Filomat 39:28 (2025), 10065–10075
https://doi.org/10.2298/FIL2528065C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new generalization of Kantorovich operators having three parameters

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Abstract. This paper mainly deals with Kantorovich variant of generalized Stancu operators with three parameters. Initially, we exhibit approximation theorems in the space of real valued continuous functions on compact interval and next L_p-space. Additionally, we obtain some estimates for the rate of convergence by making use of modulus of continuity and L_p modulus of smoothness of the first order. Ultimately, we yield some graphical examples showing the relevance of the results.

1. Introduction

One of the most important results in the Approximation Theory is the Weierstrass approximation theorem, stating that every continous function defined on a compact interval can uniformly be approximated by algebraic polynomials. After that, the most fundamental problem is how to construct such polynomials. In order to present one of the easiest and briefest proofs of the Weierstrass approximation theorem, Bernstein [3] introduced the classical Bernstein polynomials, given by

$$B_\mu(g;\xi) = \sum_{j=0}^{\mu} p_{\mu,j}(\xi)\, g\left(\frac{j}{\mu}\right), \quad \mu \in \mathbb{N}, \tag{1}$$

for $g \in C[0,1]$ and

$$p_{\mu,j}(\xi) = \begin{cases} \binom{\mu}{j}\xi^j(1-\xi)^{\mu-j}; & 0 \le j \le \mu \\ 0; & j < 0 \text{ or } j > \mu \end{cases}, \quad \xi \in [0,1]. \tag{2}$$

Since Bernstein polynomials have elegant construction and many beneficial approximation properties, the exploration of their some modifications in different ways has been an important topic. Bernstein-Durrmeyer

2020 *Mathematics Subject Classification*. Primary 41A36; Secondary 41A25.
Keywords. Kantorovich operators, Stancu operators, L_p-convergence, modulus of continuity, rate of convergence.
Received: 23 November 2024; Accepted: 18 August 2025
Communicated by Snežana Č. Živković-Zlatanović
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operators, Bernstein-Stancu operators and Bernstein-Kantorovich operators are some examples of the well-known modifications of Bernstein polynomials. In [17], Stancu modified Bernstein polynomials by using a probabilistic method as

$$L_{\mu,\tau}(g;\xi) := \sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi) g\left(\frac{j}{\mu}\right), \quad \xi \in [0,1], \tag{3}$$

where $g \in C[0,1]$, τ is a non-negative integer parameter, $\mu \in \mathbb{N}$ such that $\tau < \mu/2$, for which

$$w_{\mu,j,\tau}(\xi) := \begin{cases} (1-\xi)\, p_{\mu-\tau,j}(\xi); & 0 \le j < \tau \\ (1-\xi)\, p_{\mu-\tau,j}(\xi) + \xi p_{\mu-\tau,j-\tau}(\xi); & \tau \le j \le \mu-\tau \\ \xi p_{\mu-\tau,j-\tau}(\xi); & \mu-\tau < j \le \mu \end{cases} \tag{4}$$

is a generalization of Bernstein's fundamental functions $p_{\mu,j}(\xi)$ defined by (2). The operators $L_{\mu,\tau}$ are called as μ-th Stancu operators. We note that for the cases $\tau = 0$ and $\tau = 1$, Stancu operators expressed by (3) reduce to the Bernstein polynomials. Stancu obtained an expression for the remainder of the approximation notation by means of the second order divided differences, an asymptotic Voronovskaya result, the order of approximation by the sequence of the operators $L_{\mu,\tau}$ via modulus of continuity and also spectral properties of the operator $L_{\mu,\tau}$ in [17]. Later on, many authors have studied Stancu operators and their some generalizations in the real and complex cases that we refer the readers to [5–7, 9, 11, 13, 18, 19].

As Bernstein operators are not applicable for approximation of discontinuous functions, Kantorovich [12] constructed a linear positive operator known as Kantorovich operators in the literature given by

$$K_{\mu}(g;\xi) = \sum_{j=0}^{\mu} p_{\mu,j}(\xi) \left((\mu+1) \int_{\frac{j}{\mu+1}}^{\frac{j+1}{\mu+1}} g(l)\, dl \right), \ \mu \in \mathbb{N},$$

to obtain approximation of Lebesgue integrable function on $[0,1]$. Depending on two given real parameters, Stancu [16] introduced and studied any other generalization of Bernstein operators which is called as Bernstein-Stancu operators. On the other side, in [2], Bărbosu defined the Kantorovich shape of Stancu-type operators $K_{\mu}^{a,b} : L_1[0,1] \to C[0,1]$ as

$$K_{\mu}^{a,b}(g;\xi) = \sum_{j=0}^{\mu} p_{\mu,j}(\xi) \left((\mu+b+1) \int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} g(l)\, dl \right), \ \mu \in \mathbb{N},$$

where $\xi \in [0,1]$, $g \in L_1[0,1]$, the space of all functions defined on $[0,1]$ that is Lebesgue integrable, two real parameters a, b providing $0 \le a \le b$ and $p_{\mu,j}(\xi)$ is Bernstein's fundamental functions given in (2).

In the present paper, motivated by the earlier mentioned works, we will define a new generalization of Kantorovich-Stancu-type operators, depending upon a non-negative integer parameter τ and two real parameters a, b providing the prerequisite $0 \le a \le b$, having the form

$$K_{\mu,\tau}^{a,b}(g;\xi) := \sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\, (\mu+b+1) \int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} g(l)\, dl, \ \ \xi \in [0,1] \tag{5}$$

where $g \in L_1[0,1]$, $\mu \in \mathbb{N}$ such that $\tau < \mu/2$ and $w_{\mu,j,\tau}(\xi)$ is the Stancu's fundamental functions given by (4). We name $K_{\mu,\tau}^{a,b}$ as generalized Kantorovich-Stancu operators. By means of the definition of Stancu's fundamental functions $w_{\mu,j,\tau}(\xi)$, the generalized Kantorovich-Stancu operators can be expressed as

$$K_{\mu,\tau}^{a,b}(g;\xi)=\sum_{j=0}^{\mu-\tau}p_{\mu-\tau,j}(\xi)(\mu+b+1)\left[(1-\xi)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}g(l)\,dl+\xi\int_{\frac{j+\tau+a}{\mu+b+1}}^{\frac{j+\tau+a+1}{\mu+b+1}}g(l)\,dl\right].$$

Obviously, for the special cases $\tau=0$ and $\tau=1$, the generalized Kantorovich-Stancu operators $K_{\mu,\tau}^{a,b}$ give the Kantorovich-Stancu-type operators studied in [2]. If $\tau=0$ and $\tau=1$ with $a=b=0$, the operators $K_{\mu,\tau}^{a,b}$ become the Kantorovich operators investigated in [12]. For $a=b=0$, the operators $K_{\mu,\tau}^{a,b}$ yield the Stancu-Kantorovich operators defined in [4]. Here, at first we get approximation results in the space of real valued continuous functions on compact interval and then L_p-space. Later, we establish some estimates for the rate of convergence with the help of modulus of continuity and L_p modulus of smoothness of the first order. In conclusion, we give some numerical examples to show approximation by the new operator $K_{\mu,\tau}^{a,b}$.

2. Approximation by the generalized Kantorovich-Stancu operators

Throughout the paper, let us denote the test functions by $e_j(\xi)=\xi^j$, $j\in\mathbb{N}\cup\{0\}$ and $\varphi_\xi^i(t):=(t-\xi)^i$, $i\in\mathbb{N}$.

In the following, we have the first three moments and the central moments of the generalized Kantorovich-Stancu operators $K_{\mu,\tau}^{a,b}$, successively.

Lemma 2.1. *For the operators $K_{\mu,\tau}^{a,b}$ given by (5), one has*

$$K_{\mu,\tau}^{a,b}(e_0;\xi) = 1, \tag{6}$$

$$K_{\mu,\tau}^{a,b}(e_1;\xi) = \frac{\mu}{\mu+b+1}\xi+\frac{2a+1}{2(\mu+b+1)}, \tag{7}$$

$$K_{\mu,\tau}^{a,b}(e_2;\xi) = \frac{\mu^2}{(\mu+b+1)^2}\left[\xi^2+\left(1+\frac{\tau(\tau-1)}{\mu}\right)\frac{\xi(1-\xi)}{\mu}\right]+\frac{\mu\xi(2a+1)}{(\mu+b+1)^2}+\frac{3a^2+3a+1}{3(\mu+b+1)^2}. \tag{8}$$

Lemma 2.2. *For the central moments of the operators $K_{\mu,\tau}^{a,b}$ given by (5), one has*

$$K_{\mu,\tau}^{a,b}\left(\varphi_\xi^1;\xi\right)=-\frac{b+1}{\mu+b+1}\xi+\frac{2a+1}{2(\mu+b+1)}$$

and

$$K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)=\frac{(b+1)^2}{(\mu+b+1)^2}\xi^2+\frac{\mu+\tau(\tau-1)}{(\mu+b+1)^2}\xi(1-\xi)-\frac{(2a+1)(b+1)}{(\mu+b+1)^2}\xi+\frac{3a^2+3a+1}{3(\mu+b+1)^2}.$$

By implementing the Korovkin theorem to the sequence of $K_{\mu,\tau}^{a,b}$, from (6)-(8) we immediately obtain the following convergence theorem in $C[0,1]$.

Theorem 2.3. *If $g\in C[0,1]$, τ is a non-negative fixed integer and $0\le a\le b$, then we have*

$$\lim_{\mu\to\infty}K_{\mu,\tau}^{a,b}(g)=g$$

uniformly on $[0,1]$.

Remark 2.4. *For $\tau=0$ and $\tau=1$, the operators $K_{\mu,\tau}^{a,b}$ become the Kantorovich-Stancu-type operators studied in [2]. L_p-approximation can be obtained as a special case in [14, Theorem 2.5].*

Now, we prove L_p-approximation by the sequences of these operators only for $\tau > 1$.

Theorem 2.5. *If $g \in L_p[0,1]$, $1 \le p < \infty$, $\tau > 1$ and $0 \le a \le b$, then we have*

$$\lim_{\mu\to\infty} K_{\mu,\tau}^{a,b}(g) = g$$

in $L_p[0,1]$.

Proof. Let $\left\|K_{\mu,\tau}^{a,b}\right\|$ denote norm of $K_{\mu,\tau}^{a,b}$ acting from $L_p[0,1]$ into $L_p[0,1]$, where $\tau > 1$ and $\mu \in \mathbb{N}$ such that $\tau < \mu/2$. We show that there exists an $N > 0$ such that $\left\|K_{\mu,\tau}^{a,b}\right\| \le N$. Taking into account of the facts that $\psi(\upsilon) = |\upsilon|^p$, $1 \le p < \infty$, $\upsilon \in \mathbb{R}$, is convex and $\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi) = 1$ for $w_{\mu,j,\tau}(\xi) \ge 0$, by Jensen's inequality and integral Jensen's inequality (see, e.g., [1]) we can write

$$\begin{aligned}
\left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^p &\le \sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\left|(\mu+b+1)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} g(l)\,dl\right|^p \\
&\le \sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)(\mu+b+1)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl \\
&= \left\{\sum_{j=0}^{\mu-\tau}(1-\xi)\,p_{\mu-\tau,j}(\xi) + \sum_{j=\tau}^{\mu}\xi p_{\mu-\tau,j-\tau}(\xi)\right\}(\mu+b+1)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl,
\end{aligned}$$

which follows that

$$\begin{aligned}
\int_0^1 \left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^p d\xi &\le \sum_{j=0}^{\mu-\tau}\binom{\mu-\tau}{j}\int_0^1 \xi^j(1-\xi)^{\mu-\tau-j+1}d\xi\,(\mu+b+1)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl \\
&\quad + \sum_{j=\tau}^{\mu}\binom{\mu-\tau}{j-\tau}\int_0^1 \xi^{j-\tau+1}(1-\xi)^{\mu-j}d\xi\,(\mu+b+1)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl.
\end{aligned}$$

From the well-known Beta integral, one has

$$\int_0^1 \left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^p d\xi \le \sum_{j=0}^{\mu-\tau}\frac{(\mu-\tau-j+1)(\mu+b+1)}{(\mu-\tau+2)(\mu-\tau+1)}\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl + \sum_{j=\tau}^{\mu}\frac{(j-\tau+1)(\mu+b+1)}{(\mu-\tau+2)(\mu-\tau+1)}\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}} \left|g(l)\right|^p dl.$$

Because $\mu > 2\tau$ for $\tau > 1$, we get $\mu - \tau > \tau$. Therefore, we can write

$$\int_0^1 \left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^p d\xi$$

$$\leq \frac{\mu+b+1}{\mu-\tau+2}\left\{\sum_{j=0}^{\tau-1}+\sum_{j=\tau}^{\mu-\tau}\right\}\frac{\mu-\tau-j+1}{\mu-\tau+1}\int\limits_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left|g(l)\right|^{p}dl+\frac{\mu+b+1}{\mu-\tau+2}\left\{\sum_{j=\tau}^{\mu-\tau}+\sum_{j=\mu-\tau+1}^{\mu}\right\}\frac{j-\tau+1}{\mu-\tau+1}\int\limits_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left|g(l)\right|^{p}dl$$

$$\leq \frac{\mu+b+1}{\mu-\tau+2}\left\{\sum_{j=0}^{\tau-1}\frac{\mu-\tau-j+1}{\mu-\tau+1}+\sum_{j=\tau}^{\mu-\tau}\frac{\mu-2\tau+2}{\mu-\tau+1}+\sum_{j=\mu-\tau+1}^{\mu}\frac{j-\tau+1}{\mu-\tau+1}\right\}\int\limits_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left|g(l)\right|^{p}dl. \tag{9}$$

For $0\leq j\leq\tau-1$, we get $\mu-\tau-j+1\leq\mu-\tau+1$. When $\mu-2\tau+1<\mu-\tau+1$, we obtain $\mu-2\tau+2\leq\mu-\tau+1$. For $\mu-\tau+1\leq j\leq\mu$, we acquire $j-\tau+1\leq\mu-\tau+1$. By these arguments, (9) reduces to

$$\begin{aligned}\int\limits_{0}^{1}\left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^{p}d\xi &\leq \frac{\mu+b+1}{\mu-\tau+2}\left\{\sum_{j=0}^{\tau-1}+\sum_{j=\tau}^{\mu-\tau}+\sum_{j=\mu-\tau+1}^{\mu}\right\}\int\limits_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left|g(l)\right|^{p}dl\\ &= \frac{\mu+b+1}{\mu-\tau+2}\sum_{j=0}^{\mu}\int\limits_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left|g(l)\right|^{p}dl\\ &\leq \frac{\mu+b+1}{\mu-\tau+2}\int\limits_{0}^{1}\left|g(l)\right|^{p}dl. \end{aligned}\tag{10}$$

Taking $\sup_{\mu>2\tau}\frac{\mu+b+1}{\mu-\tau+2}$ for $\tau>1$, we get

$$\sup_{\mu>2\tau}\frac{\mu+b+1}{\mu-\tau+2}=\frac{2\tau+2+b}{\tau+3}=:C_1.$$

Hence, (10) gives

$$\int\limits_{0}^{1}\left|K_{\mu,\tau}^{a,b}(g;\xi)\right|^{p}d\xi\leq C_1\int\limits_{0}^{1}\left|g(l)\right|^{p}dl.$$

Going by the L_p-norm, we have $\left\|K_{\mu,\tau}^{a,b}(g)\right\|_{L_p[0,1]}\leq C_1^{1/p}\left\|g\right\|_{L_p[0,1]}$ for every $g\in L_p[0,1]$. It implies that $K_{\mu,\tau}^{a,b}$ is a bounded operator with $\left\|K_{\mu,\tau}^{a,b}\right\|\leq C_1^{1/p}$ for every $\mu\in\mathbb{N}$ such that $\mu>2\tau$. Let $\epsilon>0$ be given. Since the density of $C[0,1]$ in $L_p[0,1]$, there exists a $\varphi\in C[0,1]$ such that $\left\|g-\varphi\right\|_{L_p[0,1]}<\epsilon$. Then, from Theorem 2.3, there exists an $\mu_0\in\mathbb{N}$ with $\mu>\mu_0$ such that $\left\|K_{\mu,\tau}^{a,b}(\varphi)-\varphi\right\|_{C[0,1]}<\epsilon$. By the following inequality

$$\left\|K_{\mu,\tau}^{a,b}(g)-g\right\|_{L_p[0,1]} \leq \left\|K_{\mu,\tau}^{a,b}(g)-K_{\mu,\tau}^{a,b}(\varphi)\right\|_{L_p[0,1]}+\left\|K_{\mu,\tau}^{a,b}(\varphi)-\varphi\right\|_{C[0,1]}+\left\|\varphi-g\right\|_{L_p[0,1]},$$

we arrive at

$$\begin{aligned}\left\|K_{\mu,\tau}^{a,b}(g)-g\right\|_{L_p[0,1]} &\leq \left\|K_{\mu,\tau}^{a,b}\right\|\left\|g-\varphi\right\|_{L_p[0,1]}+\left\|K_{\mu,\tau}^{a,b}(\varphi)-\varphi\right\|_{C[0,1]}+\left\|\varphi-g\right\|_{L_p[0,1]}\\ &< C_1^{1/p}\epsilon+\epsilon+\epsilon=\left(2+C_1^{1/p}\right)\epsilon,\end{aligned}$$

which completes the proof. □

Next, we give the rate of convergence for the sequence $\left\{K_{\mu,\tau}^{a,b}(g)\right\}_{\mu\in\mathbb{N}}$.

Now, firstly we remember that if $J \subset \mathbb{R}$ is a given interval and g is a real valued function described on J and bounded on this interval, modulus of continuity of g, is the function ω_1 given by for any $\delta > 0$

$$\omega_1(g;\delta) = \sup\left\{\left|g(l) - g(\xi)\right| : l, \xi \in J, \quad |l-\xi| \leq \delta\right\}.$$

In the following theorem, we give an estimate regarding local approximation via modulus of continuity in $C[0,1]$.

Theorem 2.6. *If $g \in C[0,1]$, τ is a non-negative integer and $0 \leq a \leq b$, then for every $\mu \in \mathbb{N}$ such that $\tau < \mu/2$ and $\xi \in [0,1]$, we have*

$$\left|K_{\mu,\tau}^{a,b}(g;\xi) - g(\xi)\right| \leq 2\omega_1\left(g;\sqrt{\delta_{\mu,\tau}^{a,b}(\xi)}\right),$$

where

$$\delta_{\mu,\tau}^{a,b}(\xi) = K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)$$

and $K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)$ is as in Lemma 2.2.

Proof. From the trait of modulus of continuity below

$$\left|g(l) - g(\xi)\right| \leq \left(1 + \frac{(l-\xi)^2}{\delta^2}\right)\omega_1(g;\delta)$$

for $l, \xi \in [0,1]$ and $\delta > 0$, we can write

$$\begin{aligned}\left|K_{\mu,\tau}^{a,b}(g;\xi) - g(\xi)\right| &\leq K_{\mu,\tau}^{a,b}\left(\left|g(l) - g(\xi)\right|;\xi\right) \\ &\leq \left(1 + \frac{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}{\delta^2}\right)\omega_1(g;\delta).\end{aligned}$$

Choosing $\delta = \sqrt{\delta_{\mu,\tau}^{a,b}(\xi)} = \sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}$, the desired result is obtained. □

In the next theorem, we present an estimate regarding global approximation via modulus of continuity.

Corollary 2.7. *If $g \in C[0,1]$, τ is a non-negative integer and $0 \leq a \leq b$, then for every $\mu \in \mathbb{N}$ such that $\tau < \mu/2$ and $\xi \in [0,1]$, we have*

$$\left|K_{\mu,\tau}^{a,b}(g;\xi) - g(\xi)\right| \leq 2\omega_1\left(g;\sqrt{\delta_{\mu,\tau}^{a,b}}\right),$$

where

$$\delta_{\mu,\tau}^{a,b} = \max_{\xi\in[0,1]} \delta_{\mu,\tau}^{a,b}(\xi).$$

Now, we give the rate of approximation via modulus of continuity in $C^1[0,1]$, which is the space of once continuously differentiable functions on $[0,1]$.

Theorem 2.8. *If $g \in C^1[0,1]$, τ is a non-negative integer and $0 \leq a \leq b$, then for every $\mu \in \mathbb{N}$ such that $\tau < \mu/2$ and $\xi \in [0,1]$, we have*

$$\left|K_{\mu,\tau}^{a,b}(g;\xi) - g(\xi)\right| \leq \left|g'(\xi)\right|\left|\frac{2a+1-2(1+b)\xi}{2(\mu+b+1)}\right| + 2\sqrt{\delta_{\mu,\tau}^{a,b}(\xi)}\,\omega_1\left(g';\sqrt{\delta_{\mu,\tau}^{a,b}(\xi)}\right),$$

where

$$\delta_{\mu,\tau}^{a,b}(\xi) = K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right).$$

Proof. From the result in [15], we have

$$\begin{aligned}\left|K_{\mu,\tau}^{a,b}(g;\xi)-g(\xi)\right| &\leq \left|g(\xi)\right|\left|K_{\mu,\tau}^{a,b}(e_0;\xi)-1\right|+\left|g'(\xi)\right|\left|K_{\mu,\tau}^{a,b}\left(\varphi_\xi^1;\xi\right)\right| \\ &\quad+\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}\left\{\sqrt{K_{\mu,\tau}^{a,b}(e_0;\xi)}+\frac{1}{\delta}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}\right\}\omega_1(g';\delta).\end{aligned}$$

By Lemma 2.1 and Lemma 2.2, taking $\delta=\sqrt{\delta_{\mu,\tau}^{a,b}(\xi)}=\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}$, the desired result is achieved. □

In the next conclusion, we establish the order of global approximation.

Theorem 2.9. *If $g\in C^1[0,1]$, τ is a non-negative integer and $0\leq a\leq b$, then for every $\mu\in\mathbb{N}$ such that $\tau<\mu/2$ and $\xi\in[0,1]$, we have*

$$\left|K_{\mu,\tau}^{a,b}(g;\xi)-g(\xi)\right|\leq M\lambda_\mu^{a,b}+2\sqrt{\delta_{\mu,\tau}^{a,b}}\,\omega_1\left(g';\sqrt{\delta_{\mu,\tau}^{a,b}}\right),$$

where

$$M=\max_{\xi\in[0,1]}\left|g'(\xi)\right|,\ \lambda_\mu^{a,b}=\max\left\{\frac{2a+1}{2(\mu+b+1)},\frac{|-2b+2a-1|}{2(\mu+b+1)}\right\}$$

and $\delta_{\mu,\tau}^{a,b}=\max_{\xi\in[0,1]}\delta_{\mu,\tau}^{a,b}(\xi)$.

Next, we will prove a theorem providing an upper estimate for the L_p-norm of the approximation error via the operators $K_{\mu,\tau}^{a,b}$.

Firstly, let us recall that L_p modulus of smoothness of the first order is defined by for all $g\in L_p[0,1-h]$

$$\omega_1(g;\delta)_p=\sup_{0<h\leqslant\delta}\left\|g(\cdot+h)-g(\cdot)\right\|_{L_p[0,1-h]},\qquad(1\leqslant p<\infty),$$

where $\|\cdot\|_{L_p[0,1-h]}$ is the L_p-norm defined over $[0,1-h]$ (see, e.g., [8]).

Peetre's K-functional is very beneficial means for the error of the convergence in L_p-norm. Now, we give Peetre's K-functional for functions in $L_p[0,1]$. Let, $L_p^{(1)}[0,1]$, $1\leqslant p<\infty$, denote

$$\begin{aligned}L_p^{(1)}[0,1] &= \Big\{g\in L_p[0,1]: g \text{ is absolutely continuous function} \\ &\qquad \text{on } [0,1] \text{ and } g'\in L_p[0,1]\Big\}.\end{aligned}$$

For any $g\in L_p[0,1]$, $1\leqslant p<\infty$, and $\delta>0$, Peetre's K-functional is given as below

$$K_{1,p}(g;\delta):=\inf\left\{\left\|g-\psi\right\|_{L_p[0,1]}+\delta\left\|\psi'\right\|_{L_p[0,1]}:\psi\in L_p^{(1)}[0,1]\right\}.$$

The relationship between Peetre's K-functional and L_p modulus of smoothness of the first order is the following form

$$M_1\omega_1(g;\delta)_p\leq K_{1,p}(g;\delta)\leq M_2\omega_1(g;\delta)_p,$$

in which M_1,M_2 are positive constants not depending upon g and δ (see, e.g., [10],[8]).

To prove the rate of convergence in L_p-space, the following result will be useful.

Theorem 2.10. *If $\psi\in L_p^{(1)}[0,1]$, $p>1$, τ is a non-negative integer and $0\leq a\leq b$, then for every $\mu\in\mathbb{N}$ such that $\tau<\mu/2$, the operators $K_{\mu,\tau}^{a,b}$ hold the following*

$$\left\|K_{\mu,\tau}^{a,b}(\psi)-\psi\right\|_{L_p[0,1]}\leq 2^{1/p}\frac{p}{p-1}\max_{\xi\in[0,1]}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}\left\|\psi'\right\|_{L_p[0,1]},$$

where $K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)$ is given as in Lemma 2.2.

Proof. From (5), we can write

$$\begin{aligned}
\left|K_{\mu,\tau}^{a,b}(\psi;\xi)-\psi(\xi)\right| &= (\mu+b+1)\left|\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\left[\psi(l)-\psi(\xi)\right]dl\right| \\
&\leq (\mu+b+1)\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}\int_{\xi}^{l}\left|\psi'(v)\right|dv\,dl \\
&\leq \Omega_{\psi'}(\xi)(\mu+b+1)\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}|l-\xi|\,dl,
\end{aligned}$$

in which $\Omega_{\psi'}(\xi)=\sup_{l\in[0,1]}\frac{1}{l-\xi}\int_{\xi}^{l}\left|\psi'(s)\right|ds$ $(l\neq\xi)$ is the Hardy-Littlewood majorant of ψ'. By the Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
\left|K_{\mu,\tau}^{a,b}(\psi;\xi)-\psi(\xi)\right| &\leq \Omega_{\psi'}(\xi)(\mu+b+1)^{1/2}\left(\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\right)^{1/2}\left(\sum_{j=0}^{\mu} w_{\mu,j,\tau}(\xi)\int_{\frac{j+a}{\mu+b+1}}^{\frac{j+a+1}{\mu+b+1}}(l-\xi)^2\,dl\right)^{1/2} \\
&\leq \Omega_{\psi'}(\xi)\max_{\xi\in[0,1]}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_{\xi}^{2};\xi\right)}. \qquad (11)
\end{aligned}$$

From the Hardy-Littlewood theorem (see, [20]), for $p>1$ the following holds

$$\int_0^1\Omega_{\psi'}^{p}(\xi)\,d\xi\leq 2\left(\frac{p}{p-1}\right)^{p}\int_0^1\left|\psi'(\xi)\right|^{p}d\xi.$$

Hence, (11) immediately follows that

$$\left\|K_{\mu,\tau}^{a,b}(\psi)-\psi\right\|_{L_p[0,1]}\leq 2^{1/p}\frac{p}{p-1}\max_{\xi\in[0,1]}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_{\xi}^{2};\xi\right)}\left\|\psi'\right\|_{L_p[0,1]}.$$

□

In what follows, for the L_p-norm of the approximation error via the specified operators $K_{\mu,\tau}^{a,b}$, we give the following upper estimate.

Theorem 2.11. *If $g\in L_p[0,1]$, $p>1$, $\tau>1$ and $0\leq a\leq b$, then for every $\mu\in\mathbb{N}$ such that $\tau<\mu/2$, the operators $K_{\mu,\tau}^{a,b}$ satisfy*

$$\left\|K_{\mu,\tau}^{a,b}(g)-g\right\|_{L_p[0,1]}\leq\left(1+C_1^{1/p}\right)M_2\left(1+\frac{2^{1/p}}{1+C_1^{1/p}}\frac{p}{p-1}\right)\omega_1\left(g;\Gamma_{\mu,\tau}^{a,b}\right)_p,$$

where $C_1=\frac{2\tau+2+b}{\tau+3}$, $M_2>0$, and $\Gamma_{\mu,\tau}^{a,b}=\max_{\xi\in[0,1]}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_{\xi}^{2};\xi\right)}$.

Proof. We can write

$$\left\|K_{\mu,\tau}^{a,b}(g)-g\right\|_{L_p[0,1]}=\begin{cases}\left(1+C_1^{1/p}\right)\|g\|_{L_p[0,1]}; & g\in L_p[0,1]\\ 2^{1/p}\frac{p}{p-1}\Gamma_{\mu,\tau}^{a,b}\left\|g'\right\|_{L_p[0,1]}; & g\in L_p^{(1)}[0,1]\end{cases},$$

where $C_1=\frac{2\tau+2+b}{\tau+3}$ and $\Gamma_{\mu,\tau}^{a,b}=\max_{\xi\in[0,1]}\sqrt{K_{\mu,\tau}^{a,b}\left(\varphi_\xi^2;\xi\right)}$ by Theorem 2.10. For arbitrary $\psi\in L_p^{(1)}[0,1]$, one has

$$\begin{aligned}\left\|K_{\mu,\tau}^{a,b}(g)-g\right\|_{L_p[0,1]} &\leq \left(1+C_1^{1/p}\right)\|g-\psi\|_{L_p[0,1]}+2^{1/p}\frac{p}{p-1}\Gamma_{\mu,\tau}^{a,b}\left\|\psi'\right\|_{L_p[0,1]}\\ &\leq \left(1+C_1^{1/p}\right)\left\{\|g-\psi\|_{L_p[0,1]}+2^{1/p}\frac{p}{p-1}\frac{1}{1+C_1^{1/p}}\Gamma_{\mu,\tau}^{a,b}\left\|\psi'\right\|_{L_p[0,1]}\right\}\\ &\leq \left(1+C_1^{1/p}\right)K_{1,p}\left(g;\frac{2^{1/p}}{1+C_1^{1/p}}\frac{p}{p-1}\Gamma_{\mu,\tau}^{a,b}\right)\\ &\leq \left(1+C_1^{1/p}\right)M_2\left(1+\frac{2^{1/p}}{1+C_1^{1/p}}\frac{p}{p-1}\right)\omega_1\left(g;\Gamma_{\mu,\tau}^{a,b}\right)_p, \quad M_2>0.\end{aligned}$$

□

Remark 2.12. *We remark that for $\tau=0$ and $\tau=1$, the similar inequality for $K_{\mu,0}^{a,b}$ and $K_{\mu,1}^{a,b}$ was obtained as a special case in [14, Theorem 3.7].*

3. Numerical Examples

In the present section, we will give some graphical and numerical illustrations with a view to demonstrating the approximation process by the generalized Kantorovich-Stancu operators $K_{\mu,\tau}^{a,b}$ with the help of Maple.

Example 3.1. *The convergence of the generalized Kantorovich-Stancu operators $K_{\mu,\tau}^{a,b}(g)$ to $g(\xi)=\xi(\xi-1)(\xi-2)$ is shown in Figure 1 for $a=6$, $b=6.3$, $\mu=16$ and different values of the parameter τ. This example gives information about how the change of non-negative integer parameter τ affects the convergence, where μ is any natural number such that $\tau<\mu/2$. As demonstrated in Figure 1, our operators $K_{\mu,\tau}^{a,b}$ get a better approximation for some values of the parameter τ.*

Example 3.2. *The convergence of the generalized Kantorovich-Stancu operators $K_{\mu,\tau}^{a,b}(g)$ to $g(\xi)=\cos(3\pi\xi)-\sin(8\xi)$ is illustrated in Figure 2 for $a=0.2$, $b=0.3$, $\tau=2$ and different values of μ. In this example, one can see that the higher values of μ gives better approximation.*

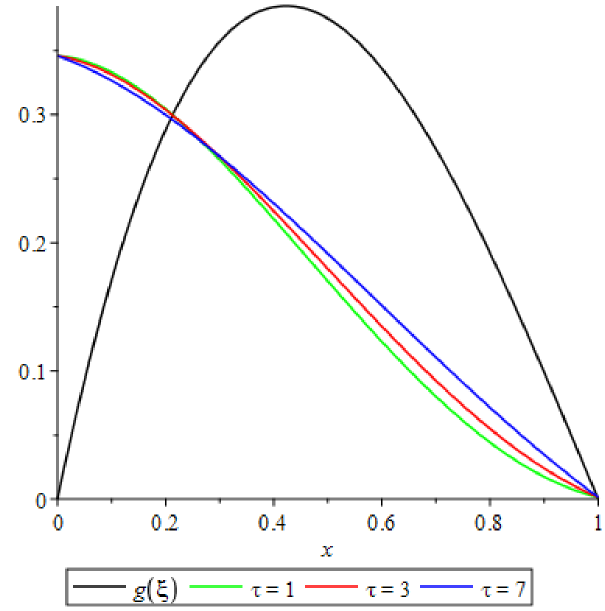

Figure 1: Approximation process of the operators $K_{\mu,\tau}^{a,b}(g)$ to $g(\xi) = \xi(\xi-1)(\xi-2)$ for $\tau = 1, 3, 7$.

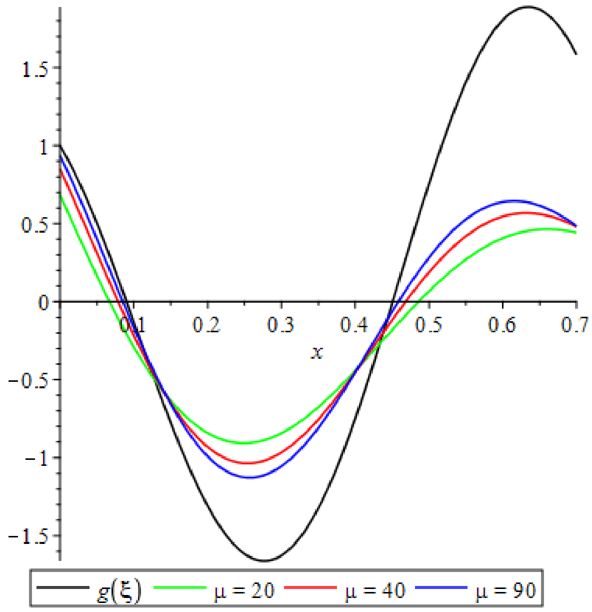

Figure 2: Approximation process of the operators $K_{\mu,\tau}^{a,b}(g)$ to $g(\xi) = \cos(3\pi\xi) - \sin(8\xi)$ for $\mu = 20, 40, 90$.

The absolute value of the difference of the new operator $K_{\mu,\tau}^{a,b}(g)$ with the function $g(\xi) = \cos(3\pi\xi) - \sin(8\xi)$ is given in Table 1 for $a = 0.2$, $b = 0.3$, $\tau = 2$ and different values of μ, ξ.

Table 1 Error estimation by the operators of $K_{\mu,\tau}^{a,b}(g)$.

μ	$\left\vert K_{\mu,\tau}^{a,b}(g;0.1)-g(0.1)\right\vert$	$\left\vert K_{\mu,\tau}^{a,b}(g;0.45)-g(0.45)\right\vert$	$\left\vert K_{\mu,\tau}^{a,b}(g;0.82)-g(0.82)\right\vert$
5	0.3851205133	0.2747460561	0.2309736595
50	0.0691645807	0.0891604983	0.2261148328
100	0.0307246684	0.0474667620	0.1780727214
200	0.0095990733	0.0227028337	0.1508059047
1000	0.0083250185	0.0006795308	0.1274005873

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