Filomat 39:28 (2025), 10077–10087
https://doi.org/10.2298/FIL2528077C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Alternative algebraic perturbation expressions for the m-weak group inverse and m-weak core inverse

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Abstract. This paper presents several novel algebraic perturbation formulas for the m-weak group inverse and the m-weak core inverse of a matrix A. The nonsingular matrix utilized in these formulas is contingent upon a basis for the nullspace of $(A^k)^*$, where k denotes the index of the matrix A. In particular, some new algebraic perturbation formulas for the Drazin inverse and core-EP inverse are presented for a special case.

1. Introduction

Throughout the paper, we adopt the standard notations. For $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices, let rank(A), A^*, $R(A)$ and $N(A)$, respectively, be its rank, conjugate transpose, range space and null space. I_n is the identity matrix of order n and the n will be omitted when the order is not ambiguous from the context. The symbol P_A represents the orthogonal projector onto a subspace $R(A)$. Let $V^* \in \mathbb{C}^{n\times(n-s)}_{n-s}$ be a matrix whose column form a basis for $N(A^k)^*$. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is defined as the unique matrix $X = A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the equations [3]

$$AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA.$$

For $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(A)$, where Ind(A) represents the index of A (i.e., the smallest nonnegative integer k such that rank(A^k) = rank(A^{k+1})), the Drazin inverse of A is the unique matrix $X = A^D \in \mathbb{C}^{n\times n}$ satisfying the equations [3]

$$XAX = X, XA^{k+1} = A^k, AX = XA.$$

In the spacial case Ind$(A) = 1$, A^D reduces to the group inverse of A. The core-EP inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X = A^{\textcircled{\dagger}} \in \mathbb{C}^{n\times n}$ which satisfies $XAX = X, R(A^k) = R(X) = R(X^*)$ [15]. If Ind$(A) = 1$, $A^{\textcircled{\#}} = A^{\textcircled{\dagger}}$ is the core inverse of A [2].

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 65F20.
Keywords. algebraic perturbation, weak group inverse, weak core inverse, Core-EP inverse.
Received: 03 July 2025; Revised: 30 July 2025; Accepted: 03 August 2025
Communicated by Dijana Mosić
Research supported by Natural Science Research of Jiangsu Higher Education Institutions of China (No.23KJB110004).
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The m-weak group inverse was defined of an element in a ring with involution by Zhou et al. [25]. For $m \in \mathbb{N}$, the m-weak group inverse of $A \in \mathbb{C}^{n\times n}$ with index k is the unique matrix $X = A^{ⓦ_m} \in \mathbb{C}^{n\times n}$ satisfying

$$AX^2 = X, XA^{k+1} = A^k, (A^k)^* A^{m+1} X = (A^k)^* A^m.$$

It was proved that the m-weak group inverse of A can be also characterized by the core-EP inverse, defined by $A^{ⓦ_m} = (A^{\textcircled{\dagger}})^{m+1} A^m$. In the case $m = 1$, the m-weak group inverse reduces to the weak group inverse [23], that is $A^{ⓦ_1} = A^{ⓦ}$. If $m = 2$, the m-weak group inverse coincides with the generalized group inverse [9]. Moreover, if $m \geq k = \mathrm{Ind}(A)$, $A^{ⓦ_m} = A^D$. Further results related to the m-weak group inverse were presented[6, 18–20].

Recently, Ferreyra and Malik [10] introduced another generalization of the core inverse by using the m-weak group inverse. If $m \in \mathbb{N}$, the m-weak core inverse $A^{\textcircled{\#}_m} \in \mathbb{C}^{n\times n}$ of $A \in \mathbb{C}^{n\times n}$ with index k is expressed as $A^{\textcircled{\#}_m} = A^{ⓦ_m} P_{A^m}$. If $m = 1$, then the m-weak core inverse coincides with the weak core inverse [8], that is $A^{\textcircled{\#}_1} = A^{ⓦ,\dagger}$. If $m \geq k = \mathrm{Ind}(A)$, then the m-weak core inverse coincides with the core-EP inverse, that is, $A^{\textcircled{\#}_m} = A^{\textcircled{\dagger}}$.

This paper focus on algebraic perturbation methods to compute the m-weak group inverse and the m-weak core inverse. These methods were initially introduced for addressing nonsingular linear systems in [1]. Subsequently, the algebraic perturbation method for the Moore-Penrose inverse of a singular square matrix A was studied [14]. Since that time, this technique has been broadened to encompass the Moore-Penrose inverse of rectangular matrices, the Drazin inverse, the core-EP inverse, and several other generalized inverses, as referenced in [7, 11, 12, 21].

As for the m-weak group inverse, algebra perturbation expressions are given [17]

$$\begin{aligned} A^{ⓦ_m} &= (A^k(A^k)^*A + V^*V)^{-1} A^k (A^k)^* (A^D)^m A^k (A^k)^\dagger A^m \\ &= (A^k + (A^D)^{m+1} A^k (A^k)^\dagger A^m V^* V)(A^{k+1} + V^*V)^{-1}. \end{aligned} \tag{1}$$

The formulas incorporate not only the index k but also the m-th power of the matrix A, in addition to the basis of the null space of $(A^k)^*$. Since V^* is of full column rank and $R(V^*) = N((A^k)^*)$, we have

$$(V^*)^\dagger = (VV^*)^{-1}V \text{ and } R(V^*) = R(A^k)^\perp,$$

giving that

$$A^k(A^k)^\dagger = P_{R(A^k)} = I - P_{R(V^*)} = I - V^*(V^*)^\dagger = I - V^*(VV^*)^{-1}V. \tag{2}$$

Considering equation (2), it is possible to eliminate the index k from the matrix A, thereby deriving the expression for the m-weak group inverse of A based on a basis of the nullspace of $(A^k)^*$, independent of the index k.

In this paper, we will further study the algebraic perturbation expressions for the m-weak group inverse of A and present several alternative formulas that do not explicitly rely on the index of A and involve a nonsingular matrix. One of these formulas is derived from the bordering technique, which effectively reorganizes the computational process to circumvent unnecessary calculations by inverting only a smaller-sized matrix. Additionally, since there is currently no established algebraic perturbation expression for the m-weak core inverse, we will also present pertinent results related to this concept.

2. Alternative expressions for the m-weak group inverse

In this section, we will present different expressions for the m-weak group inverse. We begin by introducing a different formula of the m-weak group inverse that differs from the expression presented in (1).

Proposition 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$, $m \in \mathbb{N}$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then*

$$A^{ⓦ_m} = (A^k(A^k)^*(A^D)^m A^k (A^k)^\dagger A^{m+1} + V^*V)^{-1} A^k (A^k)^* (A^D)^m A^k (A^k)^\dagger A^m.$$

Proof. Let $P = A^k(A^k)^*(A^D)^m A^k(A^k)^\dagger A^m$, $T = PA$. Firstly we observe that $R(T) \subseteq R(A^k)$. Further by $TA^{k-1}(A^k)^\dagger((A^k)^\dagger)^* = A^k$, we have $R(A^k) \subseteq R(T)$. Therefore $R(A^k) = R(T)$ and $R(T^2) = R(TA^k)$. Since $TA^k(A^D)^{m+1}(A^k)^\dagger A^{m+1} = T$, it gives $R(T) \subseteq R(T^2)$, consequently, $\mathrm{rank}(T) = \mathrm{rank}(T^2)$ and $T^{\#}$ exits. From $R(V^*) = N((A^k)^*)$, we obtain

$$N(V) = R(A^k) = R(A^{Ⓦ_m}),\ VA^{Ⓦ_m} = 0 \text{ and } VT^{\#} = VT(T^{\#})^2 = 0.$$

Let $X = T^{\#} + (V^*V)^\dagger - A^{Ⓦ_m}A(V^*V)^\dagger$. Then

$$\begin{aligned} TA^{Ⓦ_m} &= A^k(A^k)^*(A^D)^m A^k(A^k)^\dagger A^{m+1}A^{Ⓦ_m} \\ &= A^k(A^k)^*(A^D)^m((A^k)^\dagger)^*(A^k)^*A^{m+1}A^{Ⓦ_m} \\ &= A^k(A^k)^*(A^D)^m((A^k)^\dagger)^*(A^k)^*A^m = P. \end{aligned}$$

Hence,

$$\begin{aligned} (T + V^*V)X &= (T + V^*V)(T^{\#} + (V^*V)^\dagger - A^{Ⓦ_m}A(V^*V)^\dagger) \\ &= TT^{\#} + T(V^*V)^\dagger - T(V^*V)^\dagger + V^*V(V^*V)^\dagger \\ &= TT^{\#} + V^*V(V^*V)^\dagger = P_{R(T)} + P_{R(V^*V)} \\ &= P_{R(A^k)} + P_{R(V^*)} = P_{R(A^k)} + P_{R(A^k)^\perp} \\ &= I, \end{aligned}$$

which implies $T + V^*V$ is nonsingular. Since $(T + V^*V)A^{Ⓦ_m} = TA^{Ⓦ_m} = P$, we have

$$\begin{aligned} A^{Ⓦ_m} &= (T + V^*V)^{-1}P \\ &= (A^k(A^k)^*(A^D)^m A^k(A^k)^\dagger A^{m+1} + V^*V)^{-1}A^k(A^k)^*(A^D)^m A^k(A^k)^\dagger A^m. \end{aligned}$$

□

Example 2.2. *Using Proposition 2.1 to compute the m-weak group inverse of the matrix A in [8] where* $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 5 & 2 & 6 \\ 0 & -2 & -1 & -3 \end{pmatrix}$*, for this matrix,* $Ind(A) = 3$ *and perform elementary row operations on* $(A^3)^*$*,*

$$(A^3)^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \\ 18 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and take $V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *so that the columns of* V^* *form a basis for* $N((A^3)^*)$*. Using proposition 2.1, the weak group inverse of A is given by*

$$A^{Ⓦ} = (A^3(A^3)^*A^DA^3(A^3)^\dagger A^2 + V^*V)^{-1}A^3(A^3)^*A^DA^3(A^3)^\dagger A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

The generalized group inverse of A is given by

$$A^{Ⓦ_2} = (A^3(A^3)^*(A^D)^2A^3(A^3)^\dagger A^3 + V^*V)^{-1}A^3(A^3)^*(A^D)^2A^3(A^3)^\dagger A^2 = \begin{pmatrix} 1 & 7 & 4 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The m-weak group inverse of A for each $m \geq Ind(A) = 3$ is given by

$$A^{\textcircled{w}_m} = (A^3(A^3)^*(A^D)^3A^3(A^3)^\dagger A^4 + V^*V)^{-1}A^3(A^3)^*(A^D)^3A^3(A^3)^\dagger A^3$$
$$= \begin{pmatrix} 1 & 10 & 7 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^D.$$

Although different expressions were obtained, the calculation of the above formula is more complex, and not only does it involve index k, but it also involves the m-power of matrix A, as well as the basis of the nullspace of $(A^k)^*$. In the following, we remove the index k of A and obtain the expression for the m-weak group inverse of A on a basis of the null space of $(A^k)^*$. Before introducing this conclusion, let's first present the following result.

Bott and Duffin [4] defined the Bott-Duffin inverse of $A \in \mathbb{C}^{n\times n}$ by $A_L^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1} = P_L(AP_L + I - P_L)^{-1}$ when $AP_L + P_{L^\perp}$ is nonsingular. In [5, 24], the authors showed the weak group inverse by a special Bott-Duffin inverse and the different expressions for weak group inverse are given using different Bott-Duffin inverses.

Lemma 2.3. [22] *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and rank $A^k = r$. Then*

$$A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^* \tag{4}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $T \in \mathbb{C}^{r\times r}$ is nonsingular upper-triangular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent of index k. For $m \in \mathbb{N}$, it follows by [13]

$$A^{\textcircled{w}_m} = U\begin{pmatrix} T^{-1} & T^{-(m+1)}\widetilde{T}_m \\ 0 & 0 \end{pmatrix}U^*,$$

where $\widetilde{T}_m = \Sigma_{j=0}^{m-1}T^jSN^{m-1-j}$.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and $m \in \mathbb{N}$. Then*

$$A^{\textcircled{w}_m} = (A^{m+1})^{(-1)}_{R(A^k)}A^m = (P_{A^k}A^{m+1}P_{A^k})^\dagger A^m.$$

Proof. Assume that A is given by (4), we have $A^k = U\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*$, where $\widetilde{T}_k = \Sigma_{j=0}^{k-1}T^jSN^{k-1-j}$. Then $P_{R(A^k)} = A^k(A^k)^\dagger = U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}U^*$. Thus,

$$\begin{aligned}(A^{m+1})^{(-1)}_{R(A^k)}A^m &= P_{R(A^k)}(A^{m+1}P_{R(A^k)} + I - P_{R(A^k)})^{-1}A^m \\ &= U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^{m+1} & 0 \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} T^m & \widetilde{T}_m \\ 0 & N^m \end{pmatrix}U^* \\ &= U\begin{pmatrix} T^{-1} & T^{-(m+1)}\widetilde{T}_m \\ 0 & 0 \end{pmatrix}U^* \\ &= A^{\textcircled{w}_m}.\end{aligned}$$

Similarly, by a direct calculation, we can derive that $A^{\textcircled{w}_m} = (P_{A^k}A^{m+1}P_{A^k})^\dagger A^m$. □

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$, $m \in \mathbb{N}$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Define $T_1 = A^{m+1}(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$. Then T_1 is nonsingular and $A^{\textcircled{w}_m} = (I - V^*(VV^*)^{-1}V)T_1^{-1}A^m$.*

Proof. From Theorem 2.4, we have $I + (A^{m+1} - I)A^k(A^k)^\dagger$ is nonsingular, and

$$A^{\textcircled{w}_m} = A^k(A^k)^\dagger(I + (A^{m+1} - I)A^k(A^k)^\dagger)^{-1}A^m.$$

In view of (2), we can write

$$\begin{aligned} I + (A^{m+1} - I)A^k(A^k)^\dagger &= I + (A^{m+1} - I)(I - V^*(VV^*)^{-1}V) \\ &= A^{m+1}(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V, \end{aligned}$$

which immediately implies the result. □

It is shown in [13] that the m-weak group inverse of A can be extracted from the regular inverse of nonsingular bordered matrix of A when V^* and W are the full column rank such that $R(W) = N((A^k)^*A^m)$ and $R(A^k) = N(V)$,

$$\begin{pmatrix} A & W \\ V & 0 \end{pmatrix} \tag{5}$$

due to the fact that

$$\begin{pmatrix} A & W \\ V & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A^{\textcircled{w}_m} & (I - A^{\textcircled{w}_m}A)V^\dagger \\ W^\dagger(I - AA^{\textcircled{w}_m}) & W^\dagger(AA^{\textcircled{w}_m}A - A)V^\dagger \end{pmatrix}. \tag{6}$$

This bordering technique requires calculating the inverse of a larger matrix, but only a portion of that inverse is utilized for the m-weak group inverse of A. Given that the complexity of matrix inversion grows rapidly with size, a significant amount of the computation involved in this bordering method appears to be inefficient. Therefore, it is important to develop methods for constructing the (1, 1)-block of the inverse of the bordered matrix without having to compute the entire inverse directly. To support this, let's revisit famous Schur's lemma that can be easily confirmed.

Lemma 2.6. (Schur's lemma) *Let A_{22} and*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be nonsingular. Then $T = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is also nonsingular and

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} T^{-1} & -T^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}T^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}T^{-1}A_{12}A_{22}^{-1} \end{pmatrix}.$$

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and $m \in \mathbb{N}$. Let V^* and W be the matrix whose columns form bases for $N((A^k)^*)$ and $N((A^k)^*A^m)$ respectively. Define $T_2 = A - (AV^* + W)(VV^*)^{-1}V$. Then T_2 is nonsingular and $A^{\textcircled{w}_m} = (I - V^*(VV^*)^{-1}V)T_2^{-1}$.*

Proof. Obviously, VV^* is nonsingular. From the nonsingularity of bordered matrix (5),

$$\begin{pmatrix} A & W \\ V & 0 \end{pmatrix}\begin{pmatrix} I & V^* \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AV^* + W \\ V & VV^* \end{pmatrix}$$

is also nonsingular and

$$\begin{pmatrix} A & W \\ V & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I & V^* \\ 0 & I \end{pmatrix}\begin{pmatrix} A & AV^* + W \\ V & VV^* \end{pmatrix}^{-1}. \tag{7}$$

From Lemma 2.6,

$$\begin{pmatrix} A & AV^* + W \\ V & VV^* \end{pmatrix}^{-1} = \begin{pmatrix} T_2^{-1} & -T_2^{-1}(AV^* + W)(VV^*)^{-1} \\ -(VV^*)^{-1}VT_2^{-1} & (VV^*)^{-1} + (VV^*)^{-1}VT_2^{-1}(AV^* + W)(VV^*)^{-1} \end{pmatrix}. \tag{8}$$

Substitute equation (8) into (7), and then compare with (1, 1)-block of (6), we have

$$A^{\text{Ⓦ}_m} = (I - V^*(VV^*)^{-1}V)T_2^{-1}.$$

□

Similar to Theorem 2.5, the algebraic perturbation expression for the m-weak group inverse in Theorem 2.7 also uses the nonsingular matrix VV^*. The difference is that Theorem 2.7 uses two bases for calculation, making the calculation more complex. However, as can be seen from the following example, the calculation of a basis for $N((A^k)^*A^m)$ can be done using the operation process of a basis for $N((A^k)^*)$.

Example 2.8. *Use the algebraic perturbation expressions in Theorems 2.5 and 2.7 to compute the m-weak group inverse of the matrix in Example 2.2.*

(1) If the algebra perturbation method of Theorem 2.5 is chosen, then we have

$$G = V^*(VV^*)^{-1}V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the matrix $T_1 = A^{m+1}(I - G) + G$, since different values of m correspond to different generalized inverses, we will only take the case when $m = 1$, in this case $T_1 = I$, compute $(I - G)T_1^{-1}A$, obtaining the same result as in (3).

(2) If the algebra perturbation method of Theorem 2.7 is chosen and $m = 1$, then perform elementary row operations on $(A^3)^*A$,

$$(A^3)^*A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \\ 18 & 0 & 0 & 0 \end{pmatrix} A \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 5 & 2 & 6 \\ 0 & -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Take $W = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that the columns of W form a basis for $N((A^3)^*A)$. Then

$$T_2 = A - (AV^* + W)(VV^*)^{-1}V = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Compute $(I - G)T_2^{-1}$, with this W and V, the algebraic perturbation expression of Theorem 2.7 does produce accurately the weak group inverse as in (3).

For $m = 1$ and $m = 2$ in Proposition 2.1, Theorems 2.5 and 2.7, we get new expressions for the weak group inverse and the generalized group inverse respectively.

Corollary 2.9. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then $T_1 = A^2(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$ is nonsingular and*

$$\begin{aligned} A^{\text{Ⓦ}} &= (A^k(A^k)^*A^k(A^{k+1})^\dagger A^2 + V^*V)^{-1}A^k(A^k)^*A^k(A^{k+1})^\dagger A \\ &= (I - V^*(VV^*)^{-1}V)T_1^{-1}A. \end{aligned}$$

In addition, if W is the matrix whose columns form a basis for $N((A^k)^*A)$, then $T_2 = A - (AV^* + W)(VV^*)^{-1}V$ is nonsingular and $A^{\text{Ⓦ}} = (I - V^*(VV^*)^{-1}V)T_2^{-1}$.

Corollary 2.10. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then $T_1 = A^3(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$ is nonsingular and*

$$\begin{aligned} A^{\textcircled{w}_2} &= (A^k(A^k)^*A^k(A^{k+2})^\dagger A^3 + V^*V)^{-1}A^k(A^k)^*A^k(A^{k+2})^\dagger A^2 \\ &= (I - V^*(VV^*)^{-1}V)T_1^{-1}A^2. \end{aligned}$$

In addition, if W is the matrix whose columns form a basis for $N((A^k)^*A^2)$, then $T_2 = A - (AV^* + W)(VV^*)^{-1}V$ is nonsingular and $A^{\textcircled{w}_2} = (I - V^*(VV^*)^{-1}V)T_2^{-1}$.

Proposition 2.1, Theorems 2.5 and 2.7 also imply representations for Drazin inverse when $m = k$. Then we can conclude that the first formula of A^D in the following is the result of [17, Corollary 2.5].

Corollary 2.11. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then $T_1 = A^{k+1}(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$ is nonsingular and*

$$\begin{aligned} A^D &= (A^k(A^k)^*A + V^*V)^{-1}A^k(A^k)^*A^k(A^{2k})^\dagger A^k \\ &= (I - V^*(VV^*)^{-1}V)T_1^{-1}A^k. \end{aligned}$$

In addition, if W is the matrix whose columns form a basis for $N(A^k)$, then $T_2 = A - (AV^* + W)(VV^*)^{-1}V$ is nonsingular and $A^D = (I - V^*(VV^*)^{-1}V)T_2^{-1}$.

Proof. When $m = k$, from Proposition 2.1, we have

$$\begin{aligned} T + V^*V &= A^k(A^k)^*(A^D)^kA^k(A^k)^\dagger A^{k+1} + V^*V \\ &= A^k(A^k)^*A^DA^2 + V^*V \end{aligned}$$

is nonsingular, $(T + V^*V)^{-1} = T^{\textcircled{\#}} + (V^*V)^\dagger - A^{\textcircled{w}_m}A(V^*V)^\dagger$ and $(T + V^*V)^{-1}A^k(A^k)^*(A^D)^kA^k(A^k)^\dagger A^k = A^D$. Let $H = A^k(A^k)^*(A - A^DA^2)$. Then

$$\begin{aligned} A^k(A^k)^*A + V^*V &= A^k(A^k)^*A^DA^2 + V^*V + A^k(A^k)^*(A - A^DA^2) \\ &= T + V^*V + H \\ &= (T + V^*V)(I + (T + V^*V)^{-1}H) \end{aligned}$$

From $VA^k = 0$, we have $(V^*V)^\dagger H = (V^*V)^\dagger V^*V(V^*V)^\dagger H = (V^*V)^\dagger((V^*V)^\dagger)^*V^*VH = 0$. Therefore, $(T + V^*V)^{-1}H = T^{\textcircled{\#}}H + (V^*V)^\dagger H - A^{\textcircled{w}_m}A(V^*V)^\dagger H = T^{\textcircled{\#}}H$ and $(T^{\textcircled{\#}}H)^2 = T^{\textcircled{\#}}HT(T^{\textcircled{\#}})^2H = 0$. Thus, $I + (T + V^*V)^{-1}H$ is invertible. From $(I + (T + V^*V)^{-1}H)A^D = A^D$, which gives $(I + (T + V^*V)^{-1}H)^{-1}A^D = A^D$. Then

$$\begin{aligned} (A^k(A^k)^*A + V^*V)^{-1}A^k(A^k)^*A^k(A^{2k})^\dagger A^k &= (T + V^*V + H)^{-1}A^k(A^k)^*A^k(A^{2k})^\dagger A^k \\ &= ((T + V^*V)(I + (T + V^*V)^{-1}H))^{-1}A^k(A^k)^*A^k(A^{2k})^\dagger A^k \\ &= (I + (T + V^*V)^{-1}H)^{-1}(T + V^*V)^{-1}A^k(A^k)^*A^k(A^{2k})^\dagger A^k \\ &= (I + (T + V^*V)^{-1}H)^{-1}(T + V^*V)^{-1}A^k(A^k)^*(A^D)^kA^k(A^k)^\dagger A^k \\ &= (I + (T + V^*V)^{-1}H)^{-1}A^D = A^D. \end{aligned}$$

The remaining equations follow immediately from Theorems 2.5 and 2.7. □

3. Alternative expressions for the *m*-weak core inverse

In this section, we will give expressions for m-weak core inverse by utilizing the results from m-weak group inverse and the relationship between m-weak core inverse and m-weak group inverse.

Proposition 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$, $m \in \mathbb{N}$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then*

(1) $A^{\textcircled{\#}_m} = (A^k(A^k)^*A + V^*V)^{-1}A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}$.
(2) $A^{\textcircled{\#}_m} = (A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^{m+1} + V^*V)^{-1}A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}$.
(3) $A^{\textcircled{\#}_m} = (A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}A + V^*V)^{-1}A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}$.
(4) $A^{\textcircled{\#}_m} = (A^k + A^k(A^{k+m+1})^\dagger A^mP_{A^m}VV^*)(A^{k+1} + VV^*)^{-1}$.

Proof. (1) and (2) are clear from the equality (1) and Proposition 2.1 and $A^{\textcircled{\#}_m} = A^{\textcircled{w}_m}P_{A^m}$.

(3). Let $P = A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}$, $T = PA$. We firstly observe that $R(T) \subseteq R(A^k)$, further by $T(A^D)^{2m+1}A^{2m+k}(A^k)^\dagger((A^k)^\dagger)^* = A^k$, it gives $R(A^k) = R(T)$. Since $R(T^2) = R(TA^k)$ and $TA^k(A^D)^{m+1}(A^k)^\dagger A^mP_{A^m}A = T$, we have $R(T) \subseteq R(T^2)$. Therefore $\mathrm{rank}(T) = \mathrm{rank}(T^2)$ and $T^{\#}$ exits. From $R(V^*) = N((A^k)^*)$, we obtain

$$N(V) = R(A^k) = R(A^{\textcircled{\#}_m}),\ VA^{\textcircled{\#}_m} = 0 \text{ and } VT^{\#} = VT(T^{\#})^2 = 0.$$

Since $P_{A^m}(A^{\oplus})^m = A^m(A^m)^\dagger(A^D)^mA^k(A^k)^\dagger = (A^D)^mA^k(A^k)^\dagger = (A^{\oplus})^m$, then

$$\begin{aligned}
TA^{\textcircled{\#}_m} &= A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}AA^{\textcircled{\#}_m}\\
&= A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}(A^{\oplus})^mA^mP_{A^m}\\
&= A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^m(A^{\oplus})^mA^mP_{A^m}\\
&= A^k(A^k)^*(A^{\oplus})^mA^m(A^{\oplus})^mA^mP_{A^m}\\
&= A^k(A^k)^*(A^{\oplus})^mA^mP_{A^m}\\
&= A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m} = P.
\end{aligned}$$

Let $X = T^{\#} + (V^*V)^\dagger - A^{\textcircled{\#}_m}A(V^*V)^\dagger$. Then

$$\begin{aligned}
(T + V^*V)X &= (T + V^*V)(T^{\#} + (V^*V)^\dagger - A^{\textcircled{\#}_m}A(V^*V)^\dagger)\\
&= TT^{\#} + T(V^*V)^\dagger - T(V^*V)^\dagger + V^*V(V^*V)^\dagger\\
&= TT^{\#} + V^*V(V^*V)^\dagger = P_{R(T)} + P_{R(V^*V)}\\
&= P_{R(A^k)} + P_{R(V^*)} = P_{R(A^k)} + P_{R(A^k)^\perp}\\
&= I,
\end{aligned}$$

which implies $T + V^*V$ is nonsingular. Since $(T + V^*V)A^{\textcircled{\#}_m} = TA^{\textcircled{\#}_m} = P$, we have

$$\begin{aligned}
A^{\textcircled{\#}_m} &= (T + V^*V)^{-1}P\\
&= (A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}A + V^*V)^{-1}A^k(A^k)^*(A^D)^mA^k(A^k)^\dagger A^mP_{A^m}.
\end{aligned}$$

(4). According to [16, Theorem 2.4], it gives that $A^{k+1} + VV^*$ is nonsingular. Hence

$$\begin{aligned}
A^{\textcircled{\#}_m}(A^{k+1} + VV^*) &= A^{\textcircled{\#}_m}A^{k+1} + A^{\textcircled{\#}_m}VV^*\\
&= A^k + A^k(A^{k+m+1})^\dagger A^mP_{A^m}VV^*
\end{aligned}$$

implies the result. □

From $A^{\textcircled{\#}_m} = A^{\textcircled{w}_m}P_{A^m}$, Theorems 2.4 and 2.5, we have the following results.

Proposition 3.2. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and $m \in \mathbb{N}$. Then*

$$A^{\textcircled{\#}_m} = (A^{m+1})^{(-1)}_{R(A^k)}A^mP_{A^m} = (P_{A^k}A^{m+1}P_{A^k})^\dagger A^mP_{A^m}.$$

Proposition 3.3. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$, $m \in \mathbb{N}$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Define $T_1 = A^{m+1}(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$. Then T_1 is nonsingular and $A^{\textcircled{\#}_m} = (I - V^*(VV^*)^{-1}V)T_1^{-1}A^mP_{A^m}$.*

Next, we develop a relation between m-weak core inverse and a corresponding nonsingular bordered matrix.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and $m \in \mathbb{N}$. Suppose that V^* and M are the full column rank such that $R(M) = N((A^k)^*A^mP_{A^m})$ and $R(A^k) = N(V)$. Then*

$$X = \begin{pmatrix} A & M \\ V & 0 \end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{\textcircled{\#}_m} & (I - A^{\textcircled{\#}_m}A)V^\dagger \\ M^\dagger(I - AA^{\textcircled{\#}_m}) & M^\dagger(AA^{\textcircled{\#}_m}A - A)V^\dagger \end{pmatrix}. \tag{9}$$

Proof. By [10, Theorem 4.7], we have $R(A^{\textcircled{\#}_m}) = R(A^k)$, $N(A^{\textcircled{\#}_m}) = N((A^k)^*A^mP_{A^m})$, $A^{\textcircled{\#}_m}AA^{\textcircled{\#}_m} = A^{\textcircled{\#}_m}$. From

$$R(I - AA^{\textcircled{\#}_m}) = N(A^{\textcircled{\#}_m}) = N((A^k)^*A^mP_{A^m}) = R(M) = R(MM^\dagger) = N(I - MM^\dagger),$$

it gives that $(I - MM^\dagger)(I - AA^{\textcircled{\#}_m}) = 0$, that is $MM^\dagger(I - AA^{\textcircled{\#}_m}) = I - AA^{\textcircled{\#}_m}$. Further, $R(A^{\textcircled{\#}_m}) = R(A^k) = N(V)$ implies $VA^{\textcircled{\#}_m} = 0$. Let Z be the right hand side of (9). Then

$$\begin{aligned} XZ &= \begin{pmatrix} AA^{\textcircled{\#}_m} + MM^\dagger(I - AA^{\textcircled{\#}_m}) & A(I - A^{\textcircled{\#}_m}A)V^\dagger + MM^\dagger(AA^{\textcircled{\#}_m}A - A)V^\dagger \\ VA^{\textcircled{\#}_m} & V(I - A^{\textcircled{\#}_m}A)V^\dagger \end{pmatrix} \\ &= \begin{pmatrix} AA^{\textcircled{\#}_m} + I - AA^{\textcircled{\#}_m} & (I - AA^{\textcircled{\#}_m})AV^\dagger - (I - AA^{\textcircled{\#}_m})AV^\dagger \\ 0 & VV^\dagger \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I. \end{aligned}$$

Thus, the matrix X is invertible and its inverse is equal to Z. □

Utilizing the (1, 1)-block of the nonsingular bordered matrix, we derive the expression for the m-weak core inverse.

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and $m \in \mathbb{N}$.*

(1) *If W and V^* and are the matrix whose columns form bases for $N((A^k)^*A^m)$ and $N((A^k)^*)$ respectively, then $T_2 = A - (AV^* + W)(VV^*)^{-1}V$ is nonsingular and*

$$A^{\textcircled{\#}_m} = (I - V^*(VV^*)^{-1}V)T_2^{-1}P_{A^m}.$$

(2) *If M and V^* are the matrix whose columns form bases for $N((A^k)^*A^mP_{A^m})$ and $N((A^k)^*)$ respectively, then $T_3 = A - (AV^* + M)(VV^*)^{-1}V$ is nonsingular and*

$$A^{\textcircled{\#}_m} = (I - V^*(VV^*)^{-1}V)T_3^{-1}.$$

Proof. (1). It follows by $A^{\textcircled{\#}_m} = A^{\textcircled{w}_m}P_{A^m}$ and Theorem 2.7.

(2). From the proof of Theorem 2.7, replace T_3 with T_2 in (8) and compare with (1, 1)-block of (9), then we have $A^{\textcircled{\#}_m} = (I - V^*(VV^*)^{-1}V)T_3^{-1}$. □

Example 3.6. *Use the algebraic perturbation expressions in Proposition 3.3 and Theorem 3.5 to compute the m-weak core inverse of the matrix in Example 2.2.*

(1) If the algebra perturbation method of Proposition 3.3 is chosen and let $m = 1$, then take V and G as in Examples 2.2 and 2.8, by calculation, it follows that

$$A^{\textcircled{w},\dagger} = (I - G)T_1^{-1}AP_A = \begin{pmatrix} 1 & \frac{9}{11} & \frac{9}{11} & -\frac{6}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

(2) If the algebra perturbation method of Theorem 3.5(1) is chosen and let $m = 1$, then $A^{\text{Ⓦ},\dagger} = (I - G)T_2^{-1}P_A$ is the same as (10).

(3) If the algebra perturbation method of Theorem 3.5(2) is chosen and $m = 1$, then perform elementary row operations on $(A^3)^*AP_A$ which can be done using the operation process of $((A^3)^*)A$,

$$(A^3)^*AP_A \to \begin{pmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}P_A = \begin{pmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{10}{11}&-\frac{1}{11}&-\frac{3}{11}\\0&-\frac{1}{11}&\frac{10}{11}&-\frac{3}{11}\\0&-\frac{3}{11}&-\frac{3}{11}&\frac{2}{11}\end{pmatrix}$$
$$= \begin{pmatrix}1&\frac{9}{11}&\frac{9}{11}&-\frac{6}{11}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Take $M = \begin{pmatrix}6&9&9\\0&-11&0\\0&0&-11\\11&0&0\end{pmatrix}$, *so that the columns of M form a basis for $N((A^3)^*AP_A)$. Then*

$$T_3 = A - (AV^* + M)(VV^*)^{-1}V = \begin{pmatrix}1&6&-9&-9\\0&0&11&0\\0&0&0&11\\0&11&0&0\end{pmatrix}.$$

Compute $(I - G)T_3^{-1}$, the algebraic perturbation expression does produce accurately the weak core inverse as in (10).

For $m = 1$ in Propositions 3.1 and 3.3, Theorem 3.5, we get new expressions for the weak core inverse.

Corollary 3.7. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then $T_1 = A^2(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$ is nonsingular and*

$$\begin{aligned}A^{\text{Ⓦ},\dagger} &= (A^k(A^k)^*A + V^*V)^{-1}A^k(A^k)^*A^k(A^{k+1})^\dagger AP_A\\ &= (A^k(A^k)^*A^k(A^{k+1})^\dagger A^2 + V^*V)^{-1}A^k(A^k)^*A^k(A^{k+1})^\dagger AP_A\\ &= (A^k + A^k(A^{k+2})^\dagger AP_AVV^*)(A^{k+1} + VV^*)^{-1}\\ &= (I - V^*(VV^*)^{-1}V)T_1^{-1}AP_A.\end{aligned}$$

In addition,

(1) if W is the matrix whose columns form a basis for $N((A^k)^*A)$, then $T_2 = A-(AV^*+W)(VV^*)^{-1}V$ is nonsingular and $A^{\text{Ⓦ},\dagger} = (I - V^*(VV^*)^{-1}V)T_2^{-1}P_A$.

(2) if M is the matrix whose columns form a basis for $N((A^k)^*AP_A)$, then $T_3 = A - (AV^* + M)(VV^*)^{-1}V$ is nonsingular and $A^{\text{Ⓦ},\dagger} = (I - V^*(VV^*)^{-1}V)T_3^{-1}$.

For $m = k$, Propositions 3.1 and 3.3, Theorem 3.5 also impliy representations for core-EP inverse. Then we can conclude that the first formula of $A^{\textcircled{\dagger}}$ in the following is the result of [16, Theorem 2.3].

Corollary 3.8. *Let $A \in \mathbb{C}^{n\times n}$, $k = Ind(A)$ and V^* be the matrix whose columns form a basis for $N((A^k)^*)$. Then $T_1 = A^{k+1}(I - V^*(VV^*)^{-1}V) + V^*(VV^*)^{-1}V$ is nonsingular and*

$$\begin{aligned}A^{\textcircled{\dagger}} &= (A^k(A^k)^*A + V^*V)^{-1}A^k(A^k)^*\\ &= (A^k(A^k)^*A^DA^2 + V^*V)^{-1}A^k(A^k)^*\\ &= (A^k + A^k(A^{k+1})^\dagger VV^*)(A^{k+1} + VV^*)^{-1}\\ &= (I - V^*(VV^*)^{-1}V)T_1^{-1}A^kP_{A^k}\end{aligned}$$

In addition,

(1) if W is the matrix whose columns form a basis for $N((A^k)$, then $T_2 = A - (AV^* + W)(VV^*)^{-1}V$ is nonsingular and $A^{\oplus} = (I - V^*(VV^*)^{-1}V)T_2^{-1}P_{A^k}$.

(2) if M is the matrix whose columns form a basis for $N((A^k)^*A^kP_{A^k})$, then $T_3 = A - (AV^* + M)(VV^*)^{-1}V$ is nonsingular and $A^{\oplus} = (I - V^*(VV^*)^{-1}V)T_3^{-1}$.

Acknowledgments

The authors thank the editor and reviewers sincerely for their constructive comments and suggestions that have improved the quality of the paper.

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