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Conformal η-Ricci soltons on anti-invariant submanifold of LP-Kenmotsu manifold endowed with the Zamkovoy connection

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Abstract. The object of the present paper is to study anti-invariant submanifolds of Lorentzian para-Kenmotsu manifold (briefly, LP-Kenmotsu manifold) with respect to the Zamkovoy connection. We prove that if an anti-invariant submanifold M of LP-Kenmotsu manifold contains a conformal Ricci soliton with collinear Reeb vector field, then M is η-Einstein. We also study conformal η-Ricci soliton on this submanifold with the Zamkovoy connection satisfying the curvature conditions: $(\xi.)_{R^*}.S^* = 0$, $(\xi.)_{S^*}.R^*$ and $(\xi.)_{S^*}.P^* = 0$. To validate some of our results, we construct a non-trivial example of anti-invariant submanifold of 5-dimensional LP-Kenmotsu manifolds admitting conformal η-Ricci soliton with respect to the Zamkovoy connection.

1. Introduction

In 2018, the notion of LP-Kenmotsu manifold has been introduced by Haseeb and Prasad [18]. Later, Shukla and Dixit [38] examined φ-recurrent LP-Kenmotsu manifolds and discover that such type of manifolds are η-Einstein. Atceken [1] studied invariant submanifolds of LP-Kenmotsu manifolds in 2022 and he investigated the necessary and sufficient condition for the LP-Kenmotsu manifold to be total geodesic. Further, Chandra and Lal [11] also explored some special results on 3-dimensional LP-Kenmotsu manifolds. This manifold was also studied by Sai Prasad *et al.* [33], and Haseeb *et al.* [19].

In 1977, anti-invariant submanifolds of Sasakian space forms were introduced by Yano and Kon [40]. Later in 1985, Pandey and Kumar investigated properties of anti-invariant submanifolds of almost para-contact manifolds [30]. Recently, Karmakar and Bhattyacharyya [21] studied anti-invariant submanifolds of some indefinite almost contact and para-contact manifolds. Most recently, Karmakar [20] studied η-Ricci-Yamabe soliton on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection.

Let φ be a differential map from a manifold M into another manifold $\widetilde{M}$ and let the dimensions of M, $\widetilde{M}$ be m, $\widetilde{m}$ $(m < \widetilde{m})$, respectively. If rank$\varphi = m$, then φ is called an immersion of M into $\widetilde{M}$. If $\varphi(p^*) \neq \varphi(q^*)$

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for $p^* \neq q^*$, then φ is called an imbedding of M into $\widetilde{M}$. If the manifolds M and $\widetilde{M}$ satisfy the following two conditions, then M is called submanifold of $\widetilde{M}$ - (i) $M \subset \widetilde{M}$, (ii) the inclusion map from M into $\widetilde{M}$ is an imbedding of M into $\widetilde{M}$.

A submanifold M is called anti-invariant if $\omega_1 \in T_{p^*}(M) \Rightarrow \varphi\omega_1 \in T^{\perp}_{p^*}(M)$, for all $p^* \in M$, where $T_{p^*}(M)$ and $T^{\perp}_{p^*}(M)$ are, respectively, tangent space and normal space at $p^* \in M$. Thus in an anti-invariant submanifold M, we have for all $\omega_1, \omega_2 \in T_{p^*}(M)$

$$\delta(\omega_1, \varphi\omega_2) = 0.$$

In 2008, the notion of Zamkovoy canonical connection (briefly, Zamkovoy connection) was introduced by Zamkovoy [43] for a para-contact manifold. And this connection was defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold. Later, Biswas and Baishya [5, 6] studied this connection on generalized pseudo Ricci symmetric Sasakian manifolds and on almost pseudo symmetric Sasakian manifolds. This connection was further studied by Blaga [7] on para-Kenmotsu manifolds. In 2020, Mandal and Das [13, 22–24] studied in detail on various curvature tensors of Sasakian and LP-Sasakian manifolds admitting this connection. In 2021, they discussed LP-Sasakian manifolds equipped with this connection and conharmonic curvature tensor [25]. Recently, they introduced Zamkovoy connection on LP-Kenmotsu manifold [26] and study Ricci soliton on it with respect to this connection. Zamkovoy connection for an n-dimensional almost contact metric manifold M^* equipped with an almost contact metric structure $(\phi, \varsigma, \eta, \delta)$ consisting of a $(1, 1)$ tensor field ϕ, a vector field ς, a 1-form η and a Riemannian metric δ, is defined by

$$\nabla^*_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 + (\nabla_{\omega_1}\eta)(\omega_2)\,\varsigma - \eta(\omega_2)\,\nabla_{\omega_1}\varsigma + \eta(\omega_1)\,\phi\omega_2, \tag{1}$$

for all $\omega_1, \omega_2 \in \chi(M^*)$, where $\chi(M^*)$ is the set of all vector fields on M^*.

R. S. Hamilton was the first who introduced the concept of Ricci flow in differential geometry in 1982. Hamilton [17] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is an evolution equation for metrices on a Riemannian manifold defined as follows:

$$\frac{\partial\delta}{\partial t} = -2S,\ \delta(0) = \delta_0, \tag{2}$$

where δ is a Riemannian metric, S is Ricci curvatre tensor and t is time. The solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (δ, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V\delta + 2S + 2\lambda\delta = 0, \tag{3}$$

where S is Ricci curvature tensor and $L_V\delta$ denotes the Lie derivative of δ along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (δ, V, λ) is called a gradient Ricci soliton. Ricci soliton was further studied by many researchers. For instance, we see [2, 3, 29, 32, 37, 39, 42] and their references.

In 2005, Fischer [15] introduced conformal Ricci flow which is a generalization of the Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow on a manifold M is defined by the equation

$$\frac{\partial\delta}{\partial t} + 2\left(S + \frac{\delta}{n}\right) = -\nu\delta, \tag{4}$$

$$r(\delta) = -1, \tag{5}$$

where M is considered as a smooth closed connected oriented n-manifold, $r(g)$ is the scalar curvature of the manifold and ν is a scalar non-dynamical field. Corresponding to the conformal Ricci flow equation, Basu and Bhatyacharyya [4] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton given by

$$L_V\delta + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta = 0, \tag{6}$$

where λ is a constant and p is a scalar non-dynamical field..

As a generalization of Ricci soliton, the η-Ricci soliton was introduced by Cho and Kimura [12]. This notion was also studied by Cälin and Crasmarearu [10]. Later, remarkable studies on η-Ricci soliton was made by Blaga [8], Prakasha [31] and Siddiqi [36]. Let M^* be a Riemannian manifold with structure $\left(\phi, \varsigma, \eta, \delta\right)$. Consider the equation

$$L_V\delta + 2S + 2\lambda\delta + 2\beta\eta \otimes \eta = 0, \tag{7}$$

where S is Ricci curvature tensor, and λ, β are real constants. The data $(\delta, V, \lambda, \beta)$ which satisfies the equation (7) is called an η-Ricci soliton on M^*. In particular, when $\beta = 0$, the notion of η-Ricci soliton simply reduces to the notion of Ricci soliton. And when $\beta \neq 0, (\delta, V, \lambda, \beta)$ is called proper η-Ricci soliton on M^*.

In 2018, Siddiqi [35] introduced the notion of conformal η-Ricci soliton as

$$L_V\delta + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta + 2\beta\eta \otimes \eta = 0, \tag{8}$$

where $L_V\delta$ denotes the Lie derivative of δ along the vector field V, λ and β are real constants and p is a scalar non-dynamical field.

The notion of projective curvature tensor was first introduced by Yano and Bochner [41] in 1953. This curvature tensor was further studied by De and Sengupta [14], Ghosh [16]. If there exists a one-one mapping between each co-ordinate neighbourhood of a manifold $M^/$ to a domain of R^n such that any geodesic of $M^/$ corresponds to a straight line in R^n, then the manifold $M^/$ is said to be locally projectively flat. Due to [14], the projective curvature tensor P of rank three for an n-dimensional Riemannian manifold $M^/$ is given by

$$\begin{aligned} P(\omega_1, \omega_2)\omega_3 &= R(\omega_1, \omega_2)\omega_3 \\ &\quad - \frac{1}{n-1}\left[S(\omega_2, \omega_3)\omega_1 - S(\omega_1, \omega_3)\omega_2\right], \end{aligned} \tag{9}$$

for all ω_1, ω_2 and $\omega_3 \in \chi(M^/)$, where R is the Riemannianc curvature tensor and r is the scalar curvature.

Definition 1.1. *[27] A Riemannian manifold M is called an η-Einstein manifold if its Ricci curvature tensor is of the form*

$$S(\omega_2, \omega_3) = k_1\delta(\omega_2, \omega_3) + k_2\eta(\omega_2)\eta(\omega_3),$$

for all $\omega_2, \omega_3 \in \chi(M)$, where k_1, k_2 are scalars.

This paper is structured as follows: First two sections of the paper have been kept for introduction and preliminaries. In **Section-3**, we introduce Zamkovoy connection (∇^*) on anti-invariant submanifold of LP-Kenmotsu manifold. **Section-4** concerns with conformal Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to ∇^*. **Section-5** deals with conformal η-Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to ∇^*. In **Section-6**, we discuss conformal Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{R^*}.S^* = 0$. **Section-7** concerns with conformal η-Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{S^*}.R^* = 0$. **Section-8** deals with conformal η-Ricci soliton on anti-anvariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{S^*}.P^* = 0$. Finally, **Section-9** refers to an example of an anti-invariant submanifold of LP-Kenmotsu manifold admitting conformal η-Ricci soliton with respect to ∇^*.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional Lorentzian almost para-contact manifold with structure $(\phi, \varsigma, \eta, \delta)$, where η is a 1-form, ς is the structure vector field, ϕ is a $(1,1)$-tensor field and δ is a Lorentzian metric satisfying

$$\begin{aligned}
\phi^2(\omega_1) &= \omega_1 + \eta(\omega_1)\varsigma, \eta(\varsigma) = -1, &(10)\\
\delta(\omega_1, \varsigma) &= \eta(\omega_1), &(11)\\
\delta(\phi\omega_1, \phi\omega_2) &= \delta(\omega_1, \omega_2) + \eta(\omega_1)\eta(\omega_2), &(12)
\end{aligned}$$

for all vector fields ω_1, ω_2 on $\overline{M}$. A Lorentzian almost para-contact manifold is said to be Lorentzian para-contact manifold if η becomes a contact form. In a Lorentzian para-contact manifold the following relations also hold [28, 34]:

$$\begin{aligned}
\phi(\varsigma) &= 0, \eta \circ \phi = 0, &(13)\\
\delta(\omega_1, \phi\omega_2) &= \delta(\phi\omega_1, \omega_2). &(14)
\end{aligned}$$

The manifold $\overline{M}$ is called a Lorentzian para-Kenmotsu manifold if

$$(\nabla_{\omega_1}\varphi)\omega_2 = -\delta(\phi\omega_1, \omega_2)\varsigma - \eta(\omega_2)\phi\omega_1, \tag{15}$$

for all smooth vector fields ω_1, ω_2 on $\overline{M}$. In a Lorentzian para-Kenmotsu manifold the following relations also hold [18, 26]:

$$\begin{aligned}
\nabla_{\omega_1}\varsigma &= -\omega_1 - \eta(\omega_1)\varsigma, &(16)\\
(\nabla_{\omega_1}\eta)\omega_2 &= -\delta(\omega_1, \omega_2) - \eta(\omega_1)\eta(\omega_2), &(17)\\
\eta(R(\omega_1, \omega_2)\omega_3) &= \delta(\omega_2, \omega_3)\eta(\omega_1) - \delta(\omega_1, \omega_3)\eta(\omega_2), &(18)\\
R(\omega_1, \omega_2)\varsigma &= \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2, &(19)\\
R(\varsigma, \omega_1)\omega_2 &= \delta(\omega_1, \omega_2)\varsigma - \eta(\omega_2)\omega_1, &(20)\\
R(\varsigma, \omega_1)\varsigma &= \omega_1 + \eta(\omega_1)\varsigma, &(21)\\
S(\omega_1, \varsigma) &= (n-1)\eta(\omega_1) &(22)\\
S(\varsigma, \varsigma) &= -(n-1), &(23)\\
Q\varsigma &= (n-1)\varsigma, &(24)\\
S(\phi\omega_1, \phi\omega_2) &= S(\omega_1, \omega_2) + (n-1)\eta(\omega_1)\eta(\omega_2), &(25)
\end{aligned}$$

for all smooth vector fields ω_1, ω_2, ω_3 on $\overline{M}$.

3. Zamkovoy connection on anti-invariant submanifold of LP-Kenmotsu manifold

Expression of Zamkovoy connection on an n-dimensional LP-Kenmotsu manifold $\overline{M}$ [26] is

$$\nabla^*_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 - \delta(\omega_1, \omega_2)\varsigma + \eta(\omega_2)\omega_1 + \eta(\omega_1)\phi\omega_2. \tag{26}$$

Setting $\omega_2 = \varsigma$ in (26) we obtain

$$\nabla^*_{\omega_1}\varsigma = -2[\omega_1 + \eta(\omega_1)\varsigma]. \tag{27}$$

The Riemannian curvature tensor R^* with respect to Zamkovoy connection [26] on $\overline{M}$ is given by

$$\begin{aligned}
R^*(\omega_1, \omega_2)\omega_3 = {} & R(\omega_1, \omega_2)\omega_3 + 3\delta(\omega_2, \omega_3)\omega_1 - 3\delta(\omega_1, \omega_3)\omega_2\\
&+2\delta(\omega_2, \omega_3)\eta(\omega_1)\varsigma - 2\delta(\omega_1, \omega_3)\eta(\omega_2)\varsigma\\
&+2\delta(\omega_2, \phi\omega_3)\eta(\omega_1)\varsigma - 2\delta(\omega_1, \phi\omega_3)\eta(\omega_2)\varsigma\\
&+2\eta(\omega_2)\eta(\omega_3)\omega_1 - 2\eta(\omega_1)\eta(\omega_3)\omega_2\\
&-2\eta(\omega_2)\eta(\omega_3)\phi\omega_1 + 2\eta(\omega_1)\eta(\omega_3)\phi\omega_2.
\end{aligned} \tag{28}$$

For an anti-invariant submanifold M of $\overline{M}$ the Riemannian curvature tensor is given by

$$\begin{aligned} R^*(\omega_1,\omega_2)\omega_3 &= R(\omega_1,\omega_2)\omega_3 + 3\delta(\omega_2,\omega_3)\omega_1 - 3\delta(\omega_1,\omega_3)\omega_2 \\ &\quad +2\delta(\omega_2,\omega_3)\eta(\omega_1)\varsigma - 2\delta(\omega_1,\omega_3)\eta(\omega_2)\varsigma \\ &\quad +2\eta(\omega_2)\eta(\omega_3)\omega_1 - 2\eta(\omega_1)\eta(\omega_3)\omega_2 \\ &\quad -2\eta(\omega_2)\eta(\omega_3)\phi\omega_1 + 2\eta(\omega_1)\eta(\omega_3)\phi\omega_2. \end{aligned} \tag{29}$$

Writing the equation (29) by the cyclic permutations of ω_1, ω_2 and ω_3 and using the fact that $R(\omega_1,\omega_2)\omega_3 + R(\omega_2,\omega_3)\omega_1 + R(\omega_3,\omega_1)\omega_2 = 0$, we have

$$R^*(\omega_1,\omega_2)\omega_3 + R^*(\omega_2,\omega_3)\omega_1 + R^*(\omega_3,\omega_1)\omega_2 = 0. \tag{30}$$

Therefore, the Riemannian curvature tensor with respect to Zamkovoy connection on M satisfies the 1st Bianchi identity. Taking inner product of (29) with a vector field ω_0, we get

$$\begin{aligned} R^*(\omega_1,\omega_2,\omega_3,\omega_0) &= R(\omega_1,\omega_2,\omega_3,\omega_0) + 3\delta(\omega_2,\omega_3)\delta(\omega_1,\omega_0) \\ &\quad -3\delta(\omega_1,\omega_3)\delta(\omega_2,\omega_0) + 2\delta(\omega_2,\omega_3)\eta(\omega_1)\eta(\omega_0) \\ &\quad -2\delta(\omega_1,\omega_3)\eta(\omega_2)\eta(\omega_0) + 2\delta(\omega_1,\omega_0)\eta(\omega_2)\eta(\omega_3) \\ &\quad -2\eta(\omega_1)\eta(\omega_3)\delta(\omega_2,\omega_0), \end{aligned} \tag{31}$$

where $R^*(\omega_1,\omega_2,\omega_3,\omega_0) = \delta(R^*(\omega_1,\omega_2)\omega_3,\omega_0)$ and $\omega_1,\omega_2,\omega_3,\omega_0 \in \chi(M)$. Contracting (31) over ω_1 and ω_0, we get

$$S^*(\omega_2,\omega_3) = S(\omega_2,\omega_3) + (3n-5)\delta(\omega_2,\omega_3) + 2(n-2)\eta(\omega_2)\eta(\omega_3), \tag{32}$$

where S^* is the Ricci curvature tensor with respect to Zamkovoy connection.

Proposition 3.1. *The Riemannian curvature tensor with respect to Zamkovoy connection on an anti-invariant submanifold of LP-Kenmotsu manifold satisfies the 1st Bianchi identity.*

Proposition 3.2. *Ricci curvature tensor with respect to Zamkovoy connection of an anti-invariant submanifold of LP-Kenmotsu manifold is symmetric and it is given by (32).*

Lemma 3.3. *Let M be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting Zamkovoy connetion, then*

$$R^*(\omega_1,\omega_2)\varsigma = 2\left[\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2 + \eta(\omega_2)\phi\omega_1 - \eta(\omega_1)\phi\omega_2\right], \tag{33}$$

$$R^*(\varsigma,\omega_2)\omega_3 = 2\left[\delta(\omega_2,\omega_3)\varsigma - \eta(\omega_3)\omega_2 - \eta(\omega_3)\phi\omega_2\right], \tag{34}$$

$$R^*(\varsigma,\omega_2)\varsigma = 2\left[\eta(\omega_2)\varsigma + \omega_2 + \phi\omega_2\right], \tag{35}$$

$$S^*(\varsigma,\omega_3) = S^*(\omega_3,\varsigma) = 2(n-1)\eta(\omega_3), \tag{36}$$

$$Q^*\omega_2 = Q\omega_2 + (3n-5)\omega_2 + 2(n-2)\eta(\omega_2)\varsigma, \tag{37}$$

$$Q^*\varsigma = 2(n-1)\varsigma, \tag{38}$$

$$r^* = r + (n-1)(3n-4), \tag{39}$$

for all ω_1, ω_2, $\omega_3 \in \chi(M)$, where R^*, Q^* and r^* denote Riemannian curvature tensor, Ricci operator and scalar curvature of M with respect to ∇^*, respectively.

Theorem 3.4. *If an n-dimensional anti-invariant submanifold M of an LP-Kenmotsu manifold is Ricci flat with respect to Zamkovoy connection, then M is η-Einstein manifold.*

Proof. Let M be an n-dimensional anti-invariant submanifold of an LP-Kenmotsu manifold, which is Ricci flat with respect to Zamkovoy connection i.e., $S^*(\omega_2,\omega_3) = 0$, for all $\omega_2,\omega_3 \in \chi(M)$. Then from (32), we have

$$S(\omega_2,\omega_3) = -(3n-5)\delta(\omega_2,\omega_3) - 2(n-2)\eta(\omega_2)\eta(\omega_3),$$

which implies that M is an η-Einstein manifold. □

Projective curvature tensor of M with respect to Zamkovoy connection is given by

$$\begin{aligned} P^{*}(\omega_1,\omega_2)\omega_3 &= R^{*}(\omega_1,\omega_2)\omega_3 \\ &\quad -\frac{1}{n-1}\left[S^{*}(\omega_2,\omega_3)\omega_1 - S^{*}(\omega_1,\omega_3)\omega_2\right], \end{aligned} \tag{40}$$

for all $\omega_1,\omega_2,\omega_3 \in \chi(M)$, where R^{*} and P^{*} are Riemannian curvature tensor, projective curvature tensor of M with respect to ∇^{*}, respectively.

Lemma 3.5. *Let M be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting Zamkovoy connection, then*

$$\eta\left(P^{*}(\omega_1,\omega_2)\omega_3\right) = -\frac{1}{n-1}\left[S^{*}(\omega_2,\omega_3)\eta(\omega_1) - S^{*}(\omega_1,\omega_3)\eta(\omega_2)\right], \tag{41}$$

$$\eta\left(P^{*}(\omega_1,\omega_2)\varsigma\right) = 0, \eta\left(P^{*}(\omega_1,\varsigma)\varsigma\right) = 0, \eta\left(P^{*}(\varsigma,\omega_2)\varsigma\right) = 0, \tag{42}$$

for all ω_1, ω_2, $\omega_3 \in \chi(M)$.

4. Conformal Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to ∇^{*}

Theorem 4.1. *A conformal Ricci soliton (δ, V, λ) on an anti-invariant submanifold of LP-Kenmotsu manifold is invariant under Zamkovoy connection if and only if relation holds*

$$\begin{aligned} 0 &= 2\delta(\omega_1,\omega_2)\eta(V) - \delta(\omega_1,V)\eta(\omega_2) - \delta(\omega_2,V)\eta(\omega_1) \\ &\quad +2(3n-5)\delta(\omega_1,\omega_2) + 4(n-2)\eta(\omega_1)\eta(\omega_2). \end{aligned} \tag{43}$$

for arbitrary vector fields ω_1, ω_2 and V.

Proof. The equation (6) with respect to Zamkovoy connection on an anti-invariant submanifold M of LP-Kenmotsu manifold may be written as

$$(L^{*}_{V}\delta)(\omega_1,\omega_2) + 2S^{*}(\omega_1,\omega_2) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1,\omega_2) = 0, \tag{44}$$

where $L^{*}_{V}\delta$ denote Lie derivative of δ with respect to ∇^{*} along the vector field V and S^{*} is the Ricci curvature tensor of M with respect to ∇^{*}. After expanding (44) and using (26), we have

$$\begin{aligned} &(L^{*}_{V}\delta)(\omega_1,\omega_2) + 2S^{*}(\omega_1,\omega_2) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1,\omega_2) \\ =\ & \delta(\nabla^{*}_{\omega_1}V,\omega_2) + \delta(\omega_1,\nabla^{*}_{\omega_2}V) + 2S^{*}(\omega_1,\omega_2) \\ & + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1,\omega_2) \\ =\ & (L_{V}\delta)(\omega_1,\omega_2) + 2S(\omega_1,\omega_2) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1,\omega_2) \\ & +2\delta(\omega_1,\omega_2)\eta(V) - \delta(\omega_1,V)\eta(\omega_2) - \delta(\omega_2,V)\eta(\omega_1) \\ & +2(3n-5)\delta(\omega_1,\omega_2) + 4(n-2)\eta(\omega_1)\eta(\omega_2), \end{aligned}$$

which shows that the conformal Ricci soliton is invariant on M under Zamkovoy connection, if (43) holds. □

Theorem 4.2. *Let an anti-invariant submanifold $M\left(\phi,\varsigma,\eta,\delta\right)$ of LP-Kenmotsu manifold admit a conformal Ricci soliton (δ, V, λ). If the potential vector field V be collinear with ς, then M is an η-Einstein manifold.*

Proof. Setting $V = \varsigma$ in (44), we get

$$\begin{aligned} 0 &= (L^*_\varsigma \delta)(\omega_1, \omega_2) + 2S^*(\omega_1, \omega_2) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1, \omega_2) \\ &= \delta(\nabla^*_{\omega_1}\varsigma, \omega_2) + \delta(\omega_1, \nabla^*_{\omega_2}\varsigma) + 2S^*(\omega_1, \omega_2) \\ &\quad + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\delta(\omega_1, \omega_2). \end{aligned} \tag{45}$$

Using (27) in (45), we obtain

$$S^*(\omega_1, \omega_2) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda + 2\right]\delta(\omega_1, \omega_2) + 2\eta(\omega_1)\eta(\omega_2). \tag{46}$$

Using (32) in (46), we have

$$\begin{aligned} S(\omega_1, \omega_2) &= \left[\left(\frac{p}{2} + \frac{1}{n}\right) - 3n + 7 - \lambda\right]\delta(\omega_1, \omega_2) \\ &\quad -2(n-3)\eta(\omega_1)\eta(\omega_2). \end{aligned} \tag{47}$$

Therefore, M is an η-Einstein manifold. □

Corollary 4.3. *If an anti-invariant submanifold $M(\phi, \varsigma, \eta, \delta)$ of LP-Kenmotsu manifold admits conformal Ricci soliton (δ, V, λ) with respect to ∇^* and the potential vector field V be collinear with ς, then $\lambda = \frac{p}{2} + \frac{1}{n} - 2(n-1)$. Moreover, the soliton is steady if $p = 4(n-1) - \frac{2}{n}$, shrinking if $p < 4(n-1) - \frac{2}{n}$, and expanding if $p > 4(n-1) - \frac{2}{n}$.*

Proof. Setting $\omega_1 = \varsigma$ in (47), we get

$$\lambda = \frac{p}{2} + \frac{1}{n} - 2(n-1).$$

□

Corollary 4.4. *Let an anti-invariant submanifold $M(\phi, \varsigma, \eta, \delta)$ of LP-Kenmotsu manifold admit a conformal Ricci soliton $(\delta, \varsigma, \lambda)$ with respect to ∇^*. If M be Ricci flat, then $\lambda = \frac{p}{2} + \frac{1}{n} - (n-1)$. Moreover, the soliton is steady if $p = 2(n-1) - \frac{2}{n}$, shrinking if $p < 2(n-1) - \frac{2}{n}$, and expanding if $p > 2(n-1) - \frac{2}{n}$.*

Proof. Setting $S(\omega_1, \omega_2) = 0$ in (47), we have

$$\left[\left(\frac{p}{2} + \frac{1}{n}\right) - 3n + 7 - \lambda\right]\delta(\omega_1, \omega_2) = 2(n-3)\eta(\omega_1)\eta(\omega_2). \tag{48}$$

Putting $\omega_2 = \varsigma$ in (48), we obtain

$$\lambda = \frac{p}{2} + \frac{1}{n} - (n-1).$$

□

5. Conformal η-Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to ∇^*

Theorem 5.1. *If an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admits a conformal η-Ricci soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to ∇^*, then the soliton scalars are related by the following equation*

$$\beta = \lambda - \frac{p}{2} - \frac{1}{n} + 2(n-1).$$

Proof. Writing the equation (8) with respect to ∇^*, we have

$$\begin{aligned} 0 &= (L_V^*\delta)(\omega_1,\omega_2)+2S^*(\omega_1,\omega_2) \\ &\quad+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\delta(\omega_1,\omega_2)+2\beta\eta(\omega_1)\eta(\omega_2). \end{aligned} \tag{49}$$

Applying $V=\varsigma$ in (49), we get

$$\begin{aligned} 0 &= \left(L_\varsigma^*\delta\right)(\omega_1,\omega_2)+2S^*(\omega_1,\omega_2) \\ &\quad+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\delta(\omega_1,\omega_2)+2\beta\eta(\omega_1)\eta(\omega_2), \\ &= \delta(\nabla^*_{\omega_1}\varsigma,\omega_2)+\delta(\omega_1,\nabla^*_{\omega_2}\varsigma)+2S^*(\omega_1,\omega_2) \\ &\quad+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\delta(\omega_1,\omega_2)+2\beta\eta(\omega_1)\eta(\omega_2). \end{aligned} \tag{50}$$

Using (27) in (50), we get

$$S^*(\omega_1,\omega_2)=\left[\left(\frac{p}{2}+\frac{1}{n}\right)-(\lambda-2)\right]\delta(\omega_1,\omega_2)-(\beta-2)\eta(\omega_1)\eta(\omega_2). \tag{51}$$

Setting $\omega_2=\varsigma$ in (51), we get

$$\beta=\lambda-\frac{p}{2}-\frac{1}{n}+2(n-1). \tag{52}$$

□

Corollary 5.2. *If an anti-invariant submanifold $M\left(\phi,\varsigma,\eta,\delta\right)$ of LP-Kenmotsu manifold admits a conformal η-Ricci soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to ∇^*, then M is η-Einstein.*

Proof. In view of (32) and (51), we have

$$\begin{aligned} S(\omega_1,\omega_2) &= \left[\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-3n+7\right]\delta(\omega_1,\omega_2) \\ &\quad-(\beta+2n-6)\eta(\omega_1)\eta(\omega_2), \end{aligned}$$

which shows that M is M is η-Einstein. □

Corollary 5.3. *If an anti-invariant submanifold $M\left(\phi,\varsigma,\eta,\delta\right)$ of LP-Kenmotsu manifold admits a conformal η-Ricci soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to ∇^*, then the scalar curvature of M is given by*

$$r=\frac{np}{2}-n\lambda+\beta-3n^2+9n-5. \tag{53}$$

Proof. Contracting equation (51) and using (39) we get the result. □

Corollary 5.4. *If an anti-invariant submanifold $M\left(\phi,\varsigma,\eta,\delta\right)$ of LP-Kenmotsu manifold admits a conformal η-Ricci soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to ∇^* and the structure vector field ς be parallel with respect to ∇^* i.e., $\nabla^*_{\omega_1}\varsigma=0$, for all $\omega_1\in\chi(M)$, then M is η-Einstein.*

Proof. Setting $\nabla^*_{\omega_1}\varsigma=\nabla^*_{\omega_2}\varsigma=0$ in (50) and using (32), we get

$$\begin{aligned} S(\omega_1,\omega_2) &= \left[\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-3n+5\right]\delta(\omega_1,\omega_2) \\ &\quad-(\beta+2n-6)\eta(\omega_1)\eta(\omega_2), \end{aligned}$$

which shows that M is η-Einstein. □

Theorem 5.5. *If $M\left(\phi,\varsigma,\eta,\delta\right)$ be an anti-invariant submanifold of LP-Kenmotsu manifold admitting a conformal η-Ricci soliton with $(\delta,\varsigma,\lambda,\beta)$ respect to ∇^* such that $V\in D=\ker\eta$, then the relation between soliton constants is given by*

$$2(n-1)+\lambda=\beta-\left(\frac{p}{2}+\frac{1}{n}\right).$$

Proof. Consider the distribution D on $M\left(\phi,\varsigma,\eta,\delta\right)$ as $D=\ker\eta$. If $V\in D$, then

$$\eta\left(V\right)=0.$$

Taking covariant derivative with respect to ς and using $\left(\nabla_\varsigma\eta\right)V=0$, we get

$$\eta\left(\nabla_\varsigma V\right)=0. \tag{54}$$

In view of (26) and (54) it follows that

$$\eta\left(\nabla^*_\varsigma V\right)=0. \tag{55}$$

Equation (49) gives

$$\begin{aligned} 0 &= \delta(\nabla^*_{\omega_1}V,\omega_2)+\delta(\omega_1,\nabla^*_{\omega_2}V)+2S^*(\omega_1,\omega_2)\\ &\quad+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\delta(\omega_1,\omega_2)+2\beta\eta\left(\omega_1\right)\eta\left(\omega_2\right). \end{aligned} \tag{56}$$

Setting $\omega_1=\omega_2=\varsigma$ in (56), we obtain

$$\begin{aligned} 0 &= 2\eta\left(\nabla_\varsigma V\right)-4(n-1)-\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+2\beta,\\ &= (2n-2+\lambda-\beta)+\left(\frac{p}{2}+\frac{1}{n}\right), \end{aligned}$$

which gives the theorem. □

6. Conformal η-Ricci soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{R^*}.S^*=0$

Theorem 6.1. *Let $M\left(\phi,\varsigma,\eta,\delta\right)$ be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting conformal η-Ricci soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to ∇^*. If M satisfies $(\varsigma.)_{R^*}.S^*=0$, then*

$$\lambda=\frac{p}{2}+\frac{2n^2-1}{n},\beta=2.$$

Proof. The condition that must be satisfied by S^* is

$$S^*(R^*(\varsigma,\omega_1)\omega_2,\omega_3)+S^*(\omega_2,R^*(\varsigma,\omega_1)\omega_3)=0, \tag{57}$$

for all $\omega_1,\omega_2,\omega_3\in\chi\left(M\right)$. Using (34) and replacing the expression of S^* from (51) in (57), we get

$$(\beta-2)\left[\delta(\omega_1,\omega_2)\eta\left(\omega_3\right)-\delta(\omega_1,\omega_3)\eta\left(\omega_2\right)\right]=0, \tag{58}$$

For $\omega_3=\varsigma$, equation (58) becomes

$$(\beta-2)\delta(\phi\omega_1,\phi\omega_2)=0,$$

for all $\omega_1,\omega_2\in\chi\left(M\right)$, which gives

$$\beta=2. \tag{59}$$

In view of (52) and (59), we have

$$\lambda=\frac{p}{2}+\frac{2n^2-1}{n}.$$

This gives the theorem. □

7. Conformal η-Ricci soliton on anti-anvariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{S^*}.R^* = 0$

Theorem 7.1. *Let $M\left(\phi, \varsigma, \eta, \delta\right)$ be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting conformal η-Ricci soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to ∇^*. If M satisfies $(\varsigma.)_{S^*}.R^* = 0$, then*

$$\lambda = \frac{p}{2} + \frac{1}{n} + 2, \beta = 2n.$$

Proof. From (28), (33) and (35), we have

$$\eta\left(R^*\left(\omega_1, \omega_2\right)\omega_3\right) = \eta\left(R^*\left(\omega_1, \omega_2\right)\varsigma\right) = \eta\left(R^*\left(\varsigma, \omega_2\right)\varsigma\right) = 0. \tag{60}$$

The condition that must be satisfied by S^* is

$$\begin{aligned} 0 \;=\;& S^*(\omega_1, R^*(\omega_2, \omega_3)\omega_4)\varsigma - S^*(\varsigma, R^*(\omega_2, \omega_3)\omega_4)\omega_1 \\ &+S^*(\omega_1, \omega_2)R^*(\varsigma, \omega_3)\omega_4 - S^*(\varsigma, \omega_2)R^*(\omega_1, \omega_3)\omega_4 \\ &+S^*(\omega_1, \omega_3)R^*(\omega_2, \varsigma)\omega_4 - S^*(\varsigma, \omega_3)R^*(\omega_2, \omega_1)\omega_4 \\ &+S^*(\omega_1, \omega_4)R^*(\omega_2, \omega_3)\varsigma - S^*(\varsigma, \omega_4)R^*(\omega_2, \omega_3)\omega_1, \end{aligned} \tag{61}$$

for all $\omega_1, \omega_2, \omega_3$ *and* $\omega_4 \in \chi(M)$. Taking inner product with ς the above equation becomes

$$\begin{aligned} 0 \;=\;& -S^*(\omega_1, R^*(\omega_2, \omega_3)\omega_4) - S^*(\varsigma, R^*(\omega_2, \omega_3)\omega_4)\eta(\omega_1) \\ &+S^*(\omega_1, \omega_2)\eta(R^*(\varsigma, \omega_3)\omega_4) - S^*(\varsigma, \omega_2)\eta(R^*(\omega_1, \omega_3)\omega_4) \\ &+S^*(\omega_1, \omega_3)\eta(R^*(\omega_2, \varsigma)\omega_4) - S^*(\varsigma, \omega_3)\eta(R^*(\omega_2, \omega_1)\omega_4) \\ &+S^*(\omega_1, \omega_4)\eta(R^*(\omega_2, \omega_3)\varsigma) - S^*(\varsigma, \omega_4)\eta(R^*(\omega_2, \omega_3)\omega_1). \end{aligned} \tag{62}$$

Setting $\omega_4 = \varsigma$ in (62), we get

$$\begin{aligned} 0 \;=\;& -S^*(\omega_1, R^*(\omega_2, \omega_3)\varsigma) - S^*(\varsigma, R^*(\omega_2, \omega_3)\varsigma)\eta(\omega_1) \\ &+S^*(\omega_1, \omega_2)\eta(R^*(\varsigma, \omega_3)\varsigma) - S^*(\varsigma, \omega_2)\eta(R^*(\omega_1, \omega_3)\varsigma) \\ &+S^*(\omega_1, \omega_3)\eta(R^*(\omega_2, \varsigma)\varsigma) - S^*(\varsigma, \omega_3)\eta(R^*(\omega_2, \omega_1)\varsigma) \\ &+S^*(\omega_1, \varsigma)\eta(R^*(\omega_2, \omega_3)\varsigma) - S^*(\varsigma, \varsigma)\eta(R^*(\omega_2, \omega_3)\omega_1). \end{aligned} \tag{63}$$

Replacing S^* from (51) in (63) and putting $\omega_3 = \varsigma$, we get

$$\left[\left(\frac{p}{2} + \frac{1}{n}\right) - (\lambda - 2)\right]\left[\delta(\omega_1, \omega_2)\eta\left(\omega_3\right) - \delta(\omega_1, \omega_3)\eta\left(\omega_2\right)\right] = 0.$$

Putting $\omega_3 = \varsigma$ and using (12) in (64), we obtain

$$\left[\left(\frac{p}{2} + \frac{1}{n}\right) - (\lambda - 2)\right]\delta(\phi\omega_1, \phi\omega_2) = 0, \tag{64}$$

which gives

$$\lambda = \frac{p}{2} + \frac{1}{n} + 2. \tag{65}$$

From (52) and (65) it follows that

$$\beta = 2n. \tag{66}$$

This implies the theorem. □

Corollary 7.2. *Let $M\left(\phi, \varsigma, \eta, \delta\right)$ be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting conformal η-Ricci soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to ∇^*. If M satisfies $(\varsigma.)_{S^*}.P^* = 0$, then the scalar curvature of M is*

$$r = -3(n-1)(n-2).$$

Proof. By the help of (53), (65) and (66) we get the required result. □

8. Conformal η-Ricci soliton on anti-anvariant submanifold of LP-Kenmotsu manifold satisfying $(\varsigma.)_{S^*}.P^* = 0$

Theorem 8.1. *Let $M\left(\phi, \varsigma, \eta, \delta\right)$ be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting conformal η-Ricci soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to ∇^*. If M satisfies $(\varsigma.)_{S^*}.P^* = 0$, then*

$$\lambda = \frac{p}{2} + \frac{1}{n} + 2, \beta = 2n, \text{ or, } \lambda = \frac{p}{2} + \frac{1}{n} - 2(2n-3), \beta = -2(n-2).$$

Proof. The condition that must be satisfied by S^* is

$$\begin{aligned} 0 \;=\;& S^*(\omega_1, P^*(\omega_2, \omega_3)\omega_4)\varsigma - S^*(\varsigma, P^*(\omega_2, \omega_3)\omega_4)\omega_1 \\ &+S^*(\omega_1, \omega_2)P^*(\varsigma, \omega_3)\omega_4 - S^*(\varsigma, \omega_2)P^*(\omega_1, \omega_3)\omega_4 \\ &+S^*(\omega_1, \omega_3)P^*(\omega_2, \varsigma)\omega_4 - S^*(\varsigma, \omega_3)P^*(\omega_2, \omega_1)\omega_4 \\ &+S^*(\omega_1, \omega_4)P^*(\omega_2, \omega_3)\varsigma - S^*(\varsigma, \omega_4)P^*(\omega_2, \omega_3)\omega_1, \end{aligned} \tag{67}$$

for all $\omega_1, \omega_2, \omega_3$ *and* $\omega_4 \in \chi(M)$. Taking inner product with ς the above equation becomes

$$\begin{aligned} 0 \;=\;& -S^*(\omega_1, P^*(\omega_2, \omega_3)\omega_4) - S^*(\varsigma, P^*(\omega_2, \omega_3)\omega_4)\eta(\omega_1) \\ &+S^*(\omega_1, \omega_2)\eta(P^*(\varsigma, \omega_3)\omega_4) - S^*(\varsigma, \omega_2)\eta(P^*(\omega_1, \omega_3)\omega_4) \\ &+S^*(\omega_1, \omega_3)\eta(P^*(\omega_2, \varsigma)\omega_4) - S^*(\varsigma, \omega_3)\eta(P^*(\omega_2, \omega_1)\omega_4) \\ &+S^*(\omega_1, \omega_4)\eta(P^*(\omega_2, \omega_3)\varsigma) - S^*(\varsigma, \omega_4)\eta(P^*(\omega_2, \omega_3)\omega_1). \end{aligned} \tag{68}$$

Replacing ω_4 by ς and using (41), (42) in (68), we get

$$0 = S^*(\omega_1, P^*(\omega_2, \omega_3)\varsigma) + S^*(\varsigma, \varsigma)\eta(P^*(\omega_2, \omega_3)\omega_1). \tag{69}$$

Replacing S^* from (51) in (69) and putting $\omega_3 = \varsigma$, we get

$$\left[\frac{p}{2} + \frac{1}{n} - \lambda + 2\right][2(n-1) + \beta - 2]\,\delta(\phi\omega_1, \phi\omega_2) = 0,$$

which gives

$$\lambda = \frac{p}{2} + \frac{1}{n} + 2 \text{ or, } \beta = -2(n-2). \tag{70}$$

From (52) and (70) it follows that

$$\lambda = \frac{p}{2} + \frac{1}{n} + 2, \beta = 2n$$

or,

$$\lambda = \frac{p}{2} + \frac{1}{n} - 2(2n-3), \beta = -2(n-2).$$

This implies the theorem. □

9. Example of anti-invariant submanifold of 5-dimensional LP-Kenmotsu manifold admitting conformal η-Ricci soliton with respect to ∇^*

We consider a 5-dimensional manifold

$$M^5 = \left\{(x, y, z, u, v) \in R^5\right\},$$

where (x, y, z, u, v) are the standard coordinates in R^5. We choose the linearly independent vector fields

$$I_1 = z\frac{\partial}{\partial x}, I_2 = z\frac{\partial}{\partial y}, I_3 = z\frac{\partial}{\partial z}, I_4 = z\frac{\partial}{\partial u}, I_5 = z\frac{\partial}{\partial v}.$$

Let g be the Riemannian metric defined by

$$g\left(I_i, I_j\right) = 0,\ if\ i \neq j,$$

for $i, j = 1, 2, 3, 4, 5$, and

$$\begin{aligned} g(I_1, I_1) &= 1, g(I_2, I_2) = 1, \\ g(I_3, I_3) &= -1, g(I_4, I_4) = 1, g(I_5, I_5) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X, I_3)$, for any $X \in \chi\left(M^5\right)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi I_1 = -I_2, \phi I_2 = -I_1, \phi I_3 = 0, \phi I_4 = -I_5, \phi I_5 = -I_4. \tag{71}$$

Let $X, Y, Z \in \chi\left(M^5\right)$ be given by

$$\begin{aligned} X &= x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4 + x_5 I_5, \\ Y &= y_1 I_1 + y_2 I_2 + y_3 I_3 + y_4 I_4 + y_5 I_5, \\ Z &= z_1 I_1 + z_2 I_2 + z_3 I_3 + z_4 I_4 + z_5 I_5. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X, Y) &= x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5, \\ \eta(X) &= -x_3, \\ g\left(\phi X, \phi Y\right) &= x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5. \end{aligned}$$

Using the linearity of g and ϕ, $\eta(I_3) = -1, \phi^2 X = X + \eta(X) I_3$, and $g\left(\phi X, \phi Y\right) = g(X, Y) + \eta(X)\eta(Y)$, for all $X, Y \in \chi(M)$. We have

$$\begin{aligned} [I_3, I_2] &= I_2, [I_3, I_1] = I_1, [I_3, I_4] = I_4, [I_3, I_5] = I_5, \\ [I_2, I_3] &= -I_2, [I_1, I_3] = -I_1, [I_4, I_3] = -I_4, [I_5, I_3] = -I_5, \\ \left[I_i, I_j\right] &= 0 \text{ for all others } i \text{ and } j. \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following:

$$\begin{aligned} \nabla_{I_1} I_3 &= -I_1,\ \nabla_{I_1} I_2 = 0\ , \nabla_{I_1} I_1 = -I_3\ , \nabla_{I_1} I_4 = 0\ , \nabla_{I_1} I_5 = 0, \\ \nabla_{I_2} I_3 &= -I_2, \nabla_{I_2} I_2 = -I_3, \nabla_{I_2} I_1 = 0,\ \nabla_{I_2} I_4 = 0, \nabla_{I_2} I_5 = 0, \\ \nabla_{I_3} I_3 &= 0, \nabla_{I_3} I_2 = 0, \nabla_{I_3} I_1 = 0, \nabla_{I_3} I_4 = 0, \nabla_{I_3} I_5 = 0, \\ \nabla_{I_4} I_3 &= -I_4, \nabla_{I_4} I_2 = 0, \nabla_{I_4} I_1 = 0, \nabla_{I_4} I_4 = -I_3, \nabla_{I_4} I_5 = 0, \\ \nabla_{I_5} I_3 &= -I_5, \nabla_{I_5} I_2 = 0, \nabla_{I_5} I_1 = 0, \nabla_{I_5} I_4 = 0,\ \nabla_{I_5} I_5 = -I_3. \end{aligned}$$

From the above results we see that the structure $\left(\phi, \xi, \eta, g\right)$ satisfies

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for all $X, Y \in \chi\left(M^5\right)$, where $\eta(\xi) = \eta(I_1) = -1$. Hence $M^5\left(\phi, \xi, \eta, g\right)$ is a LP-Kenmotsu manifold. Let $M^{*5}\left(\phi, \xi, \eta, g\right)$ be an anti-invariant submanifold of $M^5\left(\phi, \xi, \eta, g\right)$. Then the non-zero components of Riemannian curvature of M^{*5} with respect to Levi-Civita connection ∇ are given by

$$\begin{gathered}
R(I_3, I_2) I_3 = I_2, R(I_3, I_2) I_2 = -I_3, R(I_3, I_1) I_3 = I_1, \\
R(I_3, I_1) I_1 = -I_3, R(I_3, I_4) I_3 = I_4, R(I_3, I_4) I_4 = -I_3, \\
R(I_3, I_5) I_3 = I_5, R(I_3, I_5) I_5 = -I_3, R(I_2, I_3) I_2 = I_3, \\
R(I_2, I_3) I_3 = -I_2, R(I_2, I_1) I_2 = I_1, R(I_2, I_1) I_1 = -I_2, \\
R(I_2, I_4) I_2 = I_4, R(I_2, I_4) I_4 = -I_2, R(I_2, I_5) I_2 = I_5, \\
R(I_2, I_5) I_5 = -I_2, R(I_1, I_3) I_1 = I_3, R(I_1, I_3) I_3 = -I_1, \\
R(I_1, I_2) I_1 = I_2, R(I_1, I_2) I_2 = -I_1, R(I_1, I_4) I_1 = I_4, \\
R(I_1, I_4) I_4 = -I_1, R(I_1, I_5) I_1 = I_5, R(I_1, I_5) I_5 = -I_1, \\
R(I_4, I_3) I_4 = I_3, R(I_4, I_3) I_3 = -I_4, R(I_4, I_2) I_4 = I_2, \\
R(I_4, I_2) I_2 = -I_4, R(I_4, I_1) I_4 = I_1, R(I_4, I_1) I_1 = -I_4, \\
R(I_4, I_5) I_4 = I_5, R(I_4, I_5) I_5 = -I_4, R(I_5, I_3) I_5 = I_3, \\
R(I_5, I_3) I_3 = -I_5, R(I_5, I_2) I_5 = I_2, R(I_5, I_2) I_2 = -I_5, \\
R(I_5, I_1) I_5 = I_1, R(I_5, I_1) I_1 = -I_5, R(I_5, I_4) I_5 = I_4.
\end{gathered}$$

By the help of (26) we obtain

$$\begin{aligned}
\nabla^*_{I_1} I_3 &= -2I_1, \nabla^*_{I_1} I_2 = 0, \nabla^*_{I_1} I_1 = -2I_3, \nabla^*_{I_1} I_4 = 0, \nabla^*_{I_1} I_5 = 0, \\
\nabla^*_{I_2} I_3 &= -2I_2, \nabla^*_{I_2} I_2 = -2I_3, \nabla^*_{I_2} I_1 = 0,\ \nabla^*_{I_2} I_4 = 0, \nabla^*_{I_2} I_5 = 0, \\
\nabla^*_{I_3} I_3 &= 0,\ \nabla^*_{I_3} I_2 = I_1, \nabla^*_{I_3} I_1 = I_2, \nabla^*_{I_3} I_4 = I_5, \nabla^*_{I_3} I_5 = I_4, \\
\nabla^*_{I_4} I_3 &= -2I_4,\ \nabla^*_{I_4} I_2 = 0, \nabla^*_{I_4} I_1 = 0, \nabla^*_{I_4} I_4 = -2I_3, \nabla^*_{I_4} I_5 = 0, \\
\nabla^*_{I_5} I_3 &= -2I_5, \nabla^*_{I_5} I_2 = 0, \nabla^*_{I_5} I_1 = 0, \nabla^*_{I_5} I_4 = 0,\ \nabla^*_{I_5} I_5 = -2I_3.
\end{aligned}$$

Some of the non-zero components of Riemannian curvature tensor of M^{*5} with respect to Zamkovoy connection are given by

$$\begin{aligned}
R^*(I_3, I_1) I_3 &= 2(I_2 - I_1), R^*(I_2, I_1) I_2 = -4I_1, \\
R^*(I_4, I_1) I_4 &= -4I_1, R^*(I_5, I_1) I_5 = -4I_1, \\
R^*(I_1, I_3) I_3 &= 2(I_2 - I_1), R^*(I_1, I_2) I_2 = 4I_1, \\
R^*(I_1, I_4) I_4 &= 4I_1, R^*(I_1, I_5) I_5 = 4I_4.
\end{aligned}$$

Using the above curvature tensors the Ricci curvature tensors of M^{*5} with respect to ∇ and ∇^* are:

$$\begin{aligned}
S(I_3, I_3) &= -4, S(I_2, I_2) = S(I_1, I_1) = -2, \\
S(I_4, I_4) &= S(I_5, I_5) = -2, \\
S^*(I_3, I_3) &= -8, S^*(I_2, I_2) = S^*(I_4, I_4) = 10, \\
S^*(I_5, I_5) &= S^*(I_1, I_1) = 10.
\end{aligned}$$

Therefore, the scalar curvature tensor of M^{*5} with respect to Levi-Civita connection is $r = -12$, and scalar curvature tensor with respect to Zamkovoy connection is $r^* = 32$. Setting $V = X = Y = I_3$ in (43), we have

$$\begin{aligned} 0 &= \left(L^*_{I_3} g\right)(I_3, I_3) + 2S^*(I_3, I_3) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(I_3, I_3) + 2\beta\eta(I_3)\eta(I_3) \\ &= g\left(\nabla^*_{I_3} I_3, I_3\right) + g\left(I_3, \nabla^*_{I_3} I_3\right) \\ &\quad + 2S^*(I_3, I_3) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(I_3, I_3) + 2\beta\eta(I_3)\eta(I_3) \\ &= 0 + 0 + 2(-8) + \left[2\lambda - \left(p + \frac{2}{5}\right)\right](-1) + 2\beta \\ &= \beta - \lambda + \frac{p}{2} - \frac{39}{5}, \end{aligned}$$

which gives

$$\begin{aligned} \beta &= \lambda - \frac{p}{2} + \frac{39}{5} \\ &= \lambda - \frac{p}{2} - \frac{1}{5} + 2(5-1) \\ &= \lambda - \frac{p}{2} - \frac{1}{n} + 2(n-1), \end{aligned}$$

which shows that λ and β satisfies relation (52).

10. Conlusions

In this article, we have investigated the effects of conformal Ricci soliton and conformal η-Ricci soliton on the anti-invariant submanifold of LP-Kenmotsu manifold endowed with the Zamkovoy connection. More precisely, we have characterized anti-invariant submanifold of LP-Kenmotsu manifold with respect to the Zamkovoy connection, which admits conformal Ricci soliton and conformal η-Ricci soliton in terms of Einstein and η-Einstein manifolds. We know that Einstein manifolds, Kenmotsu manifolds are very important classes of manifolds having extensive use in mathematical physics and general relativity. Also, we studied conformal η-Ricci soliton on this manifold satisfying the certain curvature conditions and obtained important results. There are many studies in the literature the study of solitons on manifolds with different connections, but since the Zamkovoy connection was introduced recently, the effect of solitons on manifolds equipped with this connection is a very current study. Furthermore, it is interesting to investigate conformal η-Ricci solitons on other contact metric manifolds and there is further scope of research in this direction within the framework of various contact manifolds.

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