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Szász-Beta operators linking general-Appell Polynomials

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Abstract. This manuscript associates with a study of general-Appell Polynomials. In this research work, we construct a new sequence of Szász-Beta type operators via general-Appell Polynomials to discuss approximation properties for the Lebesgue integrable functions i.e. $L_1[0,\infty)$. Further, estimates in view of test functions and central moments are studied. Next, rate of convergence is discussed with the aid of Korovkin theorem and Voronovskaja type theorem. Moreover, direct approximation results in terms of modulus of continuity of first and second order, Peetre's K-functional, Lipschitz type space, and the r^{th} order Lipschitz type maximal functions are investigated. In subsequent section, we present weighted approximation results and statistical approximation theorems are discussed.

1. Introduction and Preliminaries

The systematic development of operator theory began in the late 19^{th} century. An important aspect of approximation in operator theory is to find simple, computationally tractable approximations that capture the essential properties of more complex functions. These approximations can then be used for analysis, simulation, or computation in various applications, such as quantum mechanics, signal processing, and control theory. It provides powerful tools for solving problems and analyzing systems involving operators. In the last one decade, there has been continued research and development in approximation theory in operator theory, with a focus on more advanced techniques having applications in data science and machine learning. In approximation theory, Weierstrass (1885) [1] formulated a sophisticated result known as the Weierstrass approximation theorem. Proving this theorem in a more straightforward and comprehensible manner has been the focus of several prominent mathematicians.

Bernstein (1912) [2] developed a sequence of polynomials referred to as Bernstein polynomials in order to give a concise demonstration of the Weierstrass approximation theorem with the aid of binomial distribution as follows:

$$B_\kappa(\bar{g};u) = \sum_{\nu=0}^{\kappa} \bar{g}\left(\frac{\nu}{\kappa}\right)\binom{\kappa}{\nu} u^\nu (1-u)^{\kappa-\nu}, \quad u \in [0,1], \tag{1}$$

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where $\bar{g}$ is a continuous and bounded function on $[0,1]$. The sequences of operators in (1) restrict the approximation for continuous functions on bounded interval $[0,1]$. In order to discuss approximation properties on unbounded interval $[0,\infty)$, Szász [3] provided the modifications to the sequences in (1) which has a significant role to the evolution of operator theory as follows:

$$S_\kappa(\bar{g};u) = e^{-\kappa u}\sum_{\nu=0}^{\infty}\frac{(\kappa u)^\nu}{\nu!}\bar{g}\Big(\frac{\nu}{\kappa}\Big),\ \kappa\in\mathbb{N}, \tag{2}$$

where real valued function $\bar{g}\in C[0,\infty)$. As given in (2), the linear positive operators are solely limited to continuous functional space. Many integral variants of these sequences of operators are obtained in order to approximate the longer class of functions, i.e., the space of Lebesgue measurable functions, e.g., Szász-Kantorovich and Szász-Durremeyer type operators etc. Many mathematicians, e.g., Acu et al. ([4], [5]), Raza et al. ([6], [7], [8]), Mohiuddine et al. ([9], [10]), Kumar et al. [11], Mursaleen et al. ([12], [13]), Özger et al. ([14], [15]), Ayman Mursaleen et al. ([16], [18]), Braha et al. [19], and Khursheed et al. [20], Khan et al. [21], Aslan ([22], [23]), Nasiruzzaman et al. ([24], [25]), Rao et al. ([26], [27], [28], [29]) and Jha et al. [30] provided a number of generalizations for these kinds of sequences to investigate flexibility in approximation properties across several functional spaces. Recently, Raza et al. [31] provided a class of sequence of operators $G_{\kappa,A}(.;.)$, $\kappa\in\mathbb{N}$, given by the formula

$$G_{\kappa,A}(\bar{g};u) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\bar{g}\Big(\frac{\nu}{\kappa}\Big),\quad u\in\mathbb{R}_0^+, \tag{3}$$

where $A_{p,\nu}$ is the two variable Appell polynomials (see [31]).

The operators given by (3) are positive and linear. The basic information about positive linear operators, including their generalizations and applications can be observed in [34].

As the operators described in (3) are limited for continuous function only, we present a sequence of positive linear operators to provide approximations in larger class of functions, i.e., the space of Lebesgue measurable functions $L_1[0,\infty)$, which is termed as Szász-Beta operators in context of general Appell Polynomials. For $\bar{g}\in L_1[0,\infty)$,

$$D_\kappa^A(\bar{g};u) \quad = \quad \sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\bar{g}(y)dy,\quad for\quad u\in\mathbb{R}_0^+, \tag{4}$$

where

$$A_\nu^p(\kappa u,h) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\ \ and\ \ Q_\kappa^\nu(y) = \frac{1}{\beta(\nu+1,\kappa)}\Big[\frac{y^\nu}{(1+y)^{\nu+1+\kappa}}\Big],$$

with β (Beta) function, $\beta(\nu+1,\kappa) = \int_0^\infty \frac{y^\nu}{(1+y)^{\nu+1+\kappa}}dy$.

Lemma 1.1. *The sequence of operators introduced in* (4) *are linear.*

Proof. *Let $\lambda_1,\lambda_2\in\mathbb{R}$ and $\bar{g}_1,\bar{g}_2\in L_1[0,\infty)$. Then, in view of Eq* (4), *we have*

$$\begin{aligned}
D_\kappa^A(\lambda_1\bar{g}_1+\lambda_2\bar{g}_2;u) &= \sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)(\lambda_1\bar{g}_1+\lambda_2\bar{g}_2)(y)dy \\
&= \lambda_1\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\bar{g}_1(y)dy + \lambda_2\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\bar{g}_2(y)dy \\
&= \lambda_1 D_\kappa^A(\bar{g}_1;u)+\lambda_2 D_\kappa^A(\bar{g}_2;u).
\end{aligned}$$

□

Lemma 1.2. *Let $\bar{g}_s(y) = y^s$, $s \in \{0, 1, 2\}$ be the test functions. Then, we have*

$$D_\kappa^A(\bar{g}_s; u) = \sum_{\nu=0}^{\infty} A_\nu^p(\kappa u, h) \frac{(\nu + s)!(\kappa - s - 1)!}{\nu!(\kappa - 1)!}.$$

Proof. We know,

$$\begin{aligned}\int_0^\infty Q_\kappa^\nu(y) f_s(y) dy &= \frac{1}{\beta(\nu + 1, \kappa)} \int_0^\infty \frac{y^{\nu+s}}{(1 + y)^{\nu+1+\kappa}} dy \\ &= \frac{1}{\beta(\nu + 1, \kappa)} \beta(\nu + s + 1, \kappa - s) \\ &= \frac{(\nu + s)!(\kappa - s - 1)!}{\nu!(\kappa - 1)!}.\end{aligned}$$

From (4), *we have*

$$D_\kappa^A(\bar{g}_s; u) = \sum_{\nu=0}^{\infty} A_\nu^p(\kappa u, h) \frac{(\nu + s)!(\kappa - s - 1)!}{\nu!(\kappa - 1)!}.$$

□

Lemma 1.3. *As discussed by Raza et al. in [31], we can have the following equalities:*

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \tilde{\Lambda}(1) e^{\kappa u} \tilde{\xi}(h, 1); \\ \sum_{\nu=0}^{\infty} \nu \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \Big[\kappa u \tilde{\Lambda}(1) \tilde{\xi}(h, 1) + \tilde{\Lambda}(1) \tilde{\xi}'(h, 1) + \tilde{\Lambda}'(1) \tilde{\xi}(h, 1)\Big] e^{\kappa u}; \\ \sum_{\nu=0}^{\infty} \nu^2 \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \Big[(\kappa^2 u^2 + \kappa u) \tilde{\Lambda}(1) \tilde{\xi}(h, 1) + (2\kappa u + 1)[\tilde{\Lambda}(1) \tilde{\xi}'(h, 1) \\ &\quad + \tilde{\Lambda}'(1) \tilde{\xi}(h, 1)] + 2\tilde{\xi}'(h, 1) \tilde{\Lambda}'(1) + \tilde{\xi}''(h, 1) \tilde{\Lambda}(1) + \tilde{\xi}(h, 1) \tilde{\Lambda}''(1)\Big] e^{\kappa u}.\end{aligned}$$

Lemma 1.4. *Let $\bar{g}_s(y) = y^s$, $s \in \{0, 1, 2\}$ be the test functions by* (3). *Then, we have*

$$\begin{aligned}D_\kappa^A(1; u) &= 1; \\ D_\kappa^A(y; u) &= u + \frac{1}{\kappa - 1}\Big[u + \frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1\Big]; \quad \kappa > 1, \\ D_\kappa^A(y^2; u) &= u^2 + \frac{1}{(\kappa - 1)(\kappa - 2)}\Big[(3\kappa - 2)u^2 + 4\kappa u + (2\kappa u + 4)\Big[\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} \\ &+ \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big] + 2\frac{\tilde{\xi}'(h, 1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1)\tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 2\Big]; \quad \kappa > 2,\end{aligned}$$

for each $u \in \mathbb{R}_0^+$.

Proof. In the direction of (4), we have

$$D_\kappa^A(\bar{g}; u) = \sum_{\nu=0}^{\infty} A_\nu^p(\kappa u, h) \int_0^\infty Q_\kappa^\nu(y) \bar{g}(y) dy$$

$$D_\kappa^A(\bar{g}_s;u) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{k=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\frac{(\nu+s)!(\kappa-s-1)!}{\nu!(\kappa-1)!}.$$

Now, for $s = 0$,

$$D_\kappa^A(1;u) = \sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\frac{\nu!(\kappa-1)!}{\nu!(\kappa-1)!} = 1.$$

For $s = 1$,

$$\begin{aligned}
D_\kappa^A(y;u) &= \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\frac{(\nu+1)!(\kappa-2)!}{\nu!(\kappa-1)!}\\
&= \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\frac{\nu+1}{\kappa-1}\\
&= \frac{e^{-\kappa u}}{(\kappa-1)\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\Big[\tilde{\Lambda}(1)\tilde{\xi}'(h,1)+\tilde{\Lambda}'(1)\tilde{\xi}(h,1)+(\kappa u+1)\tilde{\Lambda}(1)\tilde{\xi}(h,1)\Big]e^{\kappa u}\\
&= \frac{1}{\kappa-1}\Big[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big]\\
&= u+\frac{1}{\kappa-1}\Big[u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big].
\end{aligned}$$

For $s = 2$,

$$\begin{aligned}
D_\kappa^A(y^2;u) &= \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\frac{(\nu+2)!(\kappa-3)!}{\nu!(\kappa-1)!}\\
&= \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\frac{(\kappa u+1)(\nu+2)}{(\kappa-1)(\kappa-2)}\\
&= \frac{e^{-\kappa u}}{(\kappa-1)(\kappa-2)\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\Big[(2\kappa u+4)(\tilde{\Lambda}(1)\tilde{\xi}'(h,1)\\
&+ \tilde{\Lambda}'(1)\tilde{\xi}(h,1))+2\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)+\tilde{\xi}''(h,1)\tilde{\Lambda}(1)\\
&+ \tilde{\xi}(h,1)\tilde{\Lambda}''(1)+(\kappa^2u^2+4\kappa u+2)\tilde{\Lambda}(1)\tilde{\xi}(h,1)\Big]e^{\kappa u}\\
&= u^2+\frac{1}{(\kappa-1)(\kappa-2)}\Big[(3\kappa-2)u^2+4\kappa u+(2\kappa u+4)\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}\\
&+ \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big]+2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+2\Big].
\end{aligned}$$

Hence, we complete the proof of Lemma 1.4. □

Lemma 1.5. *Let $\gamma_s(y) = (y-u)^s$, $s = 0,1,2$. Then, for the operators (4), we have central moments $D_\kappa^A(\gamma_s(y),u)$ as:*

$$\begin{aligned}
D_\kappa^A(\gamma_0(y),u) &= 1;\\
D_\kappa^A(\gamma_1(y),u) &= \frac{1}{\kappa-1}\Big[u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big];\kappa>1,\\
D_\kappa^A(\gamma_2(y),u) &= \frac{1}{(\kappa-1)(\kappa-2)}\Big[(\kappa+2)u^2+2u(\kappa+2)+4(u+1)\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}
\end{aligned}$$

$$+ \quad \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Bigg] + 2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + 2\Bigg]; \kappa > 2,$$

$$D_\kappa^A(\gamma_4(y), u) \quad = \quad o\left(\frac{1}{\kappa^2}\right);$$

for each $u \in \mathbb{R}_0^+$.

Proof. With the aid of linearity property and Lemma 1.4, we can easily prove Lemma 1.5. □

In the following sections, we examine the rate of convergence of operators and their approximation order. Specifically, we discuss direct results both locally and globally across several spaces. In the final section, we explore some results of A-Statistical approximation in various functional spaces.

2. Approximation Properties: Uniform Rate of Convergence and Order of Approximation

Definition 2.1. *[34] The modulus of smoothness for* $\bar{g} \in L_1[0,\infty), \delta > 0$ *is given by*

$$\omega(\bar{g};\delta) = \sup_{|u_1-u_2|\le\delta} |\bar{g}(u_1) - \bar{g}(u_2)|, \qquad u_1, u_2 \in [0,\infty).$$

Theorem 2.2. *Let* $D_\kappa^A(.;.)$ *be a sequence of operators described in Eq. (4) and* $C_B[0,\infty)$ *denotes a real valued functional space which acquires bounded and continuous functions. Then, on each closed and bounded subset of* $[0,\infty)$, $D_\kappa^A(\bar{g};.) \rightrightarrows \bar{g}$, *for all* $\bar{g} \in C_B[0,\infty)$, *where* $\rightrightarrows$ *denotes uniform convergence.*

Proof. Considering the classical Korovkin type theorem [32], which characterizes the uniform convergence for the sequence of positive linear operators, it is enough to note that

$$\lim_{\kappa\to\infty} D_\kappa^A(\bar{g}_s;u) = u^s,\ s = 0,1,2,$$

uniformly on all closed and bounded subsets of $[0,\infty)$. We can easily establish this result with the help of Lemma 1.4. □

Now, we show that Voronovskaja type asymptotic approximation theorem for the $D_\kappa^A(.;.)$ given in (4).

Theorem 2.3. *Let* $\bar{g} \in C_B[0,\infty)$ *and* $\bar{g}', \bar{g}''$ *exist at a fixed point* $u \in [0,\infty)$. *Then, we get*

$$\lim_{\kappa\to\infty} \kappa\Big(D_\kappa^A(\bar{g};u) - \bar{g}(u)\Big) = \bar{g}'(u)\Bigg[u + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1\Bigg] + \frac{1}{2}(u^2+2u)\bar{g}''(u).$$

Proof. In accordance with Taylor's formula for the function $\bar{g}$, we have

$$\bar{g}(y) = \bar{g}(u) + (y-u)\bar{g}'(u) + \frac{1}{2}(y-u)^2\bar{g}''(u) + t(y,u)(y-u)^2, \tag{5}$$

where $t(y,u)$ is the Peano remainder and

$$\lim_{y\to u} t(y,u) = 0.$$

Applying operators on both the sides in (5), we yield

$$(D_\kappa^A(\bar{g};u) - \bar{g}(u)) \quad = \quad \bar{g}'(u)D_\kappa^A((y-u);u) + \frac{1}{2}\bar{g}''(u)D_\kappa^A((y-u)^2;u) + D_\kappa^A(t(y,u)(y-u)^2;u).$$

In view of Lemma 1.5

$$\kappa(D_\kappa^A(\bar{g};u) - \bar{g}(u)) \quad = \quad \frac{\kappa\bar{g}'(u)}{\kappa-1}\Bigg[u + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1\Bigg]$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\frac{\kappa\bar{g}''(u)}{(\kappa-1)(\kappa-2)}\Big[(\kappa+2)u^2+2u(\kappa+2)\\
&+ \quad 4(u+1)\Big(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big)+2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\\
&+ \quad \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+2\Big]+\kappa D_\kappa^A(t(y,u)(y-u)^2;u).
\end{aligned}
$$

Operate the limits on both the sides of the above expression, we get

$$
\begin{aligned}
\lim_{\kappa\to\infty}\kappa\Big(D_\kappa^A(\bar{g};u)-\bar{g}(u)\Big) &= \bar{g}'(u)\Big[u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big]+\frac{1}{2}(u^2+2u)\bar{g}''(u)\\
&+ \lim_{\kappa\to\infty}\kappa D_\kappa^A(t(y,u)(y-u)^2;u).
\end{aligned}
$$

Now, we need to show that

$$\lim_{\kappa\to\infty}\kappa D_\kappa^A(t(y,u)(y-u)^2;u)=0.$$

In view of Cauchy-Schwarz inequality, we calculate the last term of the above expression as:

$$\kappa D_\kappa^A(t(y,u)(y-u)^2;u)\leq\sqrt{D_\kappa^A(t^2(y,u);u)}\sqrt{\kappa^2 D_\kappa^A((y-u)^4;u)}. \tag{6}$$

We see that $t^2(u,u)=0$ and $t^2(y,u)\in C_B[0,\infty)$. Thus we have

$$\lim_{\kappa\to\infty}D_\kappa^A(t^2(y,u);u)=t^2(u,u)=0. \tag{7}$$

From (6) and (7) it follows that

$$\lim_{\kappa\to\infty}\kappa D_\kappa^A(t(y,u)(y-u)^2;u)=0.$$

Hence, the proof is completed. □

In accordance with Shisha et al. [33], order of the convergence relative to Ditzian-Totik modulus of continuity can easily be proved.

Theorem 2.4. *Consider $\bar{g}\in C_B[0,\infty)$ and for the operators $D_\kappa^A(.;.)$ presented in Eq. (4), we acquire*

$$|D_\kappa^A(\bar{g};u)-\bar{g}(u)|\leq 2\omega(\bar{g};\delta),$$

where $\delta=\sqrt{D_\kappa^A((y-u)^2;u)}$.

Proof. In accordance with Lemma 1.4, 1.5 and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
|D_\kappa^A(\bar{g};u)-\bar{g}(u)| &\leq D_\kappa^A|(\bar{g}(y)-\bar{g}(u)|;u)\\
&\leq D_\kappa^A\Big(\Big(1+\frac{|y-u|}{\delta}\Big)\omega(\bar{g},\delta);u\Big)\\
&\leq \omega(\bar{g},\delta)\Big[1+\frac{1}{\delta}D_\kappa^A(|y-u|;u)\Big]\\
&\leq \omega(\bar{g},\delta)\Big[1+\frac{1}{\delta}\sqrt{D_\kappa^A((y-u)^2;u)}\Big]
\end{aligned}
$$

By selecting $\delta=\sqrt{D_\kappa^A((y-u)^2;u)}$, we obtained the desired result. □

3. Locally Approximation Results

We recall a few functional spaces and functional relations in this part. Peetre's K-functional [34] is given by

$$K_2(\bar{g},\delta)=\inf_{\bar{h}\in C_B^2[0,\infty)}\Big\{\|\bar{g}-\bar{h}\|_{C_B[0,\infty)}+\delta\|\bar{h}''\|_{C_B^2[0,\infty)}\Big\},$$

where $C_B^2[0,\infty)=\{\bar{h}\in C_B[0,\infty):\bar{h}',\bar{h}''\in C_B[0,\infty)\}$ associated with the norm $\|\bar{g}\|=\sup\limits_{0\le y<\infty}|\bar{g}(y)|$ and second order Ditzian-Totik modulus of smoothness is presented by

$$\omega_2(\bar{g};\sqrt{\delta})=\sup_{0<k\le\sqrt{\delta}}\ \sup_{y\in[0,\infty)}|\bar{g}(y+2k)-2\bar{g}(y+k)+\bar{g}(y)|.$$

We revisit a result from DeVore and Lorentz ([34] page no. 177, Theorem 2.4) as:

$$K_2(\bar{g};\delta)\le C\omega_2(\bar{g};\sqrt{\delta}),\tag{8}$$

where C is an absolute constant. To establish the next result, we consider the auxiliary operator defined as:

$$\widehat{D}_\kappa^A(\bar{g};u)=D_\kappa^A(\bar{g};u)+\bar{g}(u)-\bar{g}\left(\frac{1}{\kappa-1}\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]\right).\tag{9}$$

where $\bar{g}\in C_B[0,\infty)$, $u\ge 0$ and $\kappa>1$. From Eq. (9), one can yield

$$\widehat{D}_\kappa^A(1;u)=1,\ \widehat{D}_\kappa^A(\gamma_1(y);u)=0 \text{ and } |\widehat{D}_\kappa^A(\bar{g};u)|\le 3\|\bar{g}\|.\tag{10}$$

Lemma 3.1. *If $\kappa>2$ and $u\ge 0$, we have*

$$|\widehat{D}_\kappa^A(\bar{g};u)-\bar{g}(u)|\le\theta(u)\|\bar{g}''\|,$$

where $\bar{g}\in C_B^2[0,\infty)$ and $\theta(u)=\widehat{D}_\kappa^A(\gamma_1(y);u)+(\widehat{D}_\kappa^A(\gamma_1(y);u)^2$.

Proof. For $\bar{g}\in C_B^2[0,\infty)$ and by Taylor expansion, we get

$$\bar{g}(y)=\bar{g}(u)+(y-u)\bar{g}'(u)+\int_u^y(y-v)\bar{g}''(v)dv.\tag{11}$$

Implementing the auxiliary operators $\widehat{D}_\kappa^A(.;.)$ introduced in Eq.(9) to both sides of Eq. (11), we obtain

$$\widehat{D}_\kappa^A(\bar{g};u)-\bar{g}(u)\ =\ \bar{g}'(u)\widehat{D}_\kappa^A(\gamma_1(y);u)+\widehat{D}_\kappa^A\Big(\int_u^y(y-v)\bar{g}''(v)dv;u\Big).$$

Using the Eqs. (10) and (11), one yield

$$\begin{aligned}\widehat{D}_\kappa^A(\bar{g};u)-\bar{g}(u)&=\widehat{D}_\kappa^A\Big(\int_u^y(y-v)\bar{g}''(v)dv;u\Big)\\&=D_\kappa^A\Big(\int_u^y(y-v)\bar{g}''(v)dv;u\Big)-\int_u^{\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}}\left(\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}-v\right)\bar{g}''(v)dv,\end{aligned}$$

$$|\widehat{D}_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| \leq \left|\widehat{D}_{\kappa}^{A}\Big(\int_{u}^{y}(y-v)\bar{g}''(v)dv;u\Big)\right|$$
$$+\left|\int_{u}^{\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}}\left(\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}-v\right)\bar{g}''(v)dv\right|. \tag{12}$$

Since,

$$\left|\int_{u}^{y}(y-v)\bar{g}''(v)dv\right| \leq (y-u)^2 \parallel \bar{g}'' \parallel, \tag{13}$$

then

$$\left|\int_{u}^{\frac{1}{\kappa-1}\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}\left(\frac{1}{\kappa-1}\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]-v\right)\bar{g}''(v)dv\right| \tag{14}$$

$$\leq \left(\frac{1}{\kappa-1}\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]-u\right)^2 \parallel \bar{g}'' \parallel . \tag{15}$$

In accordance with (12), (13) and (14), we accqire

$$|\widehat{D}_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| \leq \left\{\widehat{D}_{\kappa}^{A}(\gamma_2(y);u)+\left(\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}-u\right)^2\right\}\|\bar{g}''\|$$
$$=\theta(u)\|\bar{g}''\|.$$

Which proves the required result. □

Theorem 3.2. *Let $\bar{g}\in C_B^2[0,\infty)$. Then, there corresponds a non-negative constant $\tilde{C}>0$ such that*

$$|\ D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)\ | \leq \tilde{C}\omega_2\Big(\bar{g};\ \sqrt{\theta(u)}\Big)+\omega(\bar{g};D_{\kappa}^{A}(\gamma_1(y);u),$$

where $\theta(u)$ is given by in Lemma 3.1.

Proof. For $\bar{h}\in C_B^2[0,\infty)$ and $\bar{g}\in C_B[0,\infty)$ and with the definition of $\widehat{D}_{\kappa}^{A}(.;.)$ given in (9), we get

$$|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| \leq |\widehat{D}_{\kappa}^{A}(\bar{g}-\bar{h};u)|+|(\bar{g}-\bar{h})(u)|+|\widehat{D}_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)|$$
$$+\left|\bar{g}\Big(\frac{1}{\kappa-1}\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]\Big)-\bar{g}(u)\right|.$$

In accordance with Lemma 3.1 and inequalities mentioned in Eq. (10), we acquire

$$|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| \leq 4\|\bar{g}-\bar{h}\|+|\widehat{D}_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)|+\left|\bar{g}\left(\frac{\left[\kappa u+1+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{\kappa-1}\right)-\bar{g}(u)\right|$$
$$\leq 4\|\bar{g}-\bar{h}\|+\theta(u)\|\bar{h}''\|+\omega\Big(\bar{g};D_{\kappa}^{A}(y-u);u\Big).$$

By employing Eq. (8), we established the desired result. □

Now, we address the next result in Lipschitz type space presented by [35] as:

$$Lip_{\tilde{M}}^{\zeta_1,\zeta_2}(\eta):=\Big\{\bar{g}\in C_B[0,\infty):|\bar{g}(y)-\bar{g}(u)|\leq\tilde{M}\frac{|y-u|^{\eta}}{(y+\zeta_1u+\zeta_2u^2)^{\frac{\eta}{2}}}:u,y\in(0,\infty)\Big\},$$

where $\tilde{M}>0$, $0<\eta\leq 1$ and $\zeta_1,\zeta_2>0$.

Theorem 3.3. *Consider sequence of linear positive operators in (4) and $\bar{g}\in Lip_{M}^{\zeta_1,\zeta_2}(\eta)$, one obtain*

$$|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)|\leq\tilde{M}\bigg(\frac{\lambda(u)}{\zeta_1u+\zeta_2u^2}\bigg)^{\frac{\eta}{2}},\tag{16}$$

where $0<\eta\leq 1$, $\zeta_1,\zeta_2\in(0,\infty)$ and $\lambda(y)=D_{\kappa}^{A}(\gamma_2(y);u)$.

Proof. For $\eta=1$ and $u\geq 0$, we get

$$\begin{aligned}|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| &\leq D_{\kappa}^{A}(|\bar{g}(y)-\bar{g}(u)|;u)\\ &\leq \tilde{M}D_{\kappa}^{A}\bigg(\frac{|y-u|}{(y+\zeta_1u+\zeta_2u^2)^{\frac{1}{2}}};u\bigg).\end{aligned}$$

Since $\dfrac{1}{y+\zeta_1u+\zeta_2u^2}<\dfrac{1}{\zeta_1u+\zeta_2u^2}$, for each $y\in(0,\infty)$, we acquire

$$\begin{aligned}|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| &\leq \frac{\tilde{M}}{(\zeta_1u+\zeta_2u^2)^{\frac{1}{2}}}(D_{\kappa}^{A}(\gamma_2(y);u)^{\frac{1}{2}}\\ &\leq \tilde{M}\bigg(\frac{\lambda(u)}{\zeta_1u+\zeta_2u^2}\bigg)^{\frac{1}{2}},\end{aligned}$$

which indicates that the Theorem 3.3 is valid for $\eta=1$. Next, we examine the case where $\eta\in(0,1)$ and in accordance with Hölder's inequality by selecting $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$, we obtain

$$\begin{aligned}|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)| &\leq \Big(|D_{\kappa}^{A}(|\bar{g}(y)-\bar{g}(u)|^{\frac{2}{\eta}};u)\Big)^{\frac{\eta}{2}}\\ &\leq \tilde{M}\bigg(D_{\kappa}^{A}\bigg(\frac{|y-u|^2}{(y+\zeta_1u+\zeta_2u^2)};x\bigg)\bigg)^{\frac{\eta}{2}}.\end{aligned}$$

Since $\dfrac{1}{y+\zeta_1u+\zeta_2u^2}<\dfrac{1}{\zeta_1u+\zeta_2u^2}$, for all $u\in(0,\infty)$, one get

$$|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)|\leq\tilde{M}\bigg(\frac{|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)(|y-u|^2;u)}{\zeta_1u+\zeta_2u^2}\bigg)^{\frac{\eta}{2}}\leq\tilde{M}\Big(\frac{\lambda(u)}{\zeta_1u+\zeta_2u^2}\Big)^{\frac{\eta}{2}}.$$

Thus, we yield the desired result. □

Next, we address the local approximation in terms of the r^{th} order modulus of smoothness, followed by the Lipschitz-type maximal function introduced by Lenze [35] as:

$$\widetilde{\omega}_r(\bar{g};u)=\sup_{y\neq u,y\in(0,\infty)}\frac{|\bar{g}(y)-\bar{g}(u)|}{|y-u|^r},\ u\in[0,\infty)\text{ and }r\in(0,1].\tag{17}$$

Theorem 3.4. *Assume $\bar{g}\in L_1[0,\infty)$ and $r\in(0,1]$. Then, for every $u\in[0,\infty)$, we get*

$$|D_{\kappa}^{A}(\bar{g};u)-\bar{g}(u)|\leq\widetilde{\omega}_r(\bar{g};u)\big(\lambda(u)\big)^{\frac{r}{2}}.$$

Proof. It can be observed that

$$|D_\kappa^A(\bar{g};u)-\bar{g}(u)| \leq D_\kappa^A|\bar{g}(y)-\bar{g}(u)|;u).$$

Using Eq. (17), one get

$$|D_\kappa^A(\bar{g};u)-\bar{g}(u)| \leq \widetilde{\omega}_s(\bar{g};u)D_\kappa^A(|y-u|^r;u).$$

Then by employing Hölder's inequality with $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$, we obtain

$$|D_\kappa^A(\bar{g};u)-\bar{g}(u)| \leq \widetilde{\omega}_r(\bar{g};u)\Big(D_\kappa^A(|y-u|^2;u)\Big)^{\frac{r}{2}}.$$

Thus, we concludes the proof. □

4. Approximation Properties Globally

Consider $\nu(u)=1+u^2, 0\leq u<\infty$ as weight function. Then, $B_\nu[0,\infty)=\{\bar{g}(u):|\bar{g}(u)|\leq \tilde{M}_{\bar{g}}(1+u^2)$, here the constant $\tilde{M}_{\bar{g}}$ depends on $\bar{g}$ and $C_\nu[0,\infty)$ represents the continuous functional space in $B_\nu[0,\infty)$ along with the norm $\|\bar{g}\|_\nu=\sup\limits_{u\in[0,\infty)}\frac{|\bar{g}(u)|}{\nu(u)}$ and $C_\nu^{\tilde{k}}[0,\infty)=\{\bar{g}\in C_\nu[0,\infty):\lim\limits_{u\to\infty}\frac{\bar{g}(u)}{\nu(u)}=\tilde{k}$, where constant $\tilde{k}$ depends on $\bar{g}\}$.

If $\bar{g}$ is a function defined on closed interval $[0,b]$ where $b>0$. Then, Ditzian-Totik modulus of continuity is given by

$$\omega_b(\bar{g},\delta)=\sup_{|y-u|\leq\delta}\sup_{u,y\in[0,b]}|\bar{g}(y)-\bar{g}(u)|. \tag{18}$$

It is straightforward to observe that for any $\bar{g}\in C_\nu[0,\infty)$, the modulus of smoothness defined in Eq. (18) tends to zero.

Theorem 4.1. *Let $\bar{g}\in C_\nu[0,\infty)$ and $\omega_{b+1}(\bar{g};\delta)$ denotes the modulus of smoothness defined on $[0,b+1]\subset[0,\infty)$. Then, for any $y\in[0,b]$, we obtain*

$$\|D_\kappa^A(.;.)-\bar{g}\|_{C[0,b]}\leq 4\tilde{M}_{\bar{g}}(1+b^2)\delta_s(b)+2\omega_{b+1}(\bar{g};\sqrt{\delta_s(b)}),$$

where $\delta_s(b)=\max\limits_{u\in[0,b]}D_\kappa^A(\gamma_2;u)$.

Proof. For any $u\in[0,b]$ and $y\in[0,\infty)$, one has

$$|\bar{g}(y)-\bar{g}(u)|\leq 4\tilde{M}_{\bar{g}}(1+b^2)(y-u)^2+\left(1+\frac{|y-u|}{\delta}\right)\omega_{b+1}(\bar{g};\delta).$$

Implementing operator $D_\kappa^A(.;.)$ on both the sides, we acquire

$$\begin{aligned}|D_\kappa^A(\bar{g};u)-\bar{g}(u)|&\leq 4\tilde{M}_{\bar{g}}(1+b^2)D_\kappa^A(\gamma_2;u)\\&\quad+\left(1+\frac{D_\kappa^A(|y-u|;u)}{\delta}\right)\omega_{b+1}(\bar{g};\delta).\end{aligned}$$

Now, in accordance with Lemma 1.5 and $x\in[0,b]$, one has

$$|D_\kappa^A(.;.)-\bar{g}|\leq 4\tilde{M}_{\bar{g}}(1+b^2)\delta_s(b)+\left(1+\frac{\sqrt{\delta_s(b)}}{\delta}\right)\omega_{b+1}(\bar{g};\delta).$$

Selecting $\delta=\delta_s(b)$, desired result can be obtained easily. □

Remark 4.2. *In this article, we employ the test function defined by $\bar{g}_s(y)=y^s$, $s\in\{0,1,2\}$.*

Theorem 4.3. *([36], [37]) Assume that the sequence of linear positive operators $(L_\kappa)_{\kappa\geq 1}$ mapping from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ meets the conditions*

$$\lim_{\kappa\to\infty} \|L_\kappa(\bar{g}_s;.) - \bar{g}_s\|_\nu = 0, \quad \textit{where } s = 0,1,2,$$

thus, for $\bar{g} \in C_\nu^{\tilde{k}}[0,\infty)$, we get

$$\lim_{\kappa\to\infty} \|L_\kappa(\bar{g};.) - \bar{g}\|_\nu = 0.$$

Theorem 4.4. *Let $\bar{g} \in C_\nu^{\tilde{k}}[0,\infty)$. Then, we obtain*

$$\lim_{\kappa\to\infty} \|D_\kappa^A(\bar{g};.) - \bar{g}\|_\nu = 0.$$

Proof. To prove the Theorem 4.4, it is enough to verify that

$$\lim_{\kappa\to\infty} \|D_\kappa^A(\bar{g}_s;.) - \bar{g}_s\|_\nu = 0, \textit{for } s = 0,1,2.$$

Considering the Lemma 1.4, one can see $\|D_\kappa^A(\bar{g}_0;.) - 1\|_\nu = 0$, here $\kappa\to\infty$ and restrict with $\kappa > 2$, also

$$\begin{aligned}
\|D_\kappa^A(\bar{g}_1;.) - \bar{g}_1\|_{\nu(u)} &= \sup_{u\in[0,\infty)} \frac{1}{\nu(u)} \left| \frac{1}{\kappa-1}\left[\kappa u + 1 + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right] - u \right| \\
&= \frac{1}{\kappa-1} \sup_{u\in[0,\infty)} \frac{u}{1+u^2} + \frac{1}{\kappa-1}\left[1 + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right] \sup_{u\in[0,\infty)} \frac{1}{1+u^2}.
\end{aligned}$$

For a large value of κ, we get $\|D_\kappa^A(\bar{g}_1;.) - \bar{g}_1\|_\nu \to 0$.
Also,

$$\begin{aligned}
\|D_\kappa^A(\bar{g}_2;.) - \bar{g}_2\|_\nu \leq &\left(\frac{3\kappa-2}{(\kappa-2)(\kappa-1)}\right) \sup_{u\in[0,\infty)} \frac{u^2}{1+u^2} \\
&+ \left(\frac{4\kappa}{(\kappa-2)(\kappa-1)}\right) \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\
&+ \left(\frac{1}{(\kappa-2)(\kappa-1)}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]\right) \sup_{u\in[0,\infty)} \frac{2\kappa u + 4}{1+u^2} \\
&+ \left(\frac{1}{(\kappa-2)(\kappa-1)}\left(2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)} + \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 2\right)\right) \sup_{y\in[0,\infty)} \frac{1}{1+u^2}.
\end{aligned}$$

Which implies that $\|D_\kappa^A(\bar{g}_2;.) - \bar{g}_2\|_\nu \to 0$ as $\kappa\to\infty$. Thus, we concludes the proof of the Theorem 4.4 □

Theorem 4.5. *Let $\bar{g} \in C_\nu^{\tilde{k}}[0,\infty)$ and $\zeta > 0$. Then,*

$$\lim_{\kappa\to\infty} \sup_{u\in[0,\infty)} \frac{D_\kappa^A(\bar{g};u) - \bar{g}(u)}{(1+u^2)^{1+\zeta}} = 0.$$

Proof. Since $|\bar{g}(x)| \leq \|\bar{g}\|_\nu(1+u^2)$, for any real fixed number $u_0 > 0$, we get

$$\begin{aligned}
\sup_{u\in[0,\infty)} \frac{D_\kappa^A(\bar{g};u) - \bar{g}(u)}{(1+u^2)^{1+\zeta}} &\leq \sup_{u\leq u_0} \frac{D_\kappa^A(\bar{g};u) - \bar{g}(u)}{(1+u^2)^{1+\zeta}} + \sup_{u\geq u_0} \frac{D_\kappa^A(\bar{g};u) - \bar{g}(u)}{(1+u^2)^{1+\zeta}} \\
&\leq \|D_\kappa^A(\bar{g};u) - \bar{g}(u)\|\|_{C[0,u_0]} + \|\bar{g}\|_\nu \sup_{u\geq u_0} \frac{D_\kappa^A(1+y^2;u)}{(1+u^2)^{1+\zeta}} + \sup_{u\geq u_0} \frac{|\bar{g}(u)|}{(1+u^2)^{1+\zeta}} \\
&= \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3, \quad \textit{say}. \qquad (19)
\end{aligned}$$

Now,

$$\tilde{T_3} = \sup_{u \geq u_0} \frac{|\bar{g}(u)|}{(1+u^2)^{1+\zeta}} \leq \sup_{u \geq u_0} \frac{\|\bar{g}\|_\nu (1+u^2)}{(1+u^2)^{1+\zeta}} \leq \frac{\|\bar{g}\|_\nu}{(1+u_0^2)^{\zeta}}.$$

In view of Lemma 1.4, it gives

$$\lim_{\kappa\to\infty} \sup_{u\in[u_0,\infty)} \frac{D_\kappa^A(1+y^2;u)}{1+u^2} = 1.$$

Therefore, for any arbitrary $\epsilon > 0$, there corresponds $\kappa_1 \in \mathbb{N}$ with

$$\sup_{u\in[u_0,\infty)} \frac{D_\kappa^A(1+y^2;u)}{1+u^2} \leq \frac{(1+u_0^2)^{\zeta}}{\|\bar{g}\|_\nu} \frac{\epsilon}{3} + 1, \text{ for all } \kappa \geq \kappa_1.$$

Therefore

$$\tilde{T_2} = \|\bar{g}\|_\nu \sup_{u\in[u_0,\infty)} \frac{D_\kappa^A(1+y^2;u)}{(1+u^2)^{1+\zeta}} \leq \frac{\|\bar{g}\|_\nu}{(1+u_0^2)^{\zeta}} + \frac{\epsilon}{3}, \text{ for all } \kappa \geq \kappa_1. \tag{20}$$

Hence, we get

$$\tilde{T_2} + \tilde{T_3} < 2\frac{\|\bar{g}\|_\nu}{(1+u^2)^{\zeta}} + \frac{\epsilon}{3}.$$

If we take u_0 to be so large that $\frac{\|\bar{g}\|_\nu}{(1+u^2)^{\zeta}} < \frac{\epsilon}{6}$, then, we have

$$\tilde{T_2} + \tilde{T_3} < \frac{2\epsilon}{3} \text{ for all } \kappa \geq \kappa_1. \tag{21}$$

Now, from Theorem 4.1, there corresponds $\kappa_2 > \kappa$ with

$$\tilde{T_1} = \|D_\kappa^A(\bar{g};\cdot) - \bar{g}\|_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } \kappa_2 \geq \kappa. \tag{22}$$

Let $\kappa_3 = \max(\kappa_1, \kappa_2)$. Then, using the Eqs. (19), (21) and (22), we get

$$\sup_{u\in[0,\infty)} \frac{|D_\kappa^A(\bar{g};u) - \bar{g}(u)|}{(1+u^2)^{1+\zeta}} < \epsilon,$$

which, completes the proof. □

5. A-Statistical Approximation

We revisit some notations from [38]. Suppose that $B = (b_{\kappa\mu})$ is an infinite, non-negative suitability matrix. A sequence $u := (u_\mu)$ is A-statistically convergent to L, denoted as $st_B - \lim u = L$, if for each $\epsilon > 0$

$$\lim_{\kappa} \sum_{\mu : |u_\mu - L| \geq \epsilon} b_{\kappa\mu} = 0.$$

Let $q = (q_\kappa)$ be a sequence such that the following assertions are true

$$st_B - \lim_{\kappa} q_\kappa = 1 \text{ and } st_B - \lim_{\kappa} q_\kappa^\kappa = b, \ 0 \leq b < 1. \tag{23}$$

Theorem 5.1. *Consider $B = (b_{\kappa\mu})$ be a non-negative regular suitability matrix and sequence $q = (q_\kappa)$ along with condition (23), $q_\kappa \in (0,1)$, $\kappa \in \mathbb{N}$. Then, for each $\bar{g} \in C_\nu^0[0,\infty)$, $st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g};.) - \bar{g}\|_\nu = 0$.*

Proof. In accordance with Lemma 1.4, one has

$$st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g}_0;.) - \bar{g}_0\|_\nu = 0.$$

and

$$\begin{aligned}\|D_\kappa^A(\bar{g}_1;.) - \bar{g}_1\|_\nu &= \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\left|\frac{1}{\kappa-1}\left[\kappa u + 1 + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right] - u\right| \\ &= \frac{1}{\kappa-1}\sup_{u\in[0,\infty)}\frac{u}{1+u^2} + \frac{1}{\kappa-1}\left[1 + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]\sup_{u\in[0,\infty)}\frac{1}{1+u^2}.\end{aligned}$$

Now

$$\begin{aligned}\tilde{K_1} &:= \left\{\kappa : \|D_\kappa^A(\bar{g}_1;.) - \bar{g}_1\| \geq \epsilon\right\}, \\ \tilde{K_2} &:= \left\{\kappa : \frac{1}{\kappa-1} \geq \frac{\epsilon}{2}\right\}, \\ \tilde{K_3} &:= \left\{\kappa : \frac{1}{\kappa-1}\left[1 + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right] \geq \frac{\epsilon}{2}\right\}.\end{aligned}$$

Which implies that $\tilde{K_1} \subseteq \tilde{K_2} \cup \tilde{K_3}$, this shows that $\sum_{\mu\in\tilde{K_1}} b_{\kappa\mu} \leq \sum_{\mu\in\tilde{K_2}} b_{\kappa\mu} + \sum_{\mu\in\tilde{K_3}} b_{\kappa\mu}$. Therefore, we get

$$st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g}_1;.) - \bar{g}_1\|_\nu = 0. \tag{24}$$

Now by using Lemma 1.4, we have

$$\begin{aligned}\|D_\kappa^A(\bar{g}_2;.) - \bar{g}_2\|_{1+u^2} \leq \sup_{u\in[0,\infty)} \frac{1}{\nu(u)}\Bigg|\frac{1}{(\kappa-1)(\kappa-2)}\Bigg[&(3\kappa-2)u^2 + 4\kappa u \\ &+ (2\kappa u + 4)\left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right) + 2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)} \\ &+ \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 2\Bigg]\Bigg|.\end{aligned}$$

For a given $\varepsilon > 0$, we have the following sets

$$\begin{aligned}\tilde{H_1} &:= \left\{\kappa : \left\|D_\kappa^A(\bar{g}_2;.) - \bar{g}_2\right\|_\nu \geq \epsilon\right\}, \\ \tilde{H_2} &:= \left\{\kappa : \frac{3\kappa-2}{(\kappa-2)(\kappa-1)} \geq \frac{\epsilon}{4}\right\}, \\ \tilde{H_3} &:= \left\{\kappa : \frac{4\kappa}{(\kappa-2)(\kappa-1)} \geq \frac{\epsilon}{4}\right\}, \\ \tilde{H_4} &:= \left\{\kappa : \frac{\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right]}{(\kappa-2)(\kappa-1)} \geq \frac{\epsilon}{4}\right\}, \\ \tilde{H_5} &:= \left\{\kappa : \frac{1}{(\kappa-2)(\kappa-1)}\left[2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 2\right] \geq \frac{\epsilon}{4}\right\}.\end{aligned}$$

It can be observed that $\tilde{H}_1 \subseteq \tilde{H}_2 \bigcup \tilde{H}_3 \bigcup \tilde{H}_4 \bigcup \tilde{H}_5$. Therefore, we acquire

$$\sum_{\mu\in\tilde{H}_1} b_{\kappa\mu} \leq \sum_{\mu\in\tilde{H}_2} b_{\kappa\mu} + \sum_{\mu\in\tilde{H}_3} b_{\kappa\mu} + \sum_{\mu\in\tilde{H}_4} b_{\kappa\mu} + \sum_{\mu\in\tilde{H}_5} b_{\kappa\mu}.$$

As $\kappa \to \infty$, we have

$$st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g}_2; u) - \bar{g}_2\|_\nu = 0. \tag{25}$$

Thus, we concludes the proof of the Theorem 5.1. □

Next, we will examine the convergence rate of A-Statistical approximation with respect to Peetre's K-functional for the operators $D_\kappa^A(.;.)$.

Theorem 5.2. *Let $\bar{g} \in C_B^2[0,\infty)$. Then,*

$$st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g}; \cdot) - \bar{g}\|_{C_B[0,\infty)} = 0.$$

Proof. Considering Taylor's result, we obtain

$$\bar{g}(y) = \bar{g}(u) + \bar{g}'(u)(y-u) + \frac{1}{2}\bar{g}''(\eta)(y-u)^2,$$

where $v \leq \eta \leq u$. Operating $D_\kappa^A(.;.)$, on both sides in above equation, we get

$$D_\kappa^A(\bar{g}; u) - \bar{g}(u) = \bar{g}'(u)D_\kappa^A(\eta_1; u) + \frac{1}{2}\bar{g}''(\eta)D_\kappa^A(\eta_2; u),$$

which yields that

$$\begin{aligned}\|D_\kappa^A(\bar{g}; \cdot) - \bar{g}\|_{C_B[0,\infty)} &\leq \|\bar{g}'\|_{C_B[0,\infty)}\|D_\kappa^A(\bar{g}_1-, .)\|_{C_B[0,\infty)} + \|\bar{g}''\|_{C_B[0,\infty)}\|D_\kappa^A(\bar{g}_1-, .)^2\|_{C_B[0,\infty)} \\ &= \tilde{W}_1 + \tilde{W}_2, \quad \textit{say}.\end{aligned} \tag{26}$$

Based on Eqs. (24) and (25), it follows that

$$\begin{aligned}\lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} b_{\kappa\mu} &= 0, \\ \lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_2\geq\frac{\epsilon}{2}} b_{\kappa\mu} &= 0.\end{aligned}$$

From Eq. (26), we have

$$\lim_{\kappa} \sum_{\mu\in\mathbb{N}:\|D_\kappa^A(\bar{g};\cdot)-\bar{g}\|_{C_B[0,\infty)}\geq\epsilon} b_{\kappa\mu} \leq \lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} b_{\kappa\mu} + \lim_{\kappa} \sum_{\mu\in N:\tilde{W}_2\geq\frac{\epsilon}{2}} b_{\kappa\mu}.$$

Thus $st_B - \lim_{\kappa} \|D_\kappa^A(\bar{g}; \cdot) - \bar{g}\|_{C_B[0,\infty)} \to 0$. as $\kappa \to \infty$.

Hence, we arrive the proof. □

Theorem 5.3. *Let $\bar{g} \in C_B^2[0,\infty)$. Then,*

$$\|D_\kappa^A(\bar{g}; \cdot) - \bar{g}\|_{C_B[0,\infty)} \leq M\omega_2(\bar{g}; \sqrt{\delta}),$$

where $\delta = \|D_\kappa^A(\bar{g}_1 - \cdot; \cdot)\|_{C_B[0,\infty)} + \|D_\kappa^A(\bar{g}_1 - \cdot)^2; \cdot)\|_{C_B[0,\infty)}$, and $\|\bar{g}\|_{C_B^2[0,\infty)} = \|\bar{g}\|_{C_B[0,\infty)} + \|\bar{g}'\|_{C_B[0,\infty)} + \|\bar{g}''\|_{C_B[0,\infty)}$.

Proof. Let $\bar{h} \in C_B^2[0,\infty)$. By applying Eq. (26), one obtain

$$\|D_\kappa^A(\bar{h}) - \bar{h}\|_{C_B[0,\infty)} \le \|\bar{h}'\|_{C_B[0,\infty)} \|D_\kappa^A(\bar{g}_1 - \cdot);\cdot)\|_{C_B[0,\infty)} + \frac{1}{2}\|\bar{h}''\|_{C_B[0,\infty)} D_\kappa^A(\bar{g}_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)}$$
$$\le \delta \|\bar{h}\|_{C_B^2[0,\infty)}. \tag{27}$$

For each $\bar{g} \in C_B[0,\infty)$ and $\bar{h} \in C_B^2$, Using Eq. (27), we acquire

$$\begin{aligned}\|D_\kappa^A(\bar{g};\cdot) - \bar{g}\|_{C_B[0,\infty)} &\le \|D_\kappa^A(\bar{g};\cdot) - D_\kappa^A(\bar{h};\cdot)\|_{C_B[0,\infty)} + \|D_\kappa^A(\bar{h};\cdot) - \bar{h}\|_{C_B[0,\infty)} + \|\bar{h} - \bar{g}\|_{C_B[0,\infty)}\\ &\le 2\|\bar{h} - \bar{g}\|_{C_B[0,\infty)} + \|D_\kappa^A(\bar{h};\cdot) - \bar{h}\|_{C_B[0,\infty)}\\ &\le 2\|\bar{h} - \bar{g}\|_{C_B[0,\infty)} + \delta\|\bar{h}\|_{C_B^2}.\end{aligned}$$

Considering Peetre's K-functional, one obtain

$$\|D_\kappa^A(\bar{g};\cdot) - \bar{g}\|_{C_B[0,\infty)} \le 2K_2(\bar{g};\delta)$$

and

$$\|D_\kappa^A(\bar{g};\cdot) - \bar{g}\|_{C_B[0,\infty)} \le \tilde{M}\Big\{\omega_2(\bar{g};\sqrt{\delta}) + \min(1,\delta)\|\bar{g}\|_{C_B[0,\infty)}\Big\}.$$

In view of Eq. (25), we have

$$st_B - \lim_\kappa \delta = 0, \text{ thus } st_B - \lim_\kappa \omega(\bar{g};\sqrt{\delta}) = 0,$$

which concludes the proof of desired result. □

6. Conclusion

In this paper, we present a sequence of positive linear operators using generalized Appell polynomials in the integral form. These operators are designed to approximate functions defined on a Lebesgue measurable space and are known as Szász-Beta type operators introduced in (4). Additionally, we derive estimates crucial for establishing rate of convergence and accuracy of approximation. Furthermore, we explore various aspects of approximation, including local and global results, as well as A-statistical approximation, utilizing these operators to obtain enhanced approximations across different functional spaces.

Conflicts of interest.
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