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Curvature invariants of Lagrangian Riemannian submersions from locally product spaces

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Abstract. In this paper, we obtain various inequalities which involve the Ricci and scalar curvatures of horizontal and vertical distributions of Lagrangian Riemannian submersion defined from locally product spaces onto a Riemannian manifold. Also, we obtain the Chen-Ricci inequality for the said Riemannian submersion. In the end, we give a non-trivial example.

1. Introduction

Riemannian invariants are of primary importance in Riemmanian geometry. These invariants determine the extrinsic and intrinsic properties of Riemannian manifolds which in turn affect the behaviour of the manifold in general form. The relationship between intrinsic and extrinsic invariants was established by Chen [10]. He established a link between main intrinsic invariants and main extrinsic invariants in the form some inequalities. Chen [8] also established a relationship between the squared mean curvature and Ricci curvature of a submanifold in the form of an inequality. In 2005, Chen [7] proved the generalized version of this inequality, know as Chen-Ricci inequality, for arbitrary submanifolds in an arbitrary Riemannian manifold. Later, this inequality was studied by many authors in different settings. On the other hand, the theory of Riemannian submersions was initiated by Neill [24]. A Riemannian submersion gives rise to two orthogonal complementary distibutions, horizontal and vertical, the vertical being always integrable. In addition to being of fundamental importance in Riemannian geometry, Riemannian submersions are also of great interest in many areas of theoretical physics like Yang-Mills theory [6, 38], Kaluza-Klein theory [5, 16], supergravity and string theories [19]. To explore the theory further see [18, 20, 22, 23, 29, 33–35, 37].

Chen [9] connected the theory of Riemannian submersions and minimal immersions via a simple optimal inequality. Later, Chen derived the case of equality of this inequality. Alegre et al. [2] established relationship between Riemannian submersions with totally geodesic fibers and δ-invariants. The notion of anti-invarinat Riemannian submersions from almost Hermitian manifolds was introduced by Sahin [27].

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A Lagrangian submersion is a special case of anti-invariant submersions [31]. Gunduzalp [14] studied anti-invariant Riemannian submersions from almost product manifolds. Tastan et al. [32] studied anti-invariant Riemannian submersions from locally product manifolds. Gubahar et al. [13] obtained sharp inequalities which involve the Ricci curvature of Riemannian submersions. Aytimur et al. [3] investigated the sharp inequalities of anti- invariant Riemannian submersions from Sasakian space forms. Gunduzalp [15] investigated slant submersions from almost proct manifolds. We recently studied conformal bi-slant Riemannian submersions from locally product manifolds. Our main goal is to study the optimal inequalities which involve the scalar curvature and Ricci curvature of Lagrangian Riemannian submersion from Locally product spaces and prove the Chen-Ricci inequality for the said submersion.

2. Preliminaries

In this section we mainly follow [3, 13, 25, 30].

2.1. Riemannian Submersions

Let $\psi : M_1 \longrightarrow M_2$ be a smooth map from a Riemannian manifold M_1 of dimension m onto the Riemannian manifold M_2 of dimension n where $m > n$. Then ψ is said to be a Riemannian submersion [24] if it satisfies the following conditions:

1. ψ is of maximal rank.
2. The differential map ψ_* of ψ preserves the lengths of horizontal vectors.

By a horizontal vector field Y on M_1 we mean a vector field which is orthogonal to the kernel of ψ_* at each point p of M_1 and by a vertical vector field V on M_1 we mean a vector field which is tangent to the kernel of ψ_* at each point $p \in M_1$. Denote by $\mathscr{H}_p$= {set of all horizontal vectors at p} and by $\mathscr{V}_p$= {set of all vertical vectors at p}. Thus a Riemannian submersion defines two complementary ortogonal distributions $\mathscr{H}$ and $\mathscr{V}$, called horizontal and vertical distribution respectively, on M_1. Further the vertical distribution $\mathscr{V}$ is always integrable.

O′ Neill defined two fundamental tensors T and A of a Riemannian submersion. These are (1,2)-tensors and are defined by the following formulae:

$$T_E F = \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F, \tag{1}$$

$$A_E F = \mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F + \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F, \tag{2}$$

where ∇ denotes Riemannian connection on M_1, E, F are arbitrary vector fields on M_1 and $\mathscr{V}, \mathscr{H}$ denote the projection morphisms on the distributions $\ker\psi_*$ and $(\ker\psi_*)^\perp$ respectively. These tensors are called O'Neils integrability tensors. For any $F \in \Gamma(TM_1)$, T_F and A_F are skew-symmetric operators on $\big(\Gamma(TM_1), g\big)$ and they reverse the horizontal and vertical distributions. It can be easily verified that T is vertical i.e. $T_F = T_{\mathscr{V}F}$ and A is horizontal i.e. $A_F = A_{\mathscr{H}F}$.

The tensor field T and A also satisfy:

$$T_V U = T_U V, \quad \forall \quad U, V \in \Gamma(\ker\psi_*), \tag{3}$$

$$A_X Y = -A_Y X = \frac{1}{2}\mathscr{V}[X, Y], \quad \forall \quad X, Y \in (\ker\psi_*)^\perp. \tag{4}$$

The above equations imply that T restricted over vertical distribution $\mathscr{V}$ is a symmetric operator and A restricted over horizontal distribution $\mathscr{H}$ is skew-symmetric operator. Also, operator A measures the obstruction of the horizontal distribution from being integrable.

On fibers, T acts as second fundamental form of submersion and upon restriction to vertical vectors the condition $T = 0$ translates to the condition that the fibers are totally geodesic. If T is identically equal to zero then we say that the Riemannian submersion has totally geodesic fibers. Let $V_1, V_2, ..., V_n$ be an orthonormal frame of $(\ker\psi_*)$. Then $H = \frac{1}{n}\sum_{j=1}^{n} T_{V_j}V_j$, a horizontal vector field, is called the *mean curvature* vector field of the fibre. The Riemmanian submersion is called a minimal submersion if $H=0$. A Riemannian submersion is said to have totally umbilical fibers if

$$T_U V = g_1(U,V)H.$$

Let R, R^* and $\hat{R}$ denote the Riemannian curvature tensors of M_1, M_2 and any fiber of ψ respectively. The version of Gauss-Codazzi equations for a Riemannian submersion is given by

$$R(V_1,V_2,V_3,V_4) = \hat{R}(V_1,V_2,V_3,V_4) + g(T_{V_1}V_4, T_{V_2}V_3) - g(T_{V_1}V_3, T_{V_2}V_4) \tag{5}$$

$$R(Y_1,Y_2,Y_3,Y_4) = R^*(Y_1,Y_2,Y_3,Y_4) - 2g(A_{Y_1}Y_2, A_{Y_3}Y_4) + g(A_{Y_2}Y_3, A_{Y_1}Y_4) - g(A_{Y_1}Y_3, A_{Y_2}Y_4) \tag{6}$$

$$\begin{aligned} R(Y_1,V_1,Y_2,V_2) &= g\Big((\nabla_{Y_1}T)(V_1,V_2), Y_2\Big) + g\Big((\nabla_{V_1}A)(Y_1,Y_2), V_2\Big) \\ &\quad - g(T_{V_1}Y_1, T_{V_2}Y_2) + g(A_{Y_1}V_1, A_{Y_2}V_2) \end{aligned} \tag{7}$$

where $V_1,V_2,V_3,V_4 \in \mathscr{V}(M_1)$ and $Y_1,Y_2,Y_3,Y_4 \in \mathscr{H}$.

2.2. *Locally Product Manifold*

Let M be an m-dimensional manifold with a tensor F of type $(1,1)$ such that

$$F^2 = I, (F \neq I).$$

Then, we say that M is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I+F), Q = \frac{1}{2}(I-F).$$

Then we get

$$P+Q = I, P^2 = P, Q^2 = Q, PQ = QP = 0, F = P - Q.$$

Thus P and Q define two complementary distributions P and Q. We easily see that the eigenvalues of F are +1 or -1.

If an almost product manifold M admits a Riemannian metric g such that

$$g(FX,FY) = g(X,Y)$$

for any vector fields X and Y on M, then M is called an almost product Riemannian manifold, denoted by (M,g,F). Denote the Levi-Civita connection on M with respect to g by ∇. Then, M is called a locally product Riemannian manifold if F is parallel with respect to ∇, i.e.

$$\nabla_X F = 0, X \in \Gamma(TM).$$

Let $M_1(c_1)$ and $M_2(c_2)$ be real space forms with constant sectional curvature c_1 and c_2 respectively. Then the Riemannian curvature tensor $\bar{R}$ of locally product Riemannian manifold $M{=}M_1(c_1)\times M_2(c_2)$ has the form

$$\begin{aligned} \bar{R}(X_1,X_2,X_3,X_4) &= \frac{c_1+c_2}{4}\Big[g(X_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(X_2,X_4) \\ &\quad + g(FX_1,X_4)g(FX_2,X_3) - g(FX_1,X_3)g(FX_2,X_4)\Big] \\ &\quad + \frac{c_1-c_2}{4}\Big[g(X_1,X_4)g(FX_2,X_3) - g(FX_1,X_3)g(X_2,X_4) \\ &\quad + g(FX_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(FX_2,X_4)\Big] \end{aligned} \tag{8}$$

where $X_1,X_2,X_3,X_4 \in \Gamma(TM)$.

In case of $c_1 = c_2 = c$, the Riemannian curvature tensor $\bar{R}$ of locally product Riemanian manifold $M(c) = M_1(c)\times M_2(c)$ becomes

$$\begin{aligned} \bar{R}(X_1,X_2,X_3,X_4) &= \frac{c}{2}\Big[g(X_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(X_2,X_4) \\ &\quad + g(FX_1,X_4)g(FX_2,X_3) - g(FX_1,X_3)g(FX_2,X_4)\Big] \end{aligned} \tag{9}$$

3. Main Results

3.1. Some Inequalities involing Ricci and Scalar Curvature

In this section, we will derive some inequalities which involve Ricci and Scalar curvature of horizontal and vertical distributions of the Lagrangian Riemannian submersion. The cases of equality of these inequalities will also be discussed.

Let $M(c_1, c_2)$ be a locally product space form and (N, g') be a Riemannian manifold. Let $\psi : M(c_1, c_2) \longrightarrow N$ be an Lagrangian Riemannian submersion from $M(c_1, c_2)$ onto N. For any $p \in M(c_1, c_2)$, we assume that $\{V_1, V_2, ..., V_n, Y_1, Y_2, ..., Y_m\}$ is an orthonormal basis of $T_pM(c_1, c_2)$ such that $\mathscr{V}_p$ is spanned by $\{V_1, V_2, ..., V_n\}$ and $\mathscr{H}_p$ is spanned by
$\{Y_1, Y_2, ..., Y_m\}$.

Using (5) and (8), the curvature tensor of the vertical distribution is given by

$$\begin{aligned}\hat{\hat{R}}(W_1, W_2, W_3, W_4) &= \frac{c_1 + c_2}{4}\Big[g(W_1, W_4)g(W_2, W_3) - g(W_1, W_3)g(W_2, W_4)\Big] \\ &\quad + g(T_{W_1}W_4, T_{W_2}W_3) - g(T_{W_2}W_4, T_{W_1}W_3) \qquad (10)\end{aligned}$$

for any $W_1, W_2, W_3, W_4 \in \mathscr{V}\Big(M(c_1, c_2)\Big)$.
Using (6) and (8), we get the curvature tensor of the horizontal distribution as follows:

$$\begin{aligned}\bar{R}^*(Z_1, Z_2, Z_3, Z_4) &= \frac{c_1 + c_2}{4}\Big[g(Z_1, Z_4)g(Z_2, Z_3) - g(Z_1, Z_3)g(Z_2, Z_4)\Big] \\ &\quad + 2g(A_{Z_1}Z_2, A_{Z_3}Z_4) - g(A_{Z_2}Z_3, A_{Z_1}Z_4) + g(A_{Z_1}Z_3, A_{Z_2}Z_4), \qquad (11)\end{aligned}$$

for any $Z_1, Z_2, Z_3, Z_4 \in \mathscr{H}$.

Proposition 3.1. *For an Lagrangian Riemannian submersion $\psi : M(c_1, c_2) \longrightarrow N$,*

$$R\hat{\hat{i}}c(V) \geq \frac{c_1 + c_2}{4}(n - 1)g(V, V) - ng(T_V V, H)$$

The equality holds if and only each fiber is a totally geodesic.

Proof. Since ψ is anti-invariant, ξ is vertical and T is symmetric over vertical vector fields, for $V = V_1$, using (10) we get

$$\hat{\hat{R}}(V, V_i, V_i, V) = \frac{c_1 - c_2}{4}\Big[g(V, V)g(V_i, V_i)\Big] + \Big[g(T_V V, T_{V_i}V_i) - g(T_{V_i}V, T_{V_i}V)\Big]\Big\}$$

From the above equation we get,

$$R\hat{\hat{i}}c(V) = \frac{c_1 + c_2}{4}\Big[g(V, V)\sum_{i=2}^{n} g(V_i, V_i)\Big] + \sum_{i=2}^{n}\Big[g(T_V V, T_{V_i}V_i) - g(T_{V_i}V, T_{V_i}V)\Big],$$

where $R\hat{\hat{i}}c(V) = \sum_{j=2}^{n} \hat{\hat{R}}(V, V_i, V_i, V)$.

After simplifying we immediately get,

$$R\hat{\hat{i}}c(V) = \frac{c_1 + c_2}{4}(n - 1)g(V, V) - ng(T_V V, H) + \sum_{i=1}^{n} g(T_{V_i}V, T_{V_i}V)$$

Since $\sum_{i=1}^{n} g(T_{V_i}V, T_{V_i}V) \geq 0$, the inequality holds. Also, if all the fibers are totally geodesic i.e. $T = 0$, then the equality holds in the inequality. □

The scalar curvature $\hat{\hat{\tau}}$ of the vertical distribution is given by

$$\hat{\hat{\tau}} = \sum_{1 \leq i < j \leq n} \hat{\hat{R}}(V_i, V_j, V_j, V_i) \qquad (12)$$

The next propostion gives the inequality satisfied by scalar curvature of the vertical distribution.

Proposition 3.2. *For an Lagrangian Riemannian submersion $\psi : M(c_1, c_2) \longrightarrow N$,*

$$2\hat{\tau} \geq \frac{c_1 + c_2}{4} n(n-1) - n^2 \|H\|^2$$

The case of equality for the above inequality holds when each fiber of the Riemannian submersion is a totally geodesic.

Proof. Using (10) and the facts that ψ is anti-invariant, $\xi \in \mathscr{V}(M)$ and T is symmetric over vertical vector fields, we get

$$\hat{\hat{R}}(V_i, V_j, V_j, V_i) = \frac{c_1 + c_2}{4}\Big[g(V_i, V_i) g(V_j, V_j)\Big] + \Big[g(T_{V_i} V_i, T_{V_j} V_j) - g(T_{V_i} V_j, T_{V_i} V_j)\Big]$$

Now using (12), we get

$$\hat{\tau} = \frac{c_1 + c_2}{4} \sum_{1 \leq i < j \leq n} \Big[g(V_i, V_i) g(V_j, V_j)\Big] + \sum_{1 \leq i < j \leq n} \Big[g(T_{V_i} V_i, T_{V_j} V_j) - g(T_{V_i} V_j, T_{V_i} V_j)\Big].$$

After simplification, we immediately get

$$2\hat{\tau} = \frac{c_1 + c_2}{4} n(n-1) - n^2 \|H\|^2 + \sum_{i,j=1}^{n} g(T_{V_i} V_i, T_{V_j} V_j)$$

Since $\sum_{i,j=1}^{n} g(T_{V_i} V_i, T_{V_j} V_j) \geq 0$, it is clear that the inequality holds. Also, it is obvious that the equality case holds if and only if each fiber is totally geodesic. □

Now will give an inequality involving the scalar curvature of the horizontal distribution of the Lagrangian Riemannian submersion. The scalar curvature of the horizontal distribution is given by

$$\bar{\tau}^* = \sum_{1 \leq i < j \leq m} \bar{R}^*(Y_i, Y_j, Y_j, Y_i) \tag{13}$$

Proposition 3.3. *For a Lagrangian Riemannian submersion $\psi : M(c_1, c_2) \longrightarrow N$,*

$$2\bar{\tau}^* \leq \frac{c_1 + c_2}{4} m(m-1)$$

The equality holds in above ineqaulity $\Leftrightarrow$ the horizontal distribution is integrable.

Proof. In view of (11) and using the facts that ψ is anti-invariant, ξ is horizontal and A is anti-symmetric over horizontal vectors, we have

$$\bar{R}^*(Y_i, Y_j, Y_j, Y_i) = \frac{c_1 + c_2}{4}\Big[g(Y_i, Y_i) g(Y_j, Y_j)\Big] - 3g(A_{Y_i} Y_j, A_{Y_i} Y_j).$$

Using (13), we get

$$\bar{\tau}^* = \frac{c_1 + c_2}{4} \sum_{1 \leq i < j \leq m} \Big[g(Y_i, Y_i) g(Y_j, Y_j)\Big] - 3 \sum_{1 \leq i < j \leq m} g(A_{Y_i} Y_j, A_{Y_i} Y_j)\Big\}$$

After simplifying, we immediately get

$$2\bar{\tau}^* = \frac{c_1 + c_2}{4} m(m-1) - 3 \sum_{i,j=1}^{m} g(A_{Y_i} Y_j, A_{Y_i} Y_j) \tag{14}$$

From the above equation, it is clear that the inequality holds. Also, if the horizontal distribution is integrable then A is identically zero. In this case the above inequality becomes equality. □

3.2. Chen-Ricci inequalities

Now, we will derive Chen-Ricci inequalities for the anti-invariant Riemannian submersion $\psi : M(\kappa) \longrightarrow N$. We will use the following notations:

$$T^{\alpha}_{ij} = g(T_{V_i}V_j, Y_\alpha); \quad 1 \le i, j \le n, 1 \le \alpha \le m \tag{15}$$

$$A^{s}_{ij} = g(A_{Y_i}Y_j, V_s); \quad 1 \le i, j \le m, 1 \le s \le n \tag{16}$$

$$\delta(N) = \sum_{i=1}^{m}\sum_{s=1}^{n} g\Big((\nabla_{Y_i}T)_{V_s}V_s, Y_i\Big) \tag{17}$$

From [13], we have

$$\begin{aligned}\sum_{\alpha=1}^{m}\sum_{i,j=1}^{n}(T^{\alpha}_{ij})^2 \quad = \quad & \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{\alpha=1}^{m}\Big[T^{\alpha}_{11} - T^{\alpha}_{22} - \dots - T^{\alpha}_{nn}\Big] \\ & +2\sum_{\alpha=1}^{m}\sum_{j=2}^{n}(T^{\alpha}_{1j})^2 - 2\sum_{\alpha=1}^{m}\sum_{2\le i<j\le n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj} - (T^{\alpha}_{ij})^2\Big]. \end{aligned} \tag{18}$$

Theorem 3.4. *For a Lagrangian Riemannian submersion $\psi : M(c_1, c_2) \longrightarrow N$,*

$$\hat{Ric}(V) \quad \ge \quad \frac{c_1 + c_2}{4}(n-1) - \frac{1}{4}n^2\|H\|^2$$

The equality holds $\Leftrightarrow$

$$\begin{aligned} T^{\alpha}_{11} \quad &= \quad T^{\alpha}_{22} + \dots + T^{\alpha}_{nn} \\ T^{\alpha}_{1j} \quad &= \quad 0, j = 2, \dots, n. \end{aligned}$$

Proof. Using (15) in (13) we get,

$$2\hat{\tau} \quad = \quad \frac{c_1 + c_2}{4}n(n-1) - n^2\|H\|^2 + \sum_{\alpha=1}^{m}\sum_{i,j=1}^{n}(T^{\alpha}_{ij})^2$$

Using (18) in the above equation, we get

$$2\hat{\tau} \quad = \quad \frac{c_1 + c_2}{4}n(n-1) - \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{\alpha=1}^{m}\Big[T^{\alpha}_{11} - T^{\alpha}_{22} - \dots - T^{\alpha}_{mm}\Big] + 2\sum_{\alpha=1}^{m}\sum_{j=2}^{n}(T^{\alpha}_{1j})^2 - 2\sum_{\alpha=1}^{m}\sum_{2\le i<j\le n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj} - (T^{\alpha}_{ij})^2\Big]\Big\}$$

From the above equation, we clearly have the following inequality

$$2\hat{\tau} \quad \ge \quad \frac{c_1 + c_2}{4}n(n-1) - \frac{1}{2}n^2\|H\|^2 - 2\sum_{\alpha=1}^{m}\sum_{2\le i<j\le n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj} - (T^{\alpha}_{ij})^2\Big]$$

Using (5) and (15), we have

$$2\sum_{2\le i<j\le n}\bar{R}(V_i, V_j, V_j, V_i) \quad = \quad 2\sum_{2\le i<j\le n}\hat{R}(V_i, V_j, V_j, V_i) + 2\sum_{\alpha=1}^{m}\sum_{2\le i<j\le n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj} - (T^{\alpha}_{ij})^2\Big] \tag{19}$$

In view of (19), the inequality (19) can be written as

$$2\hat{\tau} \quad \ge \quad \frac{c_1 + c_2}{4}n(n-1) - \frac{1}{2}n^2\|H\|^2 + 2\sum_{2\le i<j\le n}\hat{R}(V_i, V_j, V_j, V_i) - 2\sum_{2\le i<j\le n}\bar{R}(V_i, V_j, V_j, V_i)$$

Using (13), we can write

$$2\hat{\hat{\tau}} = 2\sum_{2\leq i<j\leq n}\hat{\hat{R}}(V_i,V_j,V_j,V_i) + 2\sum_{j=2}^{n}\hat{\hat{R}}(V,V_j,V_j,V)$$

Using the above equation in (20), we get

$$2\hat{\hat{Ric}}(U) \quad \geq \quad \frac{c_1+c_2}{4}n(n-1) - \frac{1}{2}n^2\|H\|^2 - 2\sum_{2\leq i<j\leq n}\bar{R}(V_i,V_j,V_j,V_i)$$

Now using (8), we get

$$\hat{\hat{Ric}}(U) \quad \geq \quad \frac{c_1+c_2}{4}(n-1) - \frac{1}{4}n^2\|H\|^2$$

□

Theorem 3.5. *For a Lagrangian Riemannian submersion $\psi : M(c_1,c_2) \longrightarrow N$,*

$$\bar{Ric}^*(X_1) \quad \leq \quad \frac{c_1+c_2}{4}m(m-1)$$

The case of equality hold in the above ineqaulity if and only

$$A_{1j} = 0, \quad j = 2,...,m.$$

Proof. Using (16) and the fact that A is anti-symmetric in (14), we get

$$2\bar{\tau}^* \quad = \quad \frac{c_1+c_2}{4}m(m-1) - 6\sum_{s=1}^{n}\sum_{j=2}^{m}(A^s_{1j})^2 - 6\sum_{s=1}^{n}\sum_{2\leq i<j\leq m}(A^s_{ij})^2$$

In view of (6) and (16), we have

$$2\sum_{2\leq i<j\leq m}\bar{R}(Y_i,Y_j,Y_j,Y_i) \quad = \quad 2\sum_{2\leq i<j\leq m}\bar{R}^*(Y_i,Y_j,Y_j,Y_i) + 6\sum_{s=1}^{n}\sum_{2\leq i<j\leq m}(A^s_{ij})^2.$$

Using the above expression in (20), we get

$$2\bar{\tau}^* \quad = \quad \frac{c_1+c_2}{4}m(m-1) - 6\sum_{s=1}^{n}\sum_{j=2}^{m}(A^s_{1j})^2 + 2\sum_{2\leq i<j\leq m}\bar{R}^*(Y_i,Y_j,Y_j,Y_i) - 2\sum_{2\leq i<j\leq m}\bar{R}(Y_i,Y_j,Y_j,Y_i)$$

Making use of (8) in (20), we get

$$\bar{Ric}^*(X_1) \quad = \quad \frac{c_1+c_2}{4}m(m-1) - 6\sum_{s=1}^{n}\sum_{j=2}^{m}(A^s_{1j})^2$$

Since $\sum_{j=2}^{m}(A^s_{1j})^2 \geq 0$, we clearly have the inequality. □

Denote (see [13])

$$\|T^{\mathscr{V}}\|^2 = \sum_{k=1}^{m}\sum_{i=1}^{n}\bar{g}(T_{V_i}Y_k, T_{V_i}Y_k), \tag{20}$$

$$\|A^{\mathscr{H}}\|^2 = \sum_{k=1}^{m}\sum_{i=1}^{n}\bar{g}(A_{Y_k}V_i, A_{Y_k}V_i). \tag{21}$$

Theorem 3.6. *For a Lagrangian Riemannian submersion $\psi : M(c_1, c_2) \longrightarrow N$,*

$$\frac{c_1+c_2}{4}(nm+m+n-2)+\frac{c_1-c_2}{4}\Big[3n-4-m-(n-2)\Big]$$
$$\leq \hat{\bar{R}}ic(U_1)+\bar{R}ic^{*}(X_1)+\frac{1}{4}n^2\|H\|^2+3\sum_{s=1}^{n}\sum_{\alpha=2}^{m}(A^{\alpha}_{1s})^2-\delta(N)+\|T^{\mathscr{V}}\|^2-\|A^{\mathscr{H}}\|^2\Big]$$

The case of equality holds if and only if

$$T^{\alpha}_{11}=T^{\alpha}_{22}+\ldots+T^{\alpha}_{nn}$$
$$T_{1j}=0,\ j=2,\ldots,n.$$

Proof. By the definition of scalar curvature of $M(\kappa)$, we have

$$2\bar{\tau} = 2\sum_{1\leq i<j\leq n}\bar{R}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}(Y_k,Y_r,Y_r,Y_k)+2\sum_{i=1}^{n}\sum_{k=1}^{m}\bar{R}(Y_k,V_i,V_i,Y_k) \tag{22}$$

Using (8) in (22), we get

$$2\bar{\tau} = 2\sum_{1\leq i<j\leq n}\bar{R}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}(Y_k,Y_r,Y_r,Y_k)+\frac{c_1+c_2}{4}[nm-n]-\frac{c_1-c_2}{2}n \tag{23}$$

Using (5), (6) and (7) in (22), we get

$$\begin{aligned}2\bar{\tau} &= 2\sum_{1\leq i<j\leq n}\hat{R}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}^{*}(Y_k,Y_r,Y_r,Y_k)+n^2\|H\|^2\\ &+\sum_{i,j=1}^{n}\bar{g}(T_{V_i}V_j,T_{V_i}V_j)+3\sum_{k,r=1}^{m}\bar{g}(A_{Y_k}Y_r,A_{Y_k}Y_r)-2\sum_{k=1}^{m}\sum_{i=1}^{n}\bar{g}\Big((\nabla_{Y_k}T)_{V_i}V_i,Y_k\Big)\\ &+2\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[\bar{g}(T_{V_i}Y_k,T_{V_i}Y_k)-\bar{g}(A_{Y_k}V_i,A_{Y_k}V_i)\Big].\end{aligned}$$

Making use of (3), (4)(15), (16), (17) 18, (20) and (21) in the above equation, we get

$$\begin{aligned}2\bar{\tau} &= 2\sum_{1\leq i<j\leq n}\hat{\bar{R}}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}^{*}(Y_k,Y_r,Y_r,Y_k)+\frac{1}{2}n^2\|H\|^2-\frac{1}{2}\sum_{\alpha=1}^{m}\Big[T^{\alpha}_{11}-T^{\alpha}_{22}-\ldots-T^{\alpha}_{nn}\Big]^2\\ &-2\sum_{\alpha=1}^{m}\sum_{j=2}^{n}(T^{\alpha}_{1j})^2+2\sum_{\alpha=1}^{m}\sum_{2\leq i<j\leq n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj}-(T^{\alpha}_{ij})^2\Big]+6\sum_{s=1}^{n}\sum_{r=2}^{m}(A^{s}_{1r})^2\\ &+6\sum_{s=1}^{n}\sum_{2\leq k<r\leq m}(A^{s}_{kr})^2-2\delta(N)+2\Big[\|T^{\mathscr{V}}\|^2-\|A^{\mathscr{H}}\|^2\Big].\end{aligned} \tag{24}$$

In view of (22) and (24), we get

$$\begin{aligned}&2\sum_{1\leq i<j\leq n}\bar{R}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}(Y_k,Y_r,Y_r,Y_k)+\frac{c_1+c_2}{4}[nm-n]-\frac{c_1-c_2}{2}n\\ &=2\sum_{1\leq i<j\leq n}\hat{\bar{R}}(V_i,V_j,V_j,V_i)+2\sum_{1\leq k<r\leq m}\bar{R}^{*}(Y_k,Y_r,Y_r,Y_k)+\frac{1}{2}n^2\|H\|^2\\ &-\frac{1}{2}\sum_{\alpha=1}^{m}\Big[T^{\alpha}_{11}-T^{\alpha}_{22}-\ldots-T^{\alpha}_{nn}\Big]^2-2\sum_{\alpha=1}^{m}\sum_{j=2}^{n}(T^{\alpha}_{1j})^2+2\sum_{\alpha=1}^{m}\sum_{2\leq i<j\leq n}\Big[T^{\alpha}_{ii}T^{\alpha}_{jj}-(T^{\alpha}_{ij})^2\Big]\end{aligned}$$

$$+6\sum_{s=1}^{n}\sum_{r=2}^{m}(A_{1r}^{s})^2+6\sum_{s=1}^{n}\sum_{2\le k<r\le m}(A_{kr}^{s})^2-2\delta(N)+2\Big[\|T^{\mathscr{V}}\|^2-\|A^{\mathscr{H}}\|^2\Big] \tag{25}$$

Using (5) and (6), we have the following equations.

$$\sum_{2\le i<j\le n}\bar{R}(V_i,V_j,V_j,V_i) = \sum_{2\le i<j\le n}\hat{\bar{R}}(V_i,V_j,V_j,V_i)+2\sum_{\alpha=1}^{m}\sum_{2\le i<j\le n}\Big[T_{ii}^{\alpha}T_{jj}^{\alpha}-(T_{ij}^{\alpha})^2\Big] \tag{26}$$

$$\sum_{2\le k<r\le m}\bar{R}(Y_k,Y_r,Y_r,Y_k)=\sum_{2\le k<r\le m}\bar{R}^{*}(Y_k,Y_r,Y_r,Y_k)+6\sum_{s=1}^{n}\sum_{2\le k<r\le m}(A_{kr}^{s})^2 \tag{27}$$

Now, using (26) and (27) in (25), we get

$$\frac{c_1+c_2}{4}[nm-n]-\frac{c_1-c_2}{2}n+2\sum_{j=2}^{n}\bar{R}(V_1,V_j,V_j,V_1)+2\sum_{j=2}^{n}\bar{R}(Y_1,Y_r,Y_r,Y_1)=$$

$$2\hat{\bar{Ric}}(V_1)+2\bar{Ric}^{*}(Y_1)+\frac{1}{2}n^2\|H\|^2-\frac{1}{2}\sum_{\alpha=1}^{m}\Big[T_{11}^{\alpha}-T_{22}^{\alpha}-\ldots-T_{mm}^{\alpha}\Big]^2$$

$$-2\sum_{\alpha=1}^{m}\sum_{j=2}^{n}(T_{1j}^{\alpha})^2++6\sum_{s=1}^{n}\sum_{r=2}^{m}(A_{1r}^{s})^2-2\delta(N)+2\Big[\|T^{\mathscr{V}}\|^2-\|A^{\mathscr{H}}\|^2\Big] \tag{28}$$

Using (8) in (28), we immediately get

$$\frac{c_1+c_2}{4}(nm+n+m-2)+\frac{c_1-c_2}{4}\Big(3n-4-m-(n-2)\Big)\le$$

$$\hat{\bar{Ric}}(V_1)+\bar{Ric}^{*}(Y_1)+\frac{1}{4}n^2\|H\|^2+3\sum_{s=1}^{n}\sum_{r=2}^{m}(A_{1r}^{s})^2-\delta(N)+\|T^{\mathscr{V}}\|^2-\|A^{\mathscr{H}}\|^2.$$

□

4. Example

In this section, we give a non-trivial example of Lagrangian Riemannian submersion from a locally product manifold.

Example 4.1. *Consider an 4-dimensional Euclidean space $\mathbb{R}^4$ with Euclidean metric g. We define a product structure F on $\mathbb{R}^4$ by*

$$P(y_1,y_2,y_3,y_4)=(y_1,-y_2,y_3,-y_4)$$

Then $(\mathbb{R}^4,g,F)$ forms a locally product manifold.

Now, define a map $\pi:\mathbb{R}^4\longrightarrow\mathbb{R}^2$ by

$$\pi(y_1,y_2,y_3,y_4)=(\frac{y_1+y_4}{\sqrt{2}},\frac{y_2+y_3}{\sqrt{2}})$$

By direct calculations,

$$\ker\pi_*=\mathrm{span}\Big\{V_1=\partial y_1-\partial y_4,V_2=\partial y_2-\partial x_3\Big\}$$

$$(\ker\pi_*)^{\perp}=\mathrm{span}\Big\{X_1=\partial y_1+\partial y_4,X_2=\partial y_2+\partial x_3\Big\}$$

Then it is easy to see that π is a Riemannian submersion. Moreover, $FV_1=X_2$ and $FV_2=X_1$ implies that $F(\ker\pi_*)$=$(\ker\pi_*)^{\perp}$. As a result π is a Lagrangian submersion.

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