have demonstrated significant applicability in geometric analysis. For instance, in [3, 4], Azami investigated affine generalized Ricci solitons under perturbed canonical and Kobayashi-Nomizu connections in 3-dimensional Lorentzian Lie groups. In [28], Wang classified algebraic Ricci solitons linked to these connections in 3-dimensional Lorentzian Lie groups with product structures, while in [27], Tao characterized left-invariant Ricci collineations for such connections. In [1], the authors further analyzed geodesic equations via symmetry methods for the canonical connection in related Lie groups. Additional applications include Gauss-Bonnet theorem studies in Lorentzian Heisenberg groups (see [19, 29–31]).

This paper integrates canonical and Kobayashi-Nomizu connections with Killing magnetic curves, deriving trajectory equations for these curves in the 3-dimensional Heisenberg group. The structure is as follows: Section 2 reviews fundamental concepts of magnetic curves and Killing magnetic curves. Section 3 establishes the geometric framework of $\mathbb{H}_3$ and defines the canonical and Kobayashi-Nomizu connections. Sections 4–5 derive Killing magnetic curves via Lorentz equations and present explicit formulas for these curves under the canonical and Kobayashi-Nomizu connections in $(\mathbb{H}_3, g_1)$ and $(\mathbb{H}_3, g_2)$.

2. Preliminaries

Let (M, g) be a semi-Riemannian manifold. Magnetic curves on (M, g) model the trajectories of charged particles moving under the influence of a magnetic field F, represented as a closed 2-form:

$$F(X, Y) = g(\varphi(X), Y),$$

where $X, Y \in \mathfrak{X}(M)$ and φ is a skew-symmetric $(1, 1)-$tensor field encoding the Lorentz force. For a regular curve $\gamma : I \in \mathbb{R} \to (M, g)$, γ is termed a magnetic curve if its tangent vector $\mathbf{t} = \gamma'$ satisfies the Lorentz equation:

$$\nabla^i_{\mathbf{t}}\mathbf{t} = \varphi(\mathbf{t}), \quad i \in \{0, 1\}, \tag{1}$$

where ∇^0 and ∇^1 denote the canonical connection and the Kobayashi-Nomizu connection associated to g, respectively. The magnetic curve generalizes the geodesic curve under arc-length parametrization, that is, when $\varphi = 0$ the geodesic is a particular magnetic trajectory, and the solution of the Lorentz equation is a geodesic curve.

The cross product of the vector fields $X, Y \in \mathfrak{X}(M)$ is defined via the volume form dv_g as:

$$g(X \wedge Y, Z) = dv_g(X, Y, Z),$$

for all vector fields Z on M and $X \wedge Y$ represents the skew-symmetric $(2, 0)$-tensor induced by X and Y.

Let $F_V = i_V dv_g$ denote the Killing magnetic field associated with a Killing vector field V on M, where i_V is the interior product and dv_g is the volume form. The Lorentz force φ induced by F_V is defined as:

$$\varphi(X) = V \wedge X, \quad \forall X \in \mathfrak{X}(M),$$

Substituting into (1), we can rewrite the Lorentz equation (1) as:

$$\nabla^i_{\mathbf{t}}\mathbf{t} = V \wedge \mathbf{t}, \quad i \in \{0, 1\}. \tag{2}$$

Solutions to this equation are termed Killing magnetic curves with respect to V. For brevity, these curves are referred to as $V-$magnetic curve.

3. Geometry structure of Heisenberg Spaces

The Heisenberg group $\mathbb{H}_3$ is a quasi-Abelian Lie group diffeomorphic to $\mathbb{R}^3$, represented in $GL(3, \mathbb{R})$ as:

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| (x, y, z) \in \mathbb{R}^3 \right\},$$

equipped with the group operation:

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 - x_1 y_2).$$

Every left-invariant Lorentzian metric on $\mathbb{H}_3$ is isometric to one of the following forms:

$$\begin{aligned} g_1 &= -\frac{1}{\lambda^2}dx^2 + dy^2 + (xdy + dz)^2, \\ g_2 &= \frac{1}{\lambda^2}dx^2 + dy^2 - (xdy + dz)^2, \lambda > 0, \\ g_3 &= dx^2 + (xdy + dz)^2 - ((1-x)dy - dz)^2. \end{aligned} \tag{3}$$

These metrics are pairwise non-isometric, with g_3 being flat (see [5, 24]). This work focuses on the metrics g_1 and g_2.

Let ∇ denote the Levi-Civita connection associated with g_i in (3). The tangent bundle $T\mathbb{H}_3$ is spanned by $\{e_1, e_2, e_3\}$, with $D = span\{e_1, e_2\}$ as the horizontal distribution on $\mathbb{H}_3$ and $D^\perp = span\{e_3\}$. Define a product structure J via $Je_1 = e_1$, $Je_2 = e_2$, and $Je_3 = -e_3$. Then $J^2 = id$ and $g(Je_i, e_j) = g(e_i, e_j)$ for all $e_i, e_j \in \Gamma(\mathbb{H}_3)$, and J is a product structure.

Following [16], the canonical connection ∇^0 and Kobayashi-Nomizu connection ∇^1 are defined as:

$$\begin{aligned} \nabla^0_X Y &= \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \\ \nabla^1_X Y &= \nabla^0_X Y - \frac{1}{4}[(\nabla_X J)JY - (\nabla_{JX} J)Y], \end{aligned} \tag{4}$$

for $X, Y \in \mathfrak{X}(M)$.

4. The metric g_1

An orthonormal basis for $(\mathbb{H}_3, g_1)$ is provided by:

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \lambda\frac{\partial}{\partial x}, \tag{5}$$

where e_3 is timelike. The non-zero components of the Levi-Civita connection ∇ for g_1 are:

$$\begin{aligned} \nabla_{e_1} e_2 &= \nabla_{e_2} e_1 = \frac{\lambda}{2} e_3, \\ \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = \frac{\lambda}{2} e_2, \\ \nabla_{e_2} e_3 &= -\nabla_{e_3} e_2 = \frac{\lambda}{2} e_1. \end{aligned} \tag{6}$$

Using (4) and (6), the non-vanishing components of ∇^0 and ∇^1 are:

$$\nabla^0_{e_3} e_1 = \frac{\lambda}{2} e_2, \quad \nabla^0_{e_3} e_2 = -\frac{\lambda}{2} e_1, \quad \nabla^1_{e_3} e_1 = -\lambda e_1. \tag{7}$$

A vector field V on M is termed a Killing vector field if it satisfies the Killing equation:

$$g(\nabla^i_Y V, Z) + g(\nabla^i_Z V, Y) = 0, \tag{8}$$

for all $Y, Z \in \mathfrak{X}(M)$, and $i \in 0, 1$.

Assume the Killing vector field takes the form:

$$V = f_1(x, y, z)e_1 + f_2(x, y, z)e_2 + f_3(x, y, z)e_3,$$

where $f_i \in C^\infty(\mathbb{H}_3)$ for $i = 1, 2, 3$. Substituting V into (8) with $Y = e_i$, $Z = e_j$ defined by (5), yields a system of differential equations for Killing vector fields associated with ∇^0:

$$(\mathcal{L}_V g_1)(e_1, e_1) = \frac{\partial}{\partial z} f_1 = 0, \tag{9}$$

$$(\mathcal{L}_V g_1)(e_1, e_2) = \frac{\partial}{\partial z} f_2 + \frac{\partial}{\partial y} f_1 - x\frac{\partial}{\partial z} f_1 = 0, \tag{10}$$

$$(\mathcal{L}_V g_1)(e_1, e_3) = -\frac{\partial}{\partial z} f_3 + \lambda\frac{\partial}{\partial x} f_1 - \frac{\lambda}{2} f_2 = 0, \tag{11}$$

$$(\mathcal{L}_V g_1)(e_2, e_2) = \frac{\partial}{\partial y} f_2 - x\frac{\partial}{\partial z} f_2 = 0, \tag{12}$$

$$(\mathcal{L}_V g_1)(e_2, e_3) = -\frac{\partial}{\partial y} f_3 + x\frac{\partial}{\partial z} f_3 + \frac{\lambda}{2} f_1 + \lambda\frac{\partial}{\partial x} f_2 = 0, \tag{13}$$

$$(\mathcal{L}_V g_1)(e_3, e_3) = \frac{\partial}{\partial x} f_3 = 0, \tag{14}$$

where $\mathcal{L}_V g_1$ is the Lie derivative in the direction of V.

Proposition 4.1. *The Lie algebra of infinitesimal isometries for* $(\mathbb{H}_3, g_1, \nabla^0)$ *is 5-dimensional, with a basis given by the following vector fields:*

$$\begin{aligned} &V_1 = \lambda\frac{\partial}{\partial x}, \quad V_2 = \frac{2}{\lambda}\frac{\partial}{\partial z} + \lambda y\frac{\partial}{\partial x}, \quad V_3 = \frac{x}{3}\frac{\partial}{\partial z} + (\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) - \frac{\lambda^2}{6} z\frac{\partial}{\partial x}, \\ &V_4 = \sin\frac{x}{2}\frac{\partial}{\partial z} + \cos\frac{x}{2}(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}), \quad V_5 = \cos\frac{x}{2}\frac{\partial}{\partial z} - \sin\frac{x}{2}(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}). \end{aligned} \tag{15}$$

Proof. Differentiating (11) with respect to x and z yields:

$$\frac{\partial^2 f_2}{\partial x \partial z} = 0. \tag{16}$$

Differentiating (10) with respect to x and incorporating (16) gives:

$$\frac{\partial^2 f_1}{\partial x \partial y} = 0. \tag{17}$$

Differentiating (11) with respect to x and y, and using (17), leads to:

$$\frac{\partial^2 f_2}{\partial x \partial y} = 0. \tag{18}$$

Differentiating (12) with respect to x and using (16), (18) gives:

$$\frac{\partial f_2}{\partial z} = 0. \tag{19}$$

Thus, (12) simplifies to:

$$\frac{\partial f_2}{\partial y} = 0, \tag{20}$$

implying $f_2 = f_2(x)$, then we can put $f_2(x) = B(x)$. Similarly, (10) reduces to:

$$\frac{\partial f_1}{\partial y} = 0, \tag{21}$$

so $f_1 = f_1(x)$, then we put $f_1(x) = A(x)$. Differentiating (11) with respect to x and (13) twice with respect to x yields:

$$\lambda \frac{\partial^2 f_1}{\partial x^2} - \frac{\lambda}{2}\frac{\partial f_2}{\partial x} = 0, \quad \frac{\lambda}{2}\frac{\partial^2 f_1}{\partial x^2} + 2\lambda \frac{\partial^3 f_2}{\partial x^3} = 0. \tag{22}$$

Solving these equations gives:

$$f_2(x) = 2c_1 \sin\frac{x}{2} - 2c_2 \cos\frac{x}{2} + c_3, \quad (c_1, c_2, c_3 \in \mathbb{R}). \tag{23}$$

Differentiating (11) with respect to z and y yields:

$$\frac{\partial^2 f_3}{\partial z^2} = 0, \tag{24}$$

$$\frac{\partial^2 f_3}{\partial y \partial z} = 0. \tag{25}$$

Differentiating (13) with respect to y and using (25) gives:

$$\frac{\partial^2 f_3}{\partial y^2} = 0. \tag{26}$$

So $f_3(y, z) = c_4 z + c_5 y + c_6$, where $c_4, c_5, c_6 \in \mathbb{R}$. Substituting into (13) and (11) yields:

$$f_1(x) = -2c_1 \cos\frac{x}{2} - 2c_2 \sin\frac{x}{2} - \frac{2c_4}{\lambda}x + \frac{2c_5}{\lambda}. \tag{27}$$

and:

$$\frac{\partial f_1}{\partial x} = \frac{1}{2} f_2 + \frac{c_4}{\lambda}, \tag{28}$$

implying $c_4 = -\frac{\lambda}{6}c_3$. The final expressions for f_1, f_2, f_3 are:

$$\begin{aligned} f_1(x) &= -2c_1 \cos\frac{x}{2} - 2c_2 \sin\frac{x}{2} + \frac{1}{3}c_3 x + \frac{2}{\lambda}c_5, \\ f_2(x) &= 2c_1 \sin\frac{x}{2} - 2c_2 \cos\frac{x}{2} + c_3, \\ f_3(y, z) &= -\frac{\lambda}{6}c_3 z + c_5 y + c_6. \end{aligned} \tag{29}$$

Substituting these into the vector field V and expressing in terms of the basis $\{e_1, e_2, e_3\}$ yields the stated result. □

Then, using the same process, we present the Lie algebra of the Killing vector field of $(\mathbb{H}_3, g_1, \nabla^1)$, which is generated by Killing vectors by the following proposition.

Proposition 4.2. *The Lie algebra of infinitesimal isometries for* $(\mathbb{H}_3, g_1, \nabla^1)$ *is 5-dimensional, with a basis given by the following vector fields:*

$$\begin{aligned} &V_1 = \frac{\partial}{\partial z}, \quad V_2 = \lambda\frac{\partial}{\partial x}, \quad V_3 = x\frac{\partial}{\partial z} + (\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}), \\ &V_4 = \frac{x^2}{2}\frac{\partial}{\partial z} + x(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) + \lambda^2 y\frac{\partial}{\partial x}, V_5 = (\frac{x^3}{3} - 2x)\frac{\partial}{\partial z} + x^2(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) - 2\lambda^2 z\frac{\partial}{\partial x}. \end{aligned} \tag{30}$$

Proof. Let $V = f_1e_1 + f_2e_2 + f_3e_3$ be a Killing vector field for $(\mathbb{H}_3, g_1, \nabla^1)$. Substituting V into the Killing equation with $Y = e_i$, $Z = e_j$ defined in (5) yields the system:

$$(\mathcal{L}_V g_1)(e_1, e_1) = \frac{\partial}{\partial z} f_1 = 0 \tag{31}$$

$$(\mathcal{L}_V g_1)(e_1, e_2) = \frac{\partial}{\partial z} f_2 + \frac{\partial}{\partial y} f_1 - x\frac{\partial}{\partial z} f_1 = 0, \tag{32}$$

$$(\mathcal{L}_V g_1)(e_1, e_3) = -\frac{\partial}{\partial z} f_3 + \lambda\frac{\partial}{\partial x} f_1 - \lambda f_2 = 0, \tag{33}$$

$$(\mathcal{L}_V g_1)(e_2, e_2) = \frac{\partial}{\partial y} f_2 - x\frac{\partial}{\partial z} f_2 = 0, \tag{34}$$

$$(\mathcal{L}_V g_1)(e_2, e_3) = -\frac{\partial}{\partial y} f_3 + x\frac{\partial}{\partial z} f_3 + \lambda\frac{\partial}{\partial x} f_2 = 0, \tag{35}$$

$$(\mathcal{L}_V g_1)(e_3, e_3) = \frac{\partial}{\partial x} f_3 = 0 \tag{36}$$

where $\mathcal{L}_V g_1$ is the Lie derivative in the direction of V, and $f_i \in C^\infty(\mathbb{H}_3)$ for $i = 1, 2, 3$.

Differentiating (33) with respect to x and z gives:

$$\frac{\partial^2 f_2}{\partial x \partial z} = 0. \tag{37}$$

Differentiating (32) with respect to x and incorporating (37) yields:

$$\frac{\partial^2 f_1}{\partial x \partial y} = 0. \tag{38}$$

Differentiating (33) with respect to x and y, and using (38), gives:

$$\frac{\partial^2 f_2}{\partial x \partial y} = 0. \tag{39}$$

Differentiating (34) with respect to x and using (37) and (39) leads to:

$$\frac{\partial f_2}{\partial z} = 0. \tag{40}$$

Thus, (32) and (34) simplify to:

$$\frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_2}{\partial y} = 0. \tag{41}$$

implying $f_1 = f_1(x)$ and $f_2 = f_2(x)$, then we put $f_1(x) = A(x)$ and $f_2(x) = B(x)$. Differentiating (33) with respect to y and z, and (35) with respect to y, yields:

$$\frac{\partial^2 f_3}{\partial y \partial z} = \frac{\partial^2 f_3}{\partial z^2} = \frac{\partial^2 f_3}{\partial y^2} = 0. \tag{42}$$

Thus $f_3(y, z) = c_1 y + c_2 z + c_3$ for constants $c_1, c_2, c_3 \in \mathbb{R}$. Differentiating (35) twice with respect to x gives:

$$\frac{\partial^3 f_2}{\partial x^3} = 0, \tag{43}$$

so $f_2(x) = c_4x^2 + c_5x + c_6$, where $c_4, c_5, c_6 \in \mathbb{R}$. Integrating (33) yields:

$$f_1(x) = \frac{c_4}{3}x^3 + \frac{c_5}{2}x^2 + c_6 x + \frac{1}{\lambda}c_2 x + c_7. \tag{44}$$

Using (35) to relate coefficients gives $c_1 = \lambda c_5$, $c_2 = -2\lambda c_4$, and $c_2 = \lambda c_7 - \lambda c_6$. Substituting these into f_1, f_2, f_3 results in:

$$\begin{aligned} f_1 &= \frac{c_4}{3}x^3 + \frac{c_5}{2}x^2 + (c_6 - 2c_4)x + c_8, \\ f_2 &= c_4x^2 + c_5x + c_6, \\ f_3 &= -2c_4\lambda z + c_5\lambda y + c_3. \end{aligned} \tag{45}$$

Expressing V in terms of the basis $\{e_1, e_2, e_3\}$ yields the stated result. □

Let $\gamma(t) : I \subset \mathbb{R} \to (\mathbb{H}_3, g_1)$ be a regular curve parameterized by $\gamma(t) = (x(t), y(t), z(t))$. Its velocity vector is given by:

$$\mathbf{t} = \gamma'(t) = (x'(t), y'(t), z'(t)).$$

Expressed in the basis $\{e_i\}_{i=1,2,3}$ defined in (5), the velocity vector $\mathbf{t}$ takes the form:

$$\mathbf{t} = (z' + xy')e_1 + y'e_2 + \frac{x'}{\lambda}e_3. \tag{46}$$

Using the connection formulas from (4), we compute the covariant derivatives:

$$\begin{aligned} \nabla^0_{\mathbf{t}}\mathbf{t} &= ((z' + xy')' - \frac{y'x'}{2})e_1 + (y'' + \frac{1}{2}(z' + xy')x')e_2 + \frac{x''}{\lambda}e_3, \\ \nabla^1_{\mathbf{t}}\mathbf{t} &= ((z' + xy')' - x'y')e_1 + y''e_2 + \frac{x''}{\lambda}e_3. \end{aligned} \tag{47}$$

In the subsequent sections, we derive explicit formulas for V_i-magnetic curves with respect to the canonical connection ∇^0 and Kobayashi-Nomizu connection ∇^1 on $(\mathbb{H}_3, g_1)$.

4.1. *V_1-magnetic curves associate to ∇^0.*

We consider V_1-magnetic curves correspond to the Killing vector field $V_1 = e_3$ defined in (15). Using (46), the wedge product is computed as:

$$V_1 \wedge \mathbf{t} = -y'e_1 + (z' + xy')e_2. \tag{48}$$

The Lorentz equation $\nabla_{\mathbf{t}}\mathbf{t} = V_1 \wedge \mathbf{t}$ yields the system of differential equations (S_1):

$$S_1 : \begin{cases} (z' + xy')' - \dfrac{y'x'}{2} = -y', \\ y'' + \dfrac{1}{2}(z' + xy')x' = z' + xy', \\ \dfrac{x''}{\lambda} = 0. \end{cases} \tag{49}$$

Integrating the third equation of (S_1) gives:

$$x = c_1t + c_2, \tag{50}$$

where c_1, c_2 are constants. Substituting (50) into the first two equations of (S_1) leads to:

$$y''' + (\frac{c_1 - 2}{2})^2 y' = 0.$$

1. If $c_1 = 2$, the system (S_1) reduces to:

$$S_1 : \begin{cases} x'' = 0, \\ y'' = 0, \\ (z' + xy')' = \dfrac{y'x'}{2} - y'. \end{cases} \tag{51}$$

The general solution is:

$$\begin{cases} x(t) = c_1 t + c_2, \\ y(t) = c_3 t + c_4, \\ z(t) = -(\dfrac{c_1c_3 + 2c_3}{4})t^2 - (c_2c_3 - c_5)t + c_6. \end{cases} \tag{52}$$

where $c_1, \cdots, c_6$ are constants.

2. If $c_1 \neq 2$, let $\tilde{c} = \frac{2-c_1}{2}$. The general solution of (S_1) is:

$$\begin{cases} x(t) = c_1 t + c_2, \\ y(t) = \dfrac{c_3 \sin(\tilde{c}t)}{\tilde{c}} - \dfrac{c_4 \cos(\tilde{c}t)}{\tilde{c}} + c_5, \\ z(t) = \dfrac{c_3\tilde{c} + c_1c_3 - c_2c_4\tilde{c} - c_1c_4\tilde{c}t}{\tilde{c}^2}\cos(\tilde{c}t) + \dfrac{c_4\tilde{c} + c_1c_4 + c_2c_3\tilde{c} + c_1c_3\tilde{c}t}{\tilde{c}^2}\sin(\tilde{c}t) + c_6. \end{cases} \tag{53}$$

where $c_1, \cdots, c_6$ are constants.

Theorem 4.3. *All V_1-magnetic curves of (H_3, g_1, ∇^0) are parametrized by:*

1. *If $c_1 = 2$:*

$$\gamma(t) = \left\{c_1t + c_2,\ c_3t + c_4,\ -(\tfrac{c_1c_3+2c_3}{4})t^2 - (c_2c_3 - c_5)t + c_6\right\}.$$

2. *If $c_1 \neq 2$:*

$$\gamma(t) = \begin{Bmatrix} c_1t + c_2 \\ \dfrac{c_3 \sin(\tilde{c}t)}{\tilde{c}} - \dfrac{c_4 \cos(\tilde{c}t)}{\tilde{c}} + c_5 \\ \dfrac{c_3\tilde{c} + c_1c_3 - c_2c_4\tilde{c} - c_1c_4\tilde{c}t}{\tilde{c}^2}\cos(\tilde{c}t) + \dfrac{c_4\tilde{c} + c_1c_4 + c_2c_3\tilde{c} + c_1c_3\tilde{c}t}{\tilde{c}^2}\sin(\tilde{c}t) + c_6 \end{Bmatrix}^T,$$

where $c_1, \cdots, c_6$ are real numbers.

4.2. V_2-magnetic curves associate to ∇^0.

For V_2-magnetic curves associated with the Killing vector field $V_2 = \frac{2}{\lambda}e_1 + \lambda y e_3$ (see (15)), the wedge product is computed as:

$$V_2 \wedge \mathbf{t} = yy'e_1 + (y(z' + xy') - \frac{2}{\lambda^2}x')e_2 - \frac{2}{\lambda}y'e_3. \tag{54}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_2 \wedge \mathbf{t}$ yields the system of differential equations (S_2):

$$S_2 : \begin{cases} (z' + xy')' - \dfrac{y'x'}{2} = yy', \\ y'' + \dfrac{1}{2}(z' + xy')x' = y(z' + xy') - \dfrac{2}{\lambda^2}x', \\ \dfrac{x''}{\lambda} = -\dfrac{2}{\lambda}y'. \end{cases} \tag{55}$$

Integrating the third equation of (S_2) gives:

$$x' = -2y + c_1, \tag{56}$$

where c_1 is constant. Substituting (56) into the first equation of (S_2) yields:

$$z' + xy' = \frac{c_1}{2}y + c_2, \tag{57}$$

with c_2 as another constant. The second equation of (S_2) simplifies to:

$$y'' + \frac{1}{2}(\frac{c_1}{2}y + c_2)x' = y(\frac{c_1}{2}y + c_2) - \frac{2}{\lambda^2}x'. \tag{58}$$

While the general solution of (58) is complex, we solve it for the particular case $c_1 = 0$. In this case, the system reduces to:

$$\begin{cases} x' = -2y, \\ z' + xy' = c_2, \\ y'' + \frac{c_2}{2}x' = c_2 y - \frac{2}{\lambda^2}x'. \end{cases}$$

Solving these yields:

$$y(t) = c_3 e^{\sqrt{(2c_2+\frac{4}{\lambda^2})t}} + c_4 e^{-\sqrt{(2c_2+\frac{4}{\lambda^2})t}}. \tag{59}$$

and substituting into the remaining equations of (S_2) gives:

$$z(t) = (c_2 - 4c_3c_4)t + \frac{e^{-2\sqrt{2c_2+\frac{4}{\lambda^2}}t}(c_3^2 e^{4\sqrt{2c_2+\frac{4}{\lambda^2}}} - c_4^2)}{\sqrt{2c_2+\frac{4}{\lambda^2}}} - c_3 e^{-\sqrt{2c_2+\frac{4}{\lambda^2}}t}(c_5 e^{2\sqrt{2c_2+\frac{4}{\lambda^2}}t} + c_4) + c_6, \tag{60}$$

where $c_1, \cdots, c_6$ are real numbers.

Theorem 4.4. *The parametric equations for V_2-magnetic curves in (H_3, g_1, ∇^0) are given by:*

$$\gamma(t) = \begin{Bmatrix} x(t) = c_1 t - 2\frac{c_3 e^{\sqrt{2c_2+\frac{4}{\lambda^2}}t}}{\sqrt{2c_2+\frac{4}{\lambda^2}}} + 2\frac{c_4 e^{-\sqrt{2c_2+\frac{4}{\lambda^2}}t}}{\sqrt{2c_2+\frac{4}{\lambda^2}}} \\ y(t) = c_3 e^{\sqrt{(2c_2+\frac{4}{\lambda^2})t}} + c_4 e^{-\sqrt{(2c_2+\frac{4}{\lambda^2})t}} \\ z(t) = (c_2 - 4c_3c_4)t + \frac{e^{-2\sqrt{2c_2+\frac{4}{\lambda^2}}t}(c_3^2 e^{4\sqrt{2c_2+\frac{4}{\lambda^2}}} - c_4^2)}{\sqrt{2c_2+\frac{4}{\lambda^2}}} \\ -c_3 e^{-\sqrt{2c_2+\frac{4}{\lambda^2}}t}(c_5 e^{2\sqrt{2c_2+\frac{4}{\lambda^2}}t} + c_4) + c_6 \end{Bmatrix}^T, \tag{61}$$

where $c_1, \cdots, c_6$ are real numbers.

4.3. V_3-magnetic curves associate to ∇^0.

For V_3-magnetic curves associated with the Killing vector field $V_3 = \frac{1}{2}xe_1 + e_2 - \frac{\lambda}{6}ze_3$ (see (15)), the wedge product is computed as:

$$V_3 \wedge \mathbf{t} = (\frac{x'}{\lambda} + \frac{\lambda}{6}zy')e_1 - (\frac{\lambda}{6}z(z' + xy') + \frac{1}{3\lambda}xx')e_2 + ((z' + xy') - \frac{1}{3}xy')e_3. \tag{62}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_2 \wedge \mathbf{t}$ yields the system of differential equations (S_3):

$$S_3 : \begin{cases} (z' + xy')' - \dfrac{y'x'}{2} = \dfrac{x'}{\lambda} + \dfrac{\lambda}{6} zy', \\ y'' + \dfrac{1}{2}(z' + xy')x' = -\dfrac{\lambda}{6} z(z' + xy') - \dfrac{1}{3\lambda} xx', \\ \dfrac{x''}{\lambda} = (z' + xy') - \dfrac{1}{3} xy'. \end{cases} \tag{63}$$

The general solution of (S_3) is non-trivial and not fully solvable in closed form. However, under the special assumption $x(t) = y(t) = z(t)$, the third equation in (S_3) become $x'' + (\frac{3\lambda}{2}x + \frac{5\lambda}{6}x^2 + \frac{c_1\lambda}{2})x' - (\frac{1}{\lambda} + \frac{\lambda}{6}x)x' = 0$ which contains Jacobi elliptic functions as the solution. So, we express the following theorem.

Theorem 4.5. *V_3-magnetic curves in (H_3, g_1, ∇^0) corresponding to the Killing vector field $V_3 = \frac{1}{3}xe_1 + e_2 - \frac{\lambda}{6}ze_3$ are solutions of the differential system* (63).

4.4. V_4-magnetic curves associate to ∇^0.

For V_4-magnetic curves associated with the Killing vector field $V_4 = \sin\frac{x}{2}e_1 + \cos\frac{x}{2}e_2$ (see (15)), the wedge product is computed as:

$$V_4 \wedge \mathbf{t} = \cos\frac{x}{2}\frac{x'}{\lambda}e_1 - \sin\frac{x}{2}\frac{x'}{\lambda}e_2 + ((z' + xy')\cos\frac{x}{2} - \sin\frac{x}{2}y')e_3. \tag{64}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_4 \wedge \mathbf{t}$ yields the system of differential equations (S_4):

$$S_4 : \begin{cases} (z' + xy')' - \dfrac{y'x'}{2} = \cos\dfrac{x}{2}\dfrac{x'}{\lambda}, \\ y'' + \dfrac{1}{2}(z' + xy')x' = -\sin\dfrac{x}{2}\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} = (z' + xy')\cos\dfrac{x}{2} - \sin\dfrac{x}{2}y'. \end{cases} \tag{65}$$

The general solution of the system (S_4) is highly non-trivial. We investigate special cases where at least one component is linear or constant:

1. If we assume $x = x_0$ (constant). The system reduces to:

$$S_4 : \begin{cases} (z' + xy')' = 0, \\ y'' = 0, \\ (z' + xy')\cos\dfrac{x}{2} - \sin\dfrac{x}{2}y' = 0. \end{cases} \tag{66}$$

   Solving these yields:

$$\begin{cases} x(t) = x_0, \\ y(t) = c_1 t \cot\dfrac{x_0}{2} + c_2, \\ z(t) = c_1(1 - x_0\cot\dfrac{x_0}{2})t + c_3, \end{cases} \tag{67}$$

   where $x_0, c_1, ..., c_3 \in \mathbb{R}$.
2. If we assume $y = y_0$. Then, the first two equations in (S_4) yields $z = \frac{c_1}{2}t + c_2$, substituting into the third equation in (S_4) leads to contradictions in the equations.
3. Setting $z = z_0$, does not simplify the system. We therefore seek a solution in which at least one component of the solution is linear.

- The assumption $x(t) = c_1 t + c_2$ yields a contradiction.
- Conversely, assuming $y(t) = c_1 t + c_2$ and substituting it into the first equation of (S_4) also results in a contradiction.

Theorem 4.6. *V_4-magnetic curves in (H_3, g_1, ∇^0) correspond to the Killing vector field $V_4 = \sin\frac{x}{2}e_1 + \cos\frac{x}{2}e_2$ are solutions of the system of differential equations (65). In particular, the Killing magnetic curve in (H_3, g_1, ∇^0) with at least one linear component function that corresponds to the Killing vector field $V_4 = \sin\frac{x}{2}e_1 + \cos\frac{x}{2}e_2$ are*

$$\gamma(t) = \left\{x_0,\ y(t) = c_1 t\cot\tfrac{x_0}{2} + c_2,\ c_1(1 - x_0\cot\tfrac{x_0}{2})t + c_3\right\}, \tag{68}$$

where $x_0, c_1, ..., c_3 \in \mathbb{R}$.

4.5. V_5-magnetic curves associate to ∇^0.

For V_5-magnetic curves associated with the Killing vector field $V_5 = \cos\frac{x}{2}e_1 - \sin\frac{x}{2}e_2$ (see (15)), the wedge product is computed as:

$$V_5 \wedge \mathbf{t} = -\sin\frac{x}{2}\frac{x'}{\lambda}e_1 - \cos\frac{x}{2}\frac{x'}{\lambda}e_2 - ((z' + xy')\sin\frac{x}{2} + y'\cos\frac{x}{2})e_3. \tag{69}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_5 \wedge \mathbf{t}$ yields the system of differential equations (S_5):

$$S_5 : \begin{cases} (z' + xy')' - \dfrac{y'x'}{2} = -\sin\dfrac{x}{2}\dfrac{x'}{\lambda}, \\ y'' + \dfrac{1}{2}(z' + xy')x' = -\cos\dfrac{x}{2}\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} = -(z' + xy')\sin\dfrac{x}{2} - y'\cos\dfrac{x}{2}. \end{cases} \tag{70}$$

Analogously to Subsection 4.4, we investigate special cases where components exhibit linear or constant behavior:

Theorem 4.7. *V_5-magnetic curves in (H_3, g_1, ∇^0) correspond to solutions of the differential system* (70) *associated with the Killing vector field $V_5 = \cos\frac{x}{2}e_1 - \sin\frac{x}{2}e_2$. In particular, when $x(t) = x_0$, the curves admit parametric solutions with linear components:*

$$\gamma(t) = \left\{x_0,\ c_1 t\tan\tfrac{x_0}{2} + c_2,\ c_1(1 - x_0\tan\tfrac{x_0}{2})t + c_3\right\}, \tag{71}$$

where $x_0, c_1, ..., c_3 \in \mathbb{R}$.

4.6. V_1-magnetic curves associate to ∇^1.

For V_1-magnetic curves associated with the Killing vector field $V_1 = e_1$ (see (30)), the wedge product is computed as:

$$V_1 \wedge \mathbf{t} = -\frac{x'}{\lambda}e_2 - y'e_3. \tag{72}$$

The Lorentz equation $\nabla^1_{\mathbf{t}}\mathbf{t} = V_1 \wedge \mathbf{t}$ yields the system of differential equations (S_1):

$$S_1 : \begin{cases} (z' + xy')' - x'y' = 0, \\ y'' = -\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} = -y'. \end{cases} \tag{73}$$

Solving the last two equations of (S_1) gives:

$$\begin{cases} x(t) = -\lambda c_1 e^t - \lambda c_2 e^{-t} + c_3, \\ y(t) = c_1 e^t - c_2 e^{-t} + c_4, \end{cases} \tag{74}$$

where $c_1, \cdots, c_4 \in \mathbb{R}$. Integrating the first equation of (S_1) yields:

$$z(t) = \frac{\lambda c_1^2}{4} e^{2t} - \frac{\lambda c_2^2}{4} e^{-2t} - c_1 c_3 e^t + c_2 c_3 e^{-t} + (c_5 + 2\lambda c_1 c_2)t + c_6, \tag{75}$$

where $c_5, c_6 \in \mathbb{R}$.

Theorem 4.8. *All V_1–magnetic curves on $(\mathbb{H}_3, g_1, \nabla^1)$ corresponding to the Killing vector field $V_2 = e_1$ are parametrized by:*

$$\gamma(t) = \left\{ \begin{array}{c} x(t) = -\lambda c_1 e^t - \lambda c_2 e^{-t} + c_3 \\ y(t) = c_1 e^t - c_2 e^{-t} + c_4 \\ z(t) = \frac{\lambda c_1^2}{4} e^{2t} - \frac{\lambda c_2^2}{4} e^{-2t} - c_1 c_3 e^t + c_2 c_3 e^{-t} + (c_5 + 2\lambda c_1 c_2)t + c_6 \end{array} \right\}^T, \tag{76}$$

where $c_1, \cdots, c_6 \in \mathbb{R}$ are constants.

4.7. V_2-magnetic curves associate to ∇^1.

For V_2-magnetic curves associated with the Killing vector field e_3 (see (30)), the wedge product is computed as:

$$V_2 \wedge \mathbf{t} = -y' e_1 + (z' + xy') e_2. \tag{77}$$

The Lorentz equation $\nabla^1_{\mathbf{t}} \mathbf{t} = V_2 \wedge \mathbf{t}$ yields the system of differential equations (S_2):

$$S_2 : \begin{cases} (z' + xy')' - x'y' = -y' \\ y'' = z' + xy', \\ \frac{x''}{\lambda} = 0. \end{cases} \tag{78}$$

Integrating the third equation of (S_2) gives:

$$x(t) = c_1 t + c_2, \tag{79}$$

where $c_1, c_2 \in \mathbb{R}$. Substituting (79) into the first two equations of (S_2) reduces the system to:

$$y''' + (1 - c_1) y' = 0. \tag{80}$$

1. If $c_1 = 1$. The equation simplifies to $y''' = 0$, yielding:

   $$y(t) = c_3 t^2 + c_4 t + c_5.$$

   Substituting into the first equation of (S_2), we solve for $z(t)$:

   $$z(t) = -\frac{2}{3} c_1 c_3 t^3 - \frac{c_1 c_4 + 2 c_2 c_3}{2} t^2 + (2c_3 - c_2 c_4)t + c_6,$$

   where $c_1, \cdots, c_6 \in \mathbb{R}$.

2. If $c_1 > 1$, we have $y''' + (1 - c_1)y' = 0$. The characteristic equation yields exponential solutions:

$$y(t) = \frac{c_3 e^{\sqrt{c_1 - 1}t}}{\sqrt{(c_1 - 1)}} - \frac{c_4 e^{-\sqrt{(c_1-1)}t}}{\sqrt{(c_1 - 1)}} + c_5.$$

The corresponding $z(t)$ is:

$$\begin{aligned} z(t) =& c_3 e^{\sqrt{c_1-1}t} + c_4 e^{-\sqrt{c_1-1}t} - \frac{c_3((\sqrt{c_1 - 1}t - 1)c_1 + \sqrt{c_1 - 1}c_2)}{c_1 - 1} e^{\sqrt{c_1-1}t} \\ &+ \frac{c_4(c_1 + c_1\sqrt{c_1 - 1}t + c_2\sqrt{c_1 - 1})}{c_1 - 1} e^{-\sqrt{c_1-1}t} + c_6, \end{aligned} \tag{81}$$

where $c_1, \cdots, c_6 \in \mathbb{R}$.

3. If $c_1 < 1$, we have $y''' + (1 - c_1)y' = 0$. The characteristic equation yields trigonometric solutions:

$$y(t) = c_3 \cos\sqrt{1 - c_1}t + c_4 \sin\sqrt{1 - c_1}t + c_5,$$

The corresponding $z(t)$ is:

$$\begin{aligned} z(t) =& \sin\sqrt{(c_1 - 1)}t\left(\frac{c_1c_3}{\sqrt{(c_1 - 1)}} - c_3\sqrt{(c_1 - 1)} - c_2c_4 - c_1c_4t\right) \\ &+ \cos\sqrt{(c_1 - 1)}t\left(\frac{c_1c_4}{\sqrt{(c_1 - 1)}} + c_4\sqrt{(c_1 - 1)} - c_2c_3 - c_1c_3t\right) + c_6, \end{aligned}$$

where $c_1, \cdots, c_6 \in \mathbb{R}$.

Hence, we write the following theorem.

Theorem 4.9. *All V_2−magnetic curves of $(\mathbb{H}_3, g_1, \nabla^1)$ corresponding to the Killing vector field $V_2 = e_3$ are parametrized by:*

1. *If $c_1 = 1$:*

$$\gamma(t) = \begin{Bmatrix} x(t) = c_1t + c_2 \\ y(t) = c_3 + c_4t + c_5t^2 \\ z(t) = -\frac{2}{3}c_1c_5t^3 - \frac{c_1c_4 + 2c_2c_5}{2}t^2 + (2c_5 - c_2c_4)t + c_6 \end{Bmatrix}^T. \tag{82}$$

2. *If $c_1 > 1$:*

$$\gamma(t) = \begin{Bmatrix} x(t) = c_1t + c_2, \\ y(t) = \frac{c_3 e^{\sqrt{(c_1-1)}t}}{\sqrt{c_1 - 1}} - \frac{c_4 e^{-\sqrt{(c_1-1)}t}}{\sqrt{c_1 - 1}} + c_5, \\ z(t) = c_3 e^{\sqrt{c_1-1}t} + c_4 e^{-\sqrt{c_1-1}t} - \frac{c_3((\sqrt{c_1 - 1}t - 1)c_1 + \sqrt{c_1 - 1}c_2)}{c_1 - 1} e^{\sqrt{c_1-1}t} \\ + \frac{c_4(c_1 + c_1\sqrt{c_1 - 1}t + c_2\sqrt{c_1 - 1})}{c_1 - 1} e^{-\sqrt{c_1-1}t}. \end{Bmatrix}^T. \tag{83}$$

3. *If $c_1 < 1$:*

$$\gamma(t) = \begin{Bmatrix} x(t) = c_1t + c_2, \\ y(t) = c_3 \cos\sqrt{1 - c_1}t + c_4 \sin\sqrt{1 - c_1}t + c_5, \\ z(t) = \sin\sqrt{(c_1 - 1)}t\left(\frac{c_1c_3}{\sqrt{(c_1 - 1)}} - c_3\sqrt{(c_1 - 1)} - c_2c_4 - c_1c_4t\right) \\ + \cos\sqrt{(c_1 - 1)}t\left(\frac{c_1c_4}{\sqrt{(c_1 - 1)}} + c_4\sqrt{(c_1 - 1)} - c_2c_3 - c_1c_3t\right) + c_6, \end{Bmatrix}^T, \tag{84}$$

where $c_1, \cdots, c_6 \in \mathbb{R}$ are constants.

4.8. V_3-magnetic curves associate to ∇^1.

For V_3-magnetic curves associated with the Killing vector field $V_3 = xe_1 + e_2$ (see (30)), the wedge product is computed as:

$$V_3 \wedge \mathbf{t} = \frac{x'}{\lambda}e_1 - \frac{xx'}{\lambda}e_2 + ((z' + xy') - xy')e_3. \tag{85}$$

The Lorentz equation $\nabla^1_{\mathbf{t}}\mathbf{t} = V_3 \wedge \mathbf{t}$ yields the system of differential equations (S_3):

$$S_3 : \begin{cases} (z' + xy')' - x'y' = \dfrac{x'}{\lambda}, \\ y'' = -\dfrac{xx'}{\lambda}, \\ \dfrac{x''}{\lambda} = (z' + xy') - xy'. \end{cases} \tag{86}$$

Integrating the second equation of (S_3) gives:

$$y' = -\frac{x^2}{2\lambda} + c_1, \tag{87}$$

where $c_1 \in \mathbb{R}$. Substituting (87) into (S_3) reduces the system to:

$$\begin{cases} z' + xy' = -\dfrac{x'''}{6\lambda} + (c_1 + \dfrac{1}{\lambda})x + c_2, \\ \dfrac{x''}{\lambda} - \dfrac{x^3}{3\lambda} - \dfrac{x}{\lambda} - c_2 = 0. \end{cases} \tag{88}$$

In this case, the equation $\frac{x''}{\lambda} - \frac{x^3}{3\lambda} - \frac{x}{\lambda} + c_2 = 0$ involves Jacobi elliptic functions, indicating non-elementary solutions.

Theorem 4.10. *V_3-magnetic curves in $(\mathbb{H}_3, g_1, \nabla^1)$ corresponding to the Killing vector field $V_3 = xe_1 + e_2$ are solutions of the differential system* (86)*. In particular, the system admits solutions expressed in terms of Jacobi elliptic functions when $x(t)$ satisfies the nonlinear equation $\frac{x''}{\lambda} - \frac{x^3}{3\lambda} - \frac{x}{\lambda} + c_2 = 0$.*

4.9. V_4-magnetic curves associate to ∇^1.

For V_4-magnetic curves associated with the Killing vector field $V_4 = \frac{x^2}{2}e_1 + xe_2 + \lambda ye_3$ (see (30)), the wedge product is computed as:

$$V_4 \wedge \mathbf{t} = (\frac{xx'}{\lambda} - \lambda yy')e_1 + (\lambda y(z' + xy') - \frac{x^2x'}{2\lambda})e_2 + (x(z' + xy') - \frac{x^2y'}{2})e_3. \tag{89}$$

The Lorentz equation $\nabla^1_{\mathbf{t}}\mathbf{t} = V_4 \wedge \mathbf{t}$ yields the system of differential equations (S_4):

$$S_4 : \begin{cases} (z' + xy')' - x'y' = \dfrac{xx'}{\lambda} - \lambda yy', \\ y'' = \lambda y(z' + xy') - \dfrac{x^2x'}{2\lambda}, \\ \dfrac{x''}{\lambda} = x(z' + xy') - \dfrac{x^2y'}{2}. \end{cases} \tag{90}$$

The general solution of (S_4) is non-trivial. We investigate special cases where components exhibit linear or constant behavior:

1. **Case** $x(t) = x_0$ (constant): Substituting into the third equation of (S_4) yields $z' + x_0 y' = \frac{x_0}{2} y'$. Comparing coefficients in the first two equations leads to $-\frac{x_0^2}{4} = 1$, which is a contradiction.
2. **Case** $y(t) = y_0$ (constant): Integrating the first equation of (S_4) gives $z' = \frac{x^2}{2\lambda} + c_1$. Substituting into the third equation yields $\frac{x''}{\lambda} - \frac{x^3}{2\lambda} - c_1 x$, which is a nonlinear oscillator equation involving Jacobi elliptic functions.
3. **Other linear assumptions**: Assuming $z(t) = z_0$, does not simplify the system. Assuming $x(t) = c_1 t + c_2$ or $y = c_1 t + c_2$ leads to contradictions in the equations.

Theorem 4.11. *V_4-magnetic curves in $(\mathbb{H}_3, g_1, \nabla^1)$ corresponding to the Killing vector field $V_4 = \frac{x^2}{2} e_1 + x e_2 + \lambda y e_3$ are solutions of the differential system* (90). *In particular, there exist no V_4-magnetic curves with at least one linear component function.*

4.10. V_5-magnetic curves associate to ∇^1.

For V_5-magnetic curves associated with the Killing vector field $V_5 = (\frac{x^3}{3} - 2x)e_1 + x^2 e_2 - 2\lambda z e_3$ (see (30)), the wedge product is computed as:

$$V_5 \wedge \mathbf{t} = (\frac{x'}{\lambda} x^2 + 2\lambda z y')e_1 - (2\lambda z(z' + xy') + (\frac{x^3}{3} - 2x)\frac{x'}{\lambda})e_2 + (x^2(z' + xy') - (\frac{x^3}{3} - 2x)y')e_3. \tag{91}$$

The Lorentz equation $\nabla^1_{\mathbf{t}} \mathbf{t} = V_3 \wedge \mathbf{t}$ yields the system of differential equations (S_5):

$$S_5 : \begin{cases} (z' + xy')' - x'y' = \dfrac{x'}{\lambda} x^2 + 2\lambda z y', \\ y'' = -2\lambda z(z' + xy') - (\dfrac{x^3}{3} - 2x)\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} = x^2(z' + xy') - (\dfrac{x^3}{3} - 2x)y'. \end{cases} \tag{92}$$

The general solution of (S_5) is non-trivial. We investigate the special case when $x(t) = y(t) = z(t)$. Substituting into (S_5) reduces the system to a nonlinear equation:

$$y'' + \frac{1}{3\lambda} y^3 y' + 2\lambda y^2 y' + (2\lambda - \frac{2}{\lambda}) y y' = 0$$

which admits solutions expressed in terms of Jacobi elliptic functions.

Theorem 4.12. *V_5-magnetic curves in $(\mathbb{H}_3, g_1, \nabla^1)$ corresponding to the Killing vector field $V_5 = (\frac{x^3}{3} - 2x)e_1 + x^2 e_2 - 2\lambda z e_3$ are solutions of the differential system* (92). *In particular, when $x(t) = y(t) = z(t)$, the curves are governed by a nonlinear oscillator equation solvable via Jacobi elliptic functions.*

5. The metric g_2

An orthonormal basis for the Lie algebra $(\mathbb{H}_3, g_2)$ is provided by:

$$e_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_2 = \lambda \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial z}, \tag{93}$$

where e_3 is timelike. The non-vanishing components of the Levi-Civita connection ∇ of g_2 are:

$$\begin{aligned} \nabla_{e_1} e_2 &= -\nabla_{e_2} e_1 = \frac{\lambda}{2} e_3, \\ \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = \frac{\lambda}{2} e_2, \\ \nabla_{e_2} e_3 &= \nabla_{e_3} e_2 = -\frac{\lambda}{2} e_1. \end{aligned} \tag{94}$$

Using the connection formula (4), we derive:

$$\nabla^0_{e_3} e_1 = \frac{\lambda}{2} e_2, \quad \nabla^0_{e_3} e_2 = -\frac{\lambda}{2} e_1, \quad \nabla^1_{e_i} e_j = 0. \tag{95}$$

The Kobayashi-Nomizu connection associated with g_2 vanishes identically. We focus on $(\mathbb{H}_3, g_2, \nabla^0)$ where the Killing vector fields are generated by the following proposition.

Proposition 5.1. *The Lie algebra of Killing vector fields for* $(\mathbb{H}_3, g_2, \nabla^0)$ *is 4-dimensional, with basis:*

$$\begin{aligned} &V_1 = \frac{\partial}{\partial z}, \quad V_2 = (\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) + \frac{1}{2}x\frac{\partial}{\partial z}, \quad V_3 = \lambda\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z}, \\ &V_4 = x(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}) - \lambda^2 y\frac{\partial}{\partial x} + (\frac{x^2}{4} + \frac{\lambda^2}{4}y^2)\frac{\partial}{\partial z}. \end{aligned} \tag{96}$$

Proof. Let $V = f_1 e_1 + f_2 e_2 + f_3 e_3$ be a Killing vector field of $(\mathbb{H}_3, g_2, \nabla^0)$. Substituting V into (8) with $Y = e_i$, $Z = e_j$ defined by (5), yields a system of differential equations for Killing vector fields associated with ∇^0:

$$(\mathcal{L}_V g_2)(e_1, e_1) = \frac{\partial}{\partial y} f_1 - x\frac{\partial}{\partial z} f_1 = 0, \tag{97}$$

$$(\mathcal{L}_V g_2)(e_1, e_2) = \frac{\partial}{\partial y} f_2 - x\frac{\partial}{\partial z} f_2 + \lambda\frac{\partial}{\partial x} f_1 = 0, \tag{98}$$

$$(\mathcal{L}_V g_2)(e_1, e_3) = \frac{\partial}{\partial z} f_1 - \frac{\lambda}{2} f_2 - \frac{\partial}{\partial y} f_3 = 0, \tag{99}$$

$$(\mathcal{L}_V g_2)(e_2, e_2) = \frac{\partial}{\partial x} f_2 = 0, \tag{100}$$

$$(\mathcal{L}_V g_2)(e_2, e_3) = -\lambda\frac{\partial}{\partial x} f_3 + \frac{\lambda}{2} f_1 + \frac{\partial}{\partial z} f_2 = 0, \tag{101}$$

$$(\mathcal{L}_V g_2)(e_3, e_3) = \frac{\partial}{\partial z} f_3 = 0, \tag{102}$$

where $\mathcal{L}_V g_2$ is the Lie derivative in the direction of V.

Differentiating (101) with respect to x and z yields:

$$\frac{\partial^2 f_1}{\partial x \partial z} = 0. \tag{103}$$

Differentiating (99) with respect to x and incorporating (103) gives:

$$\frac{\partial f_3}{\partial x \partial y} = 0. \tag{104}$$

Differentiationg (97) with respect to x, and using (103), (104), leads to:

$$\frac{\partial f_1}{\partial z} = 0, \tag{105}$$

implying $f_1 = f_1(x)$, then we can put $f_1(x) = A(x)$. Differentiating (98) twice with respect to x gives:

$$\frac{\partial^3 f_1}{\partial x^3} = 0, \tag{106}$$

implying $f_1(x) = c_1 x^2 + c_2 x + c_3$, where $c_1, c_2, c_3 \in \mathbb{R}$. Differentiating (99) with respect to z gives:

$$\frac{\partial f_2}{\partial z} = 0, \tag{107}$$

so $f_2 = f_2(y)$, then we put $f_2(y) = B(y)$. Differentiating (98) with respect to y leads:

$$\frac{\partial^2 f_2}{\partial y^2} = 0. \tag{108}$$

Thus, $f_2(y) = c_4 y + c_5$, where $c_4, c_5 \in \mathbb{R}$. Using (98), we deduce:

$$\frac{\partial f_2}{\partial y} = -\lambda A'(x), \tag{109}$$

implying $c_1 = 0$ and $c_4 = -\lambda c_2$. Differentiating (99) with respect to x leads to:

$$\frac{\partial^2 f_3}{\partial x \partial y} = 0. \tag{110}$$

So $f_3(x, y) = C_1(x) + C_2(y) + c_6$. From (99) and (101), we infer:

$$\begin{aligned} C_1(x) &= \frac{c_2}{4}x^2 + \frac{c_3}{2}x, \\ C_2(y) &= \frac{\lambda^2}{4}c_4 y^2 - \frac{\lambda}{2}c_5 y. \end{aligned} \tag{111}$$

Substituting these into f_1, f_2, f_3 results in:

$$\begin{aligned} f_1(x) &= c_2 x + c_3, \\ f_2(y) &= -\lambda c_4 y + c_5, \\ f_3(x, y) &= \frac{c_2}{4}x^2 + \frac{c_3}{2}x + \frac{\lambda^2}{4}c_4 y^2 - \frac{\lambda}{2}c_5 y + c_6. \end{aligned} \tag{112}$$

Expressing V in terms of the basis $\{e_1, e_2, e_3\}$ yields the announced result. □

Let $\gamma(t) : I \subset \mathbb{R} \to (\mathbb{H}_3, g_2)$ be a regular curve parameterized by $\gamma(t) = (x(t), y(t), z(t))$. Its velocity vector is given by:

$$\mathbf{t} = \gamma'(t) = (x'(t), y'(t), z'(t)). \tag{113}$$

Expressed in the basis $\{e_i\}_{i=1,2,3}$ defined in (93), the velocity vector $\mathbf{t}$ take the form:

$$\mathbf{t} = y'e_1 + \frac{x'}{\lambda}e_2 + (z' + xy')e_3. \tag{114}$$

Using the connection formulas from (95), we compute the covariant derivatives:

$$\nabla^0_{\mathbf{t}}\mathbf{t} = (y'' - \frac{x'}{2}(z' + xy'))e_1 + (\frac{x''}{\lambda} + \frac{\lambda}{2}y'(z' + xy'))e_2 + (z' + xy')'e_3. \tag{115}$$

5.1. V_1-magnetic curves associate to ∇^0.

For V_1-magnetic curves corresponding to the Killing vector field $V_1 = e_3$ (see (96)), the wedge product is computed as:

$$V_1 \wedge \mathbf{t} = -\frac{x'}{\lambda}e_1 + y'e_2. \tag{116}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_1 \wedge \mathbf{t}$ yields the system (S_1):

$$S_1 : \begin{cases} y'' - \dfrac{x'}{2}(z' + xy') = -\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} + \dfrac{\lambda}{2}y'(z' + xy') = y', \\ (z' + xy')' = 0. \end{cases} \tag{117}$$

Integrating the third equation of (S_1) gives:

$$z' + xy' = c_1, \tag{118}$$

where $c_1 \in \mathbb{R}$. Substituting (118) into the first two equations of (S_1) reduces the system to:

$$\begin{cases} y'' - \dfrac{c_1}{2}x' = -\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} + \dfrac{\lambda c_1}{2}y' = y'. \end{cases} \tag{119}$$

Differentiating the first equation of (119) yields:

$$y''' + (\frac{\lambda c_1 - 2}{2})^2 y' = 0. \tag{120}$$

1. **Case** $\lambda c_1 - 2 = 0$:
   The characteristic equation yields linear solutions:

$$\begin{cases} x(t) = c_2 t + c_3, \\ y(t) = c_4 t + c_5. \end{cases} \tag{121}$$

   Substituting into (118) gives:

$$z(t) = -\frac{c_2 c_4}{2}t^2 + (c_1 - c_3 c_4)t + c_6, \tag{122}$$

   where $c_1, \cdots, c_6 \in \mathbb{R}$.
2. **Case** $\lambda c_1 - 2 \neq 0$:
   The characteristic equation yields trigonometric solutions:

$$y(t) = c_2 \sin \frac{\lambda c_1 - 2}{2}t - c_3 \cos \frac{\lambda c_1 - 2}{2}t + c_4. \tag{123}$$

   Substituting into (S_1) gives:

$$x(t) = \frac{c_2}{\lambda} \cos \frac{\lambda c_1 - 2}{2}t + \frac{c_3}{\lambda} \sin \frac{\lambda c_1 - 2}{2}t + c_5, \tag{124}$$

   Using (118), $z(t)$ becomes:

$$\begin{aligned} z(t) =& \frac{2c_2^2 - 2c_3^2}{\lambda^2 c_1 - 2\lambda} \cos 2\frac{\lambda c_1 - 2}{2}t + \frac{2c_2c_3}{\lambda^2 c_1 - 2\lambda} \sin \frac{\lambda c_1 - 2}{2}t + \frac{2\lambda c_3 c_5 - 2c_2 c_4}{\lambda^2 c_1 - 2\lambda} \sin \frac{\lambda c_1 - 2}{2}t, \\ &- \frac{2\lambda c_2 c_5 + c_3 c_4}{\lambda^2 c_1 - 2\lambda} \cos \frac{\lambda c_1 - 2}{2}t + (c_1 - c_4 c_5)t + c_6 \end{aligned} \tag{125}$$

   where $c_1, \cdots, c_6 \in \mathbb{R}$.

Theorem 5.2. *All V_1-magnetic curves in $(\mathbb{H}_3, g_2, \nabla^0)$ corresponding to the Killing vector field $V_1 = e_3$ are parametrized by:*

1. ***Case*** $\lambda c_1 - 2 = 0$*:*

$$\gamma(t) = \left\{ c_2 t + c_3,\ c_4 t + c_5,\ -\frac{c_2 c_4}{2}t^2 + (c_1 - c_3 c_4)t + c_6 \right\}. \tag{126}$$

2. ***Case*** $\lambda c_1 - 2 \neq 0$*:*

$$\gamma(t) = \left\{\begin{array}{c} x(t) = \frac{c_2}{\lambda}\cos\frac{\lambda c_1-2}{2}t + \frac{c_3}{\lambda}\sin\frac{\lambda c_1-2}{2}t + c_5 \\ y(t) = c_2\sin\frac{\lambda c_1 - 2}{2}t - c_3\cos\frac{\lambda c_1-2}{2}t + c_4 \\ z(t) = \frac{2c_2^2 - 2c_3^2}{\lambda^2 c_1 - 2\lambda}\cos 2\frac{\lambda c_1 - 2}{2}t + \frac{2c_2c_3}{\lambda^2 c_1 - 2\lambda}\sin 2\frac{\lambda c_1-2}{2}t + \frac{2\lambda c_3c_5 - 2c_2c_4}{\lambda^2 c_1 - 2\lambda}\sin\frac{\lambda c_1 - 2}{2}t \\ -\frac{2\lambda c_2c_5 + c_3c_4}{\lambda^2 c_1 - 2\lambda}\cos\frac{\lambda c_1-2}{2}t + (c_1 - c_4c_5)t + c_6 \end{array}\right\}^T, \tag{127}$$

where $c_1, \cdots, c_6 \in \mathbb{R}$ *are constants.*

5.2. V_2-magnetic curves associate to ∇^0.

For V_2-magnetic curves corresponding to the Killing vector field $V_2 = e_1 + \frac{x}{2}e_3$ (see (96)), the wedge product is computed as:

$$V_2 \wedge \mathbf{t} = -\frac{xx'}{2\lambda}e_1 + (\frac{1}{2}xy' - (z' + xy'))e_2 - \frac{x'}{\lambda}e_3. \tag{128}$$

The Lorentz equation $\nabla^0_{\mathbf{t}}\mathbf{t} = V_2 \wedge \mathbf{t}$ yields the system (S_2):

$$S_2 : \begin{cases} y'' - \frac{x'}{2}(z' + xy') = -\frac{xx'}{2\lambda}, \\ \frac{x''}{\lambda} + \frac{\lambda}{2}y'(z' + xy') = (\frac{1}{2}x'y' - (z' + xy')), \\ (z' + xy')' = -\frac{x'}{\lambda}. \end{cases} \tag{129}$$

Integrating the third equation of (S_2) gives:

$$z' + xy' = -\frac{x}{\lambda} + c, \tag{130}$$

where $c \in \mathbb{R}$. Substituting (130) into the first two equations of (S_2) reduces the system to:

$$\begin{cases} y' = -\frac{3}{4\lambda}x^2 + cx + c_1, \\ x'' + \frac{3}{4}x^3 - \frac{11c\lambda}{8}x^2 - (1 + c_1\lambda - \frac{c^2\lambda^2}{2})x + \frac{c\lambda}{2}(2 + c_1\lambda) = 0. \end{cases} \tag{131}$$

Without loss of generality, we set $c = 0$. The system simplifies to:

$$x'' + \frac{3}{4}x^3 - (1 + c_1\lambda)x = 0,$$

which involves Jacobi elliptic functions as solution. So, we can express the following theorem.

Theorem 5.3. *V_2-magnetic curves in (H_3, g_1, ∇^0) corresponding to the Killing vector field $V_2 = e_1 + \frac{x}{2}e_3$ are solutions of the differential system* (129)*. In particular, when $c = 0$, the curves are governed by a nonlinear oscillator equation involving Jacobi elliptic functions.*

5.3. V_3-magnetic curves associate to ∇^0.

For V_3-magnetic curves corresponding to the Killing vector field $V_3 = e_2 - \frac{\lambda}{2} y e_3$ (see (96)), the wedge product is computed as:

$$V_3 \wedge \mathbf{t} = (z' + xy' + \frac{x'y}{2})e_1 - \frac{\lambda y y'}{2} e_2 + y' e_3. \tag{132}$$

The Lorentz equation $\nabla_{\mathbf{t}} \mathbf{t} = V_3 \wedge \mathbf{t}$ yields the system (S_3):

$$S_3 : \begin{cases} y'' - \frac{x'}{2}(z' + xy') = z' + xy' + \frac{xy'}{2}, \\ \frac{x''}{\lambda} + \frac{\lambda}{2} y'(z' + xy') = -\frac{\lambda y y'}{2}, \\ (z' + xy')' = y'. \end{cases} \tag{133}$$

Integrating the third equation of (S_3) gives:

$$z' + xy' = y + c, \tag{134}$$

where $c \in \mathbb{R}$. Substituting (134) into the first two equations of (S_3) reduces the system to:

$$\begin{cases} y'' - (\frac{\lambda^2}{4} y^2 + \lambda^2 c y + c_1)(y + c) = y + c + \frac{1}{2}(\frac{\lambda^2}{4} y^2 + \lambda^2 c y + c_1), \\ x' = -\frac{3\lambda^2}{4} y^2 - \frac{\lambda^2 c}{2} y + c_1. \end{cases} \tag{135}$$

Without loss of generality, set $c = 0$. The system simplifies to:

$$y'' + \frac{3\lambda^2}{4} y^3 - (\lambda c_1 + 1) y = 0,$$

which involves Jacobi elliptic functions as solution. So, we can express the following theorem.

Theorem 5.4. *V_3-magnetic curves in (H_3, g_2, ∇^0) corresponding to the Killing vector field $V_3 = e_2 - \frac{\lambda}{2} y e_3$ are solutions of the differential system* (133). *In particular, when $c = 0$, the curves are governed by a nonlinear oscillator equation involving Jacobi elliptic functions.*

5.4. V_4-magnetic curves associate to ∇^0.

For V_4-magnetic curves corresponding to the Killing vector field $V_4 = \frac{x^2}{4} e_1 + x e_2 - \frac{\lambda}{2} y e_3$ (see (96)), the wedge product is computed as:

$$V_4 \wedge \mathbf{t} = (-\lambda y(z' + xy') - (\frac{x^2}{4} + \frac{\lambda^2}{4} y^2)\frac{x'}{\lambda})e_1 + ((\frac{x^2}{4} + \frac{\lambda^2}{4} y^2)y' - x(z' + xy'))e_2 - (\lambda y y' + \frac{x x'}{\lambda})e_3. \tag{136}$$

The Lorentz equation $\nabla^0_{\mathbf{t}} \mathbf{t} = V_4 \wedge \mathbf{t}$ yields the system (S_4):

$$S_4 : \begin{cases} y'' - \frac{x'}{2}(z' + xy') = -\lambda y(z' + xy') - (\frac{x^2}{4} + \frac{\lambda^2}{4} y^2)\frac{x'}{\lambda}, \\ \frac{x''}{\lambda} + \frac{\lambda}{2} y'(z' + xy') = (\frac{x^2}{4} + \frac{\lambda^2}{4} y^2)y' - x(z' + xy'), \\ (z' + xy')' = -(\lambda y y' + \frac{x x'}{\lambda}). \end{cases} \tag{137}$$

Integrating the third equation of (S_4) gives:

$$z' + xy' = -\frac{x^2}{2\lambda} - \frac{\lambda y^2}{2} + c, \tag{138}$$

where $c \in \mathbb{R}$. Substituting into the first two equations of (S_4) yields:

$$\begin{cases} y'' - \dfrac{x'}{2}(-\dfrac{x^2}{2\lambda} - \dfrac{\lambda y^2}{2} + c) = -\lambda y(-\dfrac{x^2}{2\lambda} - \dfrac{\lambda y^2}{2} + c) - (\dfrac{x^2}{4} + \dfrac{\lambda^2}{4}y^2)\dfrac{x'}{\lambda}, \\ \dfrac{x''}{\lambda} + \dfrac{\lambda}{2}y'(-\dfrac{x^2}{2\lambda} - \dfrac{\lambda y^2}{2} + c) = (\dfrac{x^2}{4} + \dfrac{\lambda^2}{4}y^2)y' - x(-\dfrac{x^2}{2\lambda} - \dfrac{\lambda y^2}{2} + c). \end{cases} \tag{139}$$

The general solution of (S_4) is non-trivial. We investigate special cases for $c = \frac{1}{2}, \lambda = 1$. In this case, the system admits a solution:

$$x(t) = \sin\frac{t}{4}, \quad y(t) = \cos\frac{t}{4}. \tag{140}$$

Substituting into the third equation of (S_4) gives:

$$z(t) = \frac{t}{8} - \frac{1}{4}\sin\frac{t}{2} + k_1, \tag{141}$$

where $k_1 \in \mathbb{R}$.

Theorem 5.5. *The space curves parametrized by:*

$$\gamma(t) = \left\{\sin\tfrac{t}{4},\ \cos\tfrac{t}{4},\ \tfrac{t}{8} - \tfrac{1}{4}\sin\tfrac{t}{2} + k_1\right\}, \tag{142}$$

are V_4-magnetic curves in $(\mathbb{H}_3, g_2, \nabla^0)$ for arbitrary $k_1 \in \mathbb{R}$.

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