Filomat 39:28 (2025), 10115–10128
https://doi.org/10.2298/FIL2528115P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New results on interpolative metric spaces and applications

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Abstract. This paper aims to present novel fixed point results within the framework of interpolative metric spaces, which extend the concept of standard metric spaces. By leveraging the interpolative metrizability technique, which preserves completeness, we observe generalized contractions, including Banach contraction, Kannan contraction, and Chatterjea contraction. Additionally, we provide two applications to demonstrate the significance of our contributions to nonlinear analysis, particularly regarding solutions to nonlinear integral and fractional differential equations.

1. Introduction

Nonlinear analysis is a crucial domain of mathematics that examines equations and phenomena where linear approximations are inadequate to represent the inherent complexity. Nonlinear analysis techniques, including fixed-point theorems, variational approaches, and stability analysis, empower mathematicians and scientists to model, analyze, and forecast intricate dynamic systems. Its significance is seen in theoretical progress and practical applications, encompassing fluid dynamics, population models, financial systems, and neural networks. Nonlinear analysis connects abstract mathematical theory with practical problem-solving, providing profound insights into systems marked by unpredictability and complexity.

More than a century ago, Banach [2] established and proved the first metric fixed point theorem, inspired by Picard's groundbreaking discoveries [18]. Picard's innovative research resolved a particular starting value problem via consecutive approximations. It is accurate to state that Banach's fixed point theorem is primarily based on Picard's methodology. This outcome, first referred to in early literature as the Banach–Picard fixed point theorem, was further developed through significant contributions by Caccioppoli [3], who played a pivotal role in its transformation into what is currently acknowledged as Banach's Contraction Mapping Principle. In recognition of both mathematicians, specific literature refers to it as Banach–Caccioppoli's fixed point theorem.

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. interpolative metric Space, metric space, fixed point, generalized contraction.
Received: 25 February 2025; Revised: 02 October 2025; Accepted: 19 October 2025
Communicated by Erdal Karapınar
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Picard's initial article might be considered the inaugural recorded application of fixed point theory, signifying the commencement of a discipline that has since proven essential in applied mathematics. Fixed point theory has proven its applicability across the quantitative disciplines, highlighting its essential significance in multiple study fields, see, for example [20],[1]. As a result, the topic has attracted considerable interest from researchers.

Notably, contributions to the theoretical development of fixed point theory have been limited, especially in recent decades. Specific endeavors to enhance the theory have redundantly coincided with existing results, while others have restated already established conclusions. New results have occasionally emerged as direct repercussions of prior studies [8] [10]. Nevertheless, considerable advancements in metric fixed point theory have been made across several domains. The progression of the discipline has predominantly adhered to two pathways.

Initially, by mitigating, generalizing, or refining the concept of contraction to produce novel results. Secondly, by broadening the spectrum of abstract spaces where the existence of fixed points is established. There are natural and exciting extensions of conventional metric spaces, including symmetric spaces, quasi-metric spaces, b-metric spaces, ultra metric spaces, and interpolative metric spaces.

This study encompasses both techniques but in reverse order, which means primarily defining interpolative metric space, which is one of the significant extensions of traditional metric spaces [12],[13]. Subsequently, a generalized contraction is defined utilizing the c-function, also called the comparison function, resulting in novel effects.

The interpolative metric spaces are conceptually established by modifying the triangular inequality condition of traditional metric spaces to include an interpolative inequality. Recently researchers worked on the construction and generalization of fixed point theorems based on different types of interpolative contractions. Karapinar reworked Kannan-type contractions by interpolation in [9], improving classical contraction principles by giving them an interpolative structure. This method was extended in [7] by Karapinar, Fulga and Roldan Lopez de Hierro to the $(\alpha, \beta, \psi, \phi)$-interpolative contractions, giving a common framework to various known fixed point results. This concept was further developed in the work in [6], which proposed Perov-interpolative contractions of Suzuki type mappings, which provided new existence and uniqueness theorems of fixed points in metric spaces. The authors have investigated interpolative ϕ, ψ-type z-contractions in [16] where the flexibility by the use of auxiliary control functions is added. In [15], authors studied interpolative contractions of Rus-Reich-Cirica type through simulation functions, connecting classical contraction theories with modern functional techniques. Lastly, in [4] authors concentrated on interpolative forms of Boyd-Wong contractions and Matkowski contractions and generalized fixed point results to noncontinuous and generalized contractive settings. This modification facilitates more precise calculations and improved estimations in fixed-point theory applications. For further synthesis, we refer to [11]. The following sections will reevaluate the concept of interpolative metrics, positioning them as a rational extension of traditional metric spaces and their completeness. Appropriate examples demonstrate the generalized contraction as a generalization of the Banach, Kannan, and Chatterjea contractions within the interpolative metric space.

Lastly, we demonstrate the existence of solutions for a nonlinear integral equation and a nonlinear fractional differential equation within the framework of a complete interpolative metric space based on certain hypotheses, thereby underscoring the significance of the newly introduced generalized contraction.

Definition 1.1. *[13] Let X be a non-empty set. We say that $d : X \times X \to [0, +\infty)$ is (α, c)- interpolative metric if*

(m_1) $d(x, y) = 0$, *if and only if* , $x = y$ *for all* $x, y \in X$;

(m_2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$;

(m_3) *there exists an* $\alpha \in (0, 1)$ *and* $c \geq 0$ *such that*
$d(x, y) \leq d(x, z) + d(z, y) + c[(d(x, z))^{\alpha} \, (d(z, y))^{1-\alpha})]$ *for all* $x, y, z \in X$

Then, we call (X, d) *an* (α, c)- *interpolative metric space.*

Remark 1.2. *It is straightforward to see that each metric space forms an (α, c)-Interpolative metric with $c = 0$.*

Example 1.3. *[13, 14] Let X be a nonempty set of real numbers and $d : X \times X \to [0, \infty)$ as follows:*

$$d(x, y) = |x - y|(|x - y| + e)$$

for all $x, y \in X$.Then (X, d) is an interpolative metric space.

Proof. Since $|x - y|$ is itself a usual metric therefore m_1 and m_2 are obvious, we will prove (m_3) and find out the values of c and α.

$$\begin{aligned} d(x, y) &= |x - y|(|x - y| + e) \\ &\le (|x - z| + |y - z|)(|x - z| + |y - z| + e) \\ &\le |x - z|(|x - z| + e) + 2|x - z||y - z| + |y - z|(|y - z| + e) \\ &\le d(x, z) + d(y, z) + 2|x - z|^{\frac{1}{2}}|x - z|^{\frac{1}{2}}|y - z|^{\frac{1}{2}}|y - z|^{\frac{1}{2}} \\ &\le d(x, z) + d(y, z) + 2\{|x - z|^{\frac{1}{2}}(|x - z| + e)^{\frac{1}{2}}\}\{|y - z|^{\frac{1}{2}}(|y - z| + e)^{\frac{1}{2}}\} \\ &\le d(x, z) + d(y, z) + 2d(x, z)^{\frac{1}{2}}d(y, z)^{\frac{1}{2}}. \end{aligned} \tag{1}$$

Hence (X, d) is a interpolative metric space for $c = 2$ and $\alpha = \frac{1}{2}$. □

Example 1.4. *[5] Let X is a nonempty set of real numbers and define a function $d : X \times X \to [0, \infty)$ as follows*

$$d(x, y) = |x - y|^p,$$

for all $x, y \in X$, where $p > 1$ is a positive integer.

The conditions (m_1) and (m_2) are trivially observed. Regarding the condition (m_3), we have

$$\begin{aligned} d(x, y) &= |x - y|^p = |x - z + z - y|^p \\ &= |x - z|^p + {}^pC_1|x - z|^{p-1}|z - y| + {}^pC_2|x - z|^{p-2}|z - y|^2 + \\ &{}^pC_3|x - z|^{p-3}|z - y|^3 + \dots\dots + {}^pC_p|z - y|^p. \end{aligned} \tag{2}$$

Without the loss of generality, we assume that $|x - z| > |y - z|$

$$\begin{aligned} &\le d(x, z) + d(y, z) + \left({}^pC_1 + {}^pC_2 + {}^pC_3 + \cdots + {}^pC_{p-1}\right) d(x, z)^{\frac{p-1}{p}} d(z, y)^{\frac{1}{p}} \\ &\le d(x, z) + d(y, z) + \left\{\sum_{r=1}^{p-1} {}^pC_r\right\} d(x, z)^{1-\frac{1}{p}} d(z, y)^{\frac{1}{p}} \\ &\le d(x, z) + d(y, z) + c\, d(x, z)^{1-\alpha} d(z, y)^{\alpha}. \end{aligned} \tag{3}$$

Here,

$$c = \sum_{r=1}^{p-1} {}^pC_r,$$

and $\alpha = \frac{1}{p}$. Obviously $c > 0$ and $\alpha \in (0, 1)$. Hence, (X, d) is an (α, c) -interpolative metric space.

Suppose that $r > 0$ and $x \in X$. Denote

$$\beta(x, r) = \{y \in X : d(x, y) < r\},$$

as an open ball in (α, c)- interpolative metric space (X, d).

Definition 1.5. *Let (X, d) be a (α, c)- interpolative metric space and let x_n be a sequence in X. We say that x_n converges to x in X, if and only if, $d(x_n, x) \to 0$, as $n \to \infty$.*

Definition 1.6. *Let (X, d) be a (α, c)- interpolative metric space and let x_n be a sequence in X. We say that x_n is a Cauchy sequence in X, if and only if, $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$.*

Definition 1.7. *Let (X, d) be a (α, c)- interpolative metric space. We say that (X, d) is a complete (α, c)- interpolative metric space if every Cauchy sequence converges in X.*

Our paper's fundamental concept is laid forth in the next section, which is based on generalized contraction T and is defined within the context of interpolative metric spaces using comparison functions. Within the context of a complete interpolative metric space, the unique fixed point of T is demonstrated by Theorem 2.3. Suitable examples demonstrate that the generalized contraction T extends the Banach, Kannan, and Chatterjea contractions in the interpolative metric spaces.

2. Main Results

We begin this section by defining the comparison function Ψ, which is the integral part of our main definition.
Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

a) ψ is non decreasing;

b)
$$\sum_{n=1}^{+\infty} \psi^n(t) < \infty,$$
for all $t > 0$, where ψ^n is the *nth* iterate of ψ.

Lemma 2.1. *If $\psi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

1 . Each iterate ψ^k of Ψ, $k \geq 1$, is also a comparison function;

2 . ψ is continuous at 0;

3 . $\psi(t) < t$, for any $t > 0$.

In this section, first, we state the fixed point theorem in the setting of an (α, c)- interpolative metric space.

Definition 2.2. *Let (X, d) be an (α, c)- interpolative metric space and let $T : X \to X$ be a mapping. Suppose that there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(M_d(x, y))), \tag{4}$$

for all $x, y \in X$, in which,

$$M_d(x, y) = max\Big\{\alpha_1 d(x, y), \alpha_2\{d(x, Tx) + d(y, Ty)\}, \alpha_3\{d(x, Ty) + d(y, Tx)\}\Big\} \tag{5}$$

where $\alpha_i \geq 0$ for $i = 1, 2, 3$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$ and $\alpha_3 < \frac{1}{2+c}$.
Then T is called a generalized contraction in (α, c)-interpolative metric space X.

Theorem 2.3. *Let (X, d) be an (α, c)-interpolative metric space and $T : X \to X$ be a generalized contraction defined in Definition 2.2, then T has a unique fixed point in X.*

Proof. First of all, it is straightforward to notice that the uniqueness of the fixed point is derived from inequality (4) easily. Next, let us start the proof by constructing a sequence by taking an arbitrary point $x \in X$. We assume it is as a first term of the desired sequence, that is, $x_0 = x$. The rule of the constructed sequence $\{x_n\}$ is defined as follows:

$$x_{n+1} = Tx_n \text{ for all } n \in N_0.$$

Before continuing with the proof, we examine and eliminate the trivial case:
If $x_{n_0} = x_{n_0+1}$ for any $n_0 \in N_0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$.

In other words, x_{n_0} forms the desired fixed point of the given mapping T, and we are done in this case. Accordingly, throughout the proof, we shall suppose that $x_n \neq x_{n+1}$ for all $n \in N_0$. In conclusion, we observed that $d(x_n, x_{n+1}) > 0$ for all $n \in N_0$. On account of inequality (4) and the fact that $\psi(t) < t$ for all $t > 0$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi(M_d(x_n, x_{n-1})) \\ &= \psi(max\{\alpha_1 d(x_n, x_{n-1}), \alpha_2\{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})\}, \alpha_3\{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)\}\}) \\ &= \psi(max\{\alpha_1 d(x_n, x_{n-1}), \alpha_2\{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\}, \alpha_3\{d(x_n, x_n) + d(x_{n-1}, x_{n+1})\}\}) \\ &= \psi(max\{\alpha_1 d(x_n, x_{n-1}), \alpha_2\{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\}, \alpha_3\{d(x_{n-1}, x_{n+1})\}\}). \end{aligned} \tag{6}$$

$$M_d(x_n, x_{n-1}) = max\{\alpha_1 d(x_n, x_{n-1}), \alpha_2\{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\}, \alpha_3\{d(x_{n-1}, x_{n+1})\}\}.$$

$$\begin{aligned} M_d(x_n, x_{n-1}) = max\{&\alpha_1 d(x_n, x_{n-1}), \alpha_2\{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\}, \\ &\alpha_3\{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + c(d(x_{n-1}, x_n)^\alpha d(x_n, x_{n+1})^{1-\alpha})\}\}. \end{aligned} \tag{7}$$

Now, let us assume that $d(x_n, x_{n+1}) > d(x_n, x_{n-1})$, we infer

$$\begin{aligned} &M_d(x_n, x_{n-1}) < max\{\alpha_1 d(x_n, x_{n+1}), \alpha_2\{d(x_n, x_{n+1}) + d(x_{n+1}, x_n)\}, \\ &\alpha_3\{d(x_{n+1}, x_n) + d(x_n, x_{n+1}) + c(d(x_{n+1}, x_n)^\alpha d(x_n, x_{n+1})^{1-\alpha})\}\}. \end{aligned}$$

We conclude

$$M_d(x_n, x_{n-1}) < max\{\alpha_1 d(x_n, x_{n+1}), 2\alpha_2 d(x_n, x_{n+1}), \alpha_3(2 + c)d(x_n, x_{n+1})\}$$

Since $\alpha_3 < \frac{1}{2+c}$, we obtain

$$M_d(x_n, x_{n-1}) < max\{\alpha_1 d(x_n, x_{n+1}), 2\alpha_2 d(x_n, x_{n+1}), d(x_n, x_{n+1})\}.$$

This amounts to saying that $M_d(x_n, x_{n-1}) < d(x_n, x_{n+1})$.

Since we have $\psi(t) < t$, from equation (4), we have

$$d(x_n, x_{n+1}) \leq \psi\Big(M_d(x_n, x_{n-1})\Big) < M_d(x_n, x_{n-1}) < d(x_n, x_{n+1}),$$

for all $n \in N_0$.
Which gives a contradiction. Therefore, our assumption is wrong and we must have

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n-1}),$$

for all $n \in N_0$.
So the sequence $\{d(x_n, x_{n+1})\}$ is a non increasing sequence.
Further, we can easily show that

$$M_d(x_n, x_{n-1}) < d(x_n, x_{n-1}).$$

From equation (4) we have

$$d(x_n, x_{n+1}) \le \psi(M_d(x_n, x_{n-1})) < \psi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}).$$

By iteration we find that

$$0 < d(x_n, x_{n+1}) \le \psi^n\Big(d(x_0, x_1)\Big), \tag{8}$$

for all $n \in N_0$.

Letting $n \to \infty$ both side of (8) yields that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

Since the limit in (9) tends to zero, we deduce that

$$d(x_n, x_{n+1}) \le 1, \tag{10}$$

for all $n \ge 1$. For some large enough $M \in N$. In what follows, we shall prove that the constructed sequence is Cauchy. For this purpose, we presume that $m, n \in N$ and $m > n > M$. Before we show that the sequence is Cauchy, we shall eliminate the simple case: $x_n = x_m$. If we have $T^m(x_0) = T^n(x_0)$. By a simple elaboration, we get $T^{m-n}(T^n(x_0))) = T^n(x_0)$. As a result, we deduce that $T^n(x_0)$ is the fixed point of T^{m-n}. In addition, we have

$$T(T^{m-n}(T^n(x_0)))) = T^{m-n}(T(T^n(x_0)))) = T(T^n(x_0)).$$

In other words, $T(T^n(x_0))$ forms the desired fixed point of T^{m-n}. In conclusion, $T(T^n(x_0)) = T^n(x_0)$ and hence $T^n(x_0)$ is the fixed point of T. Consequently, without loss of generality, we assume that $x_n \ne x_m$. In what follows, we assert the constructed iterative sequence $\{x_n\}$ forms Cauchy. For this purpose, we claim that

$$\lim_{n\to\infty} d(x_n, x_{n+r+1}) = 0. \tag{11}$$

We shall use elementary induction to demonstrate this. We start with the following limit:

$$d(x_n, x_{n+2}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + c[d(x_n, x_{n+1})^{\alpha}(d(x_{n+1}, x_{n+2})^{1-\alpha}]. \tag{12}$$

By taking limit as $n \to \infty$ in the above inequality and taking the inequality (8) into account, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{13}$$

In addition, we have

$$d(x_n, x_{n+3}) \le d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + c[d(x_n, x_{n+2})^{\alpha}(d(x_{n+1}, x_{n+3})^{1-\alpha}]. \tag{14}$$

Taking (8) and (14) into account together by taking $n \to \infty$ in the above inequality we find that

$$\lim_{n\to\infty} d(x_n, x_{n+3}) = 0. \tag{15}$$

Now we suppose the general case of our assertion holds; that is, we have

$$\lim_{n\to\infty} d(x_n, x_{n+r}) = 0, \tag{16}$$

for some $r \in N$. Therefore, by the hypothesis of the Theorem, we arrive at

$$d(x_n, x_{n+r+1}) \le d(x_n, x_{n+r}) + d(x_{n+r}, x_{n+r+1}) + c[d(x_n, x_{n+r})^{\alpha}(d(x_{n+r}, x_{n+r+1})^{1-\alpha}]. \tag{17}$$

On account of limits (8) and (16), by taking the limit of the above inequality as $n \to \infty$, we get that

$$\lim_{n\to\infty} d(x_n, x_{n+r+1}) = 0. \tag{18}$$

Consequently, we deduce that the recursively constructed sequence $\{x_n\}$ is Cauchy.

Regarding that (X, d) is a complete (α, c)- interpolative metric space, the sequence $\{x_n\}$ converges to $x^* \in X$. We assert that x^* is the fixed point of T. Suppose to the contrary, $d(x^*, Tx^*) > 0$. Note that

$$\begin{aligned} d(x_{n+1}, Tx^*) &= d(Tx_n, Tx^*) \\ &\leq \psi[max\{\alpha_1 d(x_n, x^*), \alpha_2\{d(x_n, x_{n+1}) + d(x^*, Tx^*)\}, \alpha_3\{d(Tx^*, x_n) + d(x^*, Tx_n)\}] \\ &\leq \psi[max\{\alpha_1 d(x_n, x^*), \alpha_2\{d(x_n, x_{n+1}) + d(x^*, Tx^*)\}, \\ &\quad \alpha_3\{\{d(Tx^*, x^*) + d(x^*, x_n) + cd(Tx^*, x^*)^{\alpha} d(x^*, x_n)^{1-\alpha}\} \\ &\quad + d(x^*, x_n) + d(x_n, Tx_n) + cd(x^*, x_n)^{\alpha} d(x_n, Tx_n)^{1-\alpha}\}\}]. \end{aligned} \tag{19}$$

By taking lim sup of both sides of the inequality (19), we obtain

$$d(Tx^*, x^*) \leq \psi(d(Tx^*, x^*)) < d(Tx^*, x^*). \tag{20}$$

Since we assume $d(x^*, Tx^*) > 0$,

which is a contradiction. As a result $Tx^* = x^*$ is the fixed point of T in X.

Now we shall prove that x^* is unique. If possible suppose there exists one more fixed point of T say $y*$ such that $x^* \neq y^*$.

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \psi(M_d(x^*, y^*)) \\ &\leq \psi(max\{\alpha_1 d(x^*, y^*), \alpha_3\{d(x^*, Ty^*) + d(Tx^*, y^*)\}, \alpha_2\{d(x^*, Tx^*) + d(y^*, Ty^*)\}\}) \\ &\leq \psi(max\{\alpha_1 d(x^*, y^*), 2\alpha_3 d(x^*, y^*)\} \\ &< \psi(d(x^*, y^*)) \\ &< d(x^*, y^*), \end{aligned}$$

which gives a contradiction. Hence, our assumption is wrong, and $x^* = y^*$, authenticating that the fixed point of T is unique. □

Example 2.4. *Let (X, d) be an interpolative metric space where $X = [0, 1]$ and d is an $(\frac{1}{2}, 2)$-interpolative metric defined in 1.3.*
i.e. $d(x, y) = |x - y|(|x - y| + e)$, where e is the Euler number. Let $T : X \to X$ be a function define by $T(x) = \frac{x}{2}$.

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x}{2} - \frac{y}{2}\right|\left(\left|\frac{x}{2} - \frac{y}{2}\right| + e\right) \\ &= \frac{|x - y|}{2}\left(\frac{|x - y|}{2} + e\right) \\ &\leq \frac{1}{2}\{|x - y|(|x - y| + e)\} \\ &\leq q\{|x - y|(|x - y| + e)\}, \quad \frac{1}{2} < q < 1 \\ &\leq qd(x, y) \\ &\leq \psi(d(x, y)), \quad \textit{where } \psi(t) = qt \\ &\leq \psi\Big(\max\big\{\alpha_1 d(x, y), \alpha_2\{d(Tx, x) + d(Ty, y)\}, \alpha_3\{d(Ty, x) + d(y, Tx)\}\big\}\Big) \\ &\leq \psi(M_d(x, y)), \end{aligned} \tag{21}$$

where $\frac{1}{2} < q < \alpha_1 < 1, \alpha_2 = 0, \alpha_3 = 0 < \frac{1}{2+c} = \frac{1}{4}$.
Here, T satisfies all the conditions of Theorem 2.3, and hence T has a unique fixed point in $X = [0,1]$ which is $x = 0$. Here T is also an example of ***Banach contraction*** *on the given interpolative metric space.*

Remark 2.5. *Let (X,d) where $X = [0,1]$ be an (α,c)- interpolative metric space where d is defined by Example (4) i.e. $d(x,y) = |x-y|^p$ where $p > 1$ is an positive integer here $\alpha = \frac{1}{p}$ and $c = \sum_{r=1}^{p-1} {}^pC_r$. Here, we note that the metric defined above is also a $b-$ metric for $s = p$.*

Here, if we define $T : X \to X$ such that $T(x) = \frac{1-x}{3}$, then we claim that T is a ***Kannan contraction*** *on given interpolative metric space (X,d).*

Proof.

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{(1-x)}{3} - \frac{(1-y)}{3}\right|^p = \frac{|x-y|^p}{3^p} \\ d(x,Tx) &= \left|x - \frac{1-x}{3}\right|^p = \left|\frac{4x-1}{3}\right|^p \\ d(y,Ty) &= \left|y - \frac{1-y}{3}\right|^p = \left|\frac{4y-1}{3}\right|^p \\ d(x,Tx) + d(y,Ty) &= \left|\frac{4x-1}{3}\right|^p + \left|\frac{4y-1}{3}\right|^p . \end{aligned} \tag{22}$$

Obviously,

$$d(Tx,Ty) \leq \frac{|2x-1|^p + |2y-1|^p}{3^p} \leq \frac{d(x,Tx) + d(y,Ty)}{2^p},$$

hence

$$d(Tx,Ty) \leq q(d(x,Tx) + d(y,Ty)), \tag{23}$$

□

where $q = \frac{1}{2^p} < \frac{1}{2}$ for positive integer $p > 1$.
Hence T is a **Kannan Contraction** on (X,d). Equation (23) can also be written as

$$d(Tx,Ty) \leq q\{(d(x,Tx) + d(y,Ty))\} \leq \psi(M_d(x,y)),$$

for $\psi(t) = qt = \frac{t}{2^p}$ and

$$M_d(x,y) = max\{\alpha_1 d(x,y), \alpha_2\{d(x,Tx) + d(y,Ty)\}, \alpha_3\{d(x,Ty) + d(y,Tx)\},$$

where $\alpha_1 = 0, 0 < q < \alpha_2 < \frac{1}{2}$ and $\alpha_3 = 0 < \frac{1}{2+c}$.
Here ψ fulfills all the requirement of the c function. Hence, from Theorem 2.3, T has a unique fixed point in X which is $x = \frac{1}{4}$.

Example 2.6. *Let (X,d) be an interpolative metric space where $X = [0,1]$, and d is an $(\frac{1}{2}, 2)$-interpolative metric defined in Example 1.3.*

Let $T : X \to X$ be a function define by

$$T(x) = \frac{x}{3}.$$

$$
\begin{aligned}
d(Tx, Ty) &= \left|\frac{x}{3} - \frac{y}{3}\right|\left(\left|\frac{x}{3} - \frac{y}{3}\right| + e\right) \\
&= \frac{|x-y|}{3}\left(\frac{|x-y|}{3} + e\right) \\
&\leq \frac{1}{3}\{|x-y|(|x-y| + e)\}
\end{aligned} \tag{24}
$$

Again from definition of $d(x, y)$, we have

$$
d(x, Ty) = \left|x - \frac{y}{3}\right|\left(\left|x - \frac{y}{3}\right| + e\right). \tag{25}
$$

Now, since $y \in [0, 1]$, we can write

$$
\begin{aligned}
\frac{y}{3} &\leq y \\
-\frac{y}{3} &\geq -y \\
x - \frac{y}{3} &\geq x - y \\
\frac{1}{3}(x - \frac{y}{3}) &\geq \frac{1}{3}(x - y) \\
\frac{1}{3}(x - \frac{y}{3}) &\geq \frac{1}{3}(x - y) \\
\frac{1}{3}\left(x - \frac{y}{3}\right)\left\{(x - \frac{y}{3}) + e\right\} &\geq \frac{1}{3}\left(x - y\right)\left\{\frac{1}{3}(x - y) + e\right\} \\
\frac{1}{3}d(x, Ty) &\geq d(Tx, Ty), x \neq y
\end{aligned} \tag{26}
$$

on similar lines, we can prove that

$$
\frac{1}{3}d(y, Tx) \geq d(Tx, Ty),
$$

for all values of x and y in $[0, 1]$, where $x \neq y$.
From (26) and (2.6) we have

$$
\begin{aligned}
\frac{1}{3}\{d(x, Ty) + d(y, Tx)\} &\geq 2d(Tx, Ty), \\
d(Tx, Ty) &\leq \frac{1}{6}\{\{d(x, Ty) + d(y, Tx)\}, \\
d(Tx, Ty) &\leq q\{\{d(x, Ty) + d(y, Tx)\},
\end{aligned}
$$

where $q = \frac{1}{6} < \frac{1}{2}$. Thus T is a ***Chatterjea Contraction*** *on the $(\frac{1}{2}, 2)$interpolative metric space X.*
Moreover,

$$
d(Tx, Ty) \leq q\{\{d(y, Tx) + d(y, Tx)\} \leq \psi(M_d(x, y)), \tag{27}
$$

where

$$
M_d(x, y) = max\{\alpha_1 d(x, y), \alpha_2\{d(x, Tx) + d(y, Ty)\}, \alpha_3\{d(x, Ty) + d(y, Tx)\},
$$

for $\alpha_1 = 0, \alpha_2 = 0$ and $\frac{1}{6} < \alpha_3 < \frac{1}{4}$.

Here, ψ fulfills all the requirement of the c function. Hence, from Theorem 2.3, T there is a unique fixed point in X which is $x = 0$.

Next section, demonstrates the existence of solutions for a nonlinear integral equation and a nonlinear fractional differential equation within the framework of a complete interpolative metric space based on certain hypotheses, thereby underscoring the significance of the newly introduced generalized contraction.

3. Some Applications

3.1. An application to non-linear integral equations

Let $M = C[\alpha, \gamma]$ be the set of all continuous real valued functions defined on $[\alpha, \gamma]$ where $0 \leq \alpha < \gamma$. Let $d : M \times M \to R^+$ be defined by

$$d(x, y) = \sup_{\alpha \leq t \leq \gamma} |x(t) - y(t)|^p,$$

for all $x, y \in M$ and the positive integer $p > 1$. Clearly, (M, d) is a complete (α, c)- interpolative metric space where $\alpha = \frac{1}{p}$ and $c = \sum_{r=1}^{p-1} {}^pC_r$.
Our aim is to find a function $x(t) \in M$, $t \in [\alpha, \gamma]$ such that for $f : [\alpha, \gamma] \to R, g : [\alpha, \gamma] \times [\alpha, \gamma] \to R$ and $A : [\alpha, \gamma] \times [\alpha, \gamma] \times R \to R$ it satisfies the non linear Integral equation

$$x(t) = f(t) + \int_\alpha^\gamma g(t, \tau) A(t, \tau, x(\tau)) d\tau. \tag{28}$$

Theorem 3.1. *The non linear integral equation* (28) *has a unique solution in M provided that the following hypotheses hold.*

(i) The functions $f : [\alpha, \gamma] \to \mathbb{R}$, $g : [\alpha, \gamma] \times [\alpha, \gamma] \to \mathbb{R}$, and $A : [\alpha, \gamma] \times [\alpha, \gamma] \times \mathbb{R} \to \mathbb{R}$ are continuous on $[\alpha, \gamma]$, $[\alpha, \gamma]^2$, and $[\alpha, \gamma]^2 \times \mathbb{R}$, respectively.

(ii) For all $t, \tau \in [\alpha, \gamma]$ and for all $x, y \in M$, there exists $\sigma > 2$ such that

$$|A(t, \tau, x(\tau)) - A(t, \tau, y(\tau))| \leq e^{-\frac{p}{\sigma}} Q(x, y),$$

where

$$Q(x, y) = \max\Big\{|x - y|, |x - Tx|, |y - Ty|, |x - Ty|, |y - Tx|\Big\}.$$

(iii) For all $t, z \in [\alpha, \gamma]$,

$$\sup_{\alpha \leq t \leq \gamma} \int_\alpha^\gamma |g(t, z)|^p dz \leq \frac{1}{2^{p-1}(\gamma - \alpha)}.$$

Proof. Let $T : (C[\alpha, \gamma], R) \to C([\alpha, \gamma], R)$ be defined by

$$T(x(t)) = f(t) + \int_\alpha^\gamma g(t, \tau) A(t, \tau, x(\tau))\, d\tau. \tag{29}$$

It is easy to see that the existence of a unique solution of the nonlinear integral equation (28) is equivalent to the existence of a fixed point of T in (29). Now we will prove that T is a generalized contraction.

$$
\begin{aligned}
|Tx(t) - Ty(t)|^p &= \left| f(t) + \int_\alpha^\gamma g(t,\tau)A(t,\tau,x(\tau))d\tau - f(t) - \int_\alpha^\gamma g(t,\tau)A(t,\tau,y(\tau))d\tau \right|^p \\
&= \left| \int_\alpha^\gamma g(t,\tau)A(t,\tau,x(\tau))d\tau - \int_\alpha^\gamma g(t,\tau)A(t,\tau,y(\tau))d\tau \right|^p \\
&= \left| \int_\alpha^\gamma g(t,\tau)(A(t,\tau,x(\tau) - A(t,\tau,y(\tau))d\tau \right|^p \\
&\le \int_\alpha^\gamma |g(t,\gamma)|^p d\tau \int_\alpha^\gamma |(A(t,\tau,x(\tau) - A(t,\tau,y(\tau))|^p d\tau \\
&\le \int_\alpha^\gamma |g(t,\gamma)|^p d\tau \int_\alpha^\gamma |e^{\frac{-\sigma}{p}} Q(x(\tau),y(\tau))|^p d\tau \\
&\le \int_\alpha^\gamma |g(t,\gamma)|^p d\tau \int_\alpha^\gamma e^{-\sigma} max\Big\{ |x(\tau) - y(\tau)|^p, |x(\tau) - Tx(\tau)|^p, \\
&\quad |y(\tau) - Ty(\tau)|^p, |Tx(\tau) - y(\tau)|^p, |x(\tau) - Ty(\tau)|^p \Big\} d\tau.
\end{aligned}
$$

On account of above inequality and taking $\sup_{t\in[\alpha,\gamma]}$ on both sides, we infer

$$
\begin{aligned}
\sup_{t\in[\alpha,\gamma]} |Tx(t) - Ty(t)|^p &\le \frac{1}{2^{p-1}(\gamma-\alpha)} \int_\alpha^\gamma e^{-\sigma} \max\Big\{ \sup_{t\in[\alpha,\gamma]} |x(\tau) - y(\tau)|^p, \sup_{t\in[\alpha,\gamma]} |x(\tau) - Tx(\tau)|^p, \\
&\quad \sup_{t\in[\alpha,\gamma]} |y(\tau) - Ty(\tau)|^p, \sup_{t\in[\alpha,\gamma]} |y(\tau) - Tx(\tau)|^p, \sup_{t\in[\alpha,\gamma]} |x(\tau) - Ty(\tau)|^p \Big\} d\tau \\
&\le \frac{1}{2^{p-1}(\gamma-\alpha)} \int_\alpha^\gamma \max\Big\{ e^{-\sigma} \sup_{t\in[\alpha,\gamma]} |x(\tau) - y(\tau)|^p, \\
&\quad e^{-\sigma}\Big(\sup_{t\in[\alpha,\gamma]} |x(\tau) - Tx(\tau)|^p + \sup_{t\in[\alpha,\gamma]} |y(\tau) - Ty(\tau)|^p \Big), \\
&\quad e^{-\sigma}\Big(\sup_{t\in[\alpha,\gamma]} |y(\tau) - Tx(\tau)|^p + \sup_{t\in[\alpha,\gamma]} |x(\tau) - Ty(\tau)|^p \Big) \Big\} d\tau \\
&\le \frac{\max\Big\{ e^{-\sigma} d(x,y), e^{-\sigma}\Big(d(x,Tx) + d(y,Ty)\Big), e^{-\sigma}\Big(d(y,Tx) + d(x,Ty)\Big) \Big\}}{2^{p-1}(\gamma-\alpha)} \int_\alpha^\gamma d\tau \\
&\le \frac{1}{2^{p-1}} M_d(x,y).
\end{aligned}
$$

where $M_d(x,y) = \max\Big\{ \alpha_1 d(x,y), \alpha_2(d(x,Tx) + d(y,Ty)), \alpha_3(d(y,Tx) + d(x,Ty)) \Big\}$.
It may be noted $\alpha_1 = \alpha_2 = \alpha_3 = e^{-\sigma}$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 5e^{-\sigma}$ and $5e^{-\sigma} < 1$ i.e. $e^{\sigma} > 5$ because σ is a real number which is greater than 2.
Also for any value of $p > 1$, $\alpha_3 < \frac{1}{2 + \sum_{n=1}^{p-1} {}^pC_n}$, we must have

$$
d(T(x), T(y)) = \|Tx(t) - Ty(t)\|_{\infty,p} = \sup_{t\in[\alpha,\gamma]} |Tx(t) - Ty(t)|^p \le \frac{1}{2^{p-1}} M_d(x,y),
$$

$$
d(T(x), T(y)) \le \psi(M_d(x,y))
$$

where $\psi(t) = \frac{t}{2^{p-1}}$ is obviously a c function.

Hence, from Theorem 4, the operator T is a contraction on interpolative metric space and has a unique fixed point in $X = C[\alpha, \gamma]$ which will be the solution of the integral equation (28). □

3.2. An Application to Fractional Differential Equations

Let $(C[0,1])$ be the set of all continuous functions on $[0,1]$ and $\sigma : C[0,1] \times C[0,1] \to R$ be the $(\frac{1}{2}, 2)$ metric define by $(u, v) = \|u - v\|_\infty^2 = max_{t\in[0,1]}|u(t) - v(t)|^2$.
Now, we recall some notation of and [19] and [17]. The Caputo derivative of fractional order β for a continuous function $h : [0, +\infty) \to R$ is define as

$$^{c}D^{\beta}(h(t)) = \frac{1}{\Gamma(m-\beta)} \int_0^1 (t-s)^{m-\beta-1} g^{(m)}(s)ds, (m-1 < \beta < n, m = [\beta]+1). \tag{30}$$

Where Γ the gamma function and $[\beta]$ denotes the integer part of a real numbers.
In this work, we present the existence of the solution of nonlinear fractional differential equation.

$$^{c}D^{\beta}(u(t)) + f(t, u(t)) = 0, (0 \le t \le 1, \beta < 1) \tag{31}$$

with $u(0) = u(1) = 0$ and $f : [0,1] \times R \to R$ being a continuous function, and Green's function associated with Problem (31) is given by

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & if 0 \le t \le s \le 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(a)} & if 0 \le s \le t \le 1. \end{cases} \tag{32}$$

Assume that the following conditions hold :
1. $|f(t, u) - f(t, v)| \le e^{-\tau} W(u, v)$ where τ is a real number such that $\tau > 2$ each $t \in [0, 1]$ and $a, b \in R$, where

$$W(u, v) = max\{|u - v|, |u - Tu|, |v - Tv|, |u - Tv|, |v - Tu|\}.$$

Theorem 3.2. *Under the the assumption of condition 1,* (31) *has a solution.*

Proof. It is well known that u is a solution of (31) if and only if $u \in X$ is a solution of the integral equation:

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad \text{for all } t \in [0,1]. \tag{33}$$

Now consider:

$$
\begin{aligned}
|Tu(x) - Tv(x)|^2 &= \left| \int_0^1 G(x,s) f(s,u(s))\,ds - \int_0^1 G(x,s) f(s,v(s))\,ds \right|^2 \\
&\le \int_0^1 \left| G(x,s)\big(f(s,u(s)) - f(s,v(s))\big) \right|^2 ds \\
&\le \left(\int_0^1 |G(x,s)|\,|f(s,u(s)) - f(s,v(s))|\,ds \right)^2 \\
&\le \left(\int_0^1 |G(x,s)|\, e^{-\tau} W(u,v)\,ds \right)^2 \\
&\le e^{-2\tau} W(u,v)^2 \left(\int_0^1 |G(x,s)|\,ds \right)^2 \\
&\le e^{-2\tau} \max\{|u-v|, |u-Tu|, |v-Tv|, |u-Tv|, |v-Tu|\}^2 \left(\int_0^1 |G(x,s)|\,ds \right)^2 \\
&\le e^{-2\tau} \max\{|u-v|^2, |u-Tu|^2, |v-Tv|^2, |u-Tv|^2, |v-Tu|^2\} \left(\int_0^1 |G(x,s)|\,ds \right)^2 \\
&\le e^{-2\tau} \max\{|u-v|^2, |u-Tu|^2 + |v-Tv|^2, |u-Tv|^2 + |v-Tu|^2\} \left(\int_0^1 |G(x,s)|\,ds \right)^2 \\
&\le e^{-\tau} \max\{e^{-\tau}|u-v|^2, e^{-\tau}(|u-Tu|^2 + |v-Tv|^2), e^{-\tau}(|u-Tv|^2 + |v-Tu|^2)\} \\
&\quad \cdot \left(\sup_{x\in[0,1]} \int_0^1 |G(x,s)|\,ds \right)^2 .
\end{aligned} \tag{34}
$$

Since $\sup_{x\in[0,1]} \int_0^1 |G(x,s)|\,ds \le 1$, we have:

$$
\begin{aligned}
\max_{x\in[0,1]} |Tu(x) - Tv(x)|^2 &\le e^{-\tau} \max\{e^{-\tau}|u-v|^2, e^{-\tau}(|u-Tu|^2 + |v-Tv|^2), \\
&\qquad e^{-\tau}(|u-Tv|^2 + |v-Tu|^2)\} \\
&\le e^{-\tau} \max\{\alpha_1|u-v|^2, \alpha_2(|u-Tu|^2 + |v-Tv|^2), \\
&\qquad \alpha_3(|u-Tv|^2 + |v-Tu|^2)\}.
\end{aligned} \tag{35}
$$

Here, $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 5e^{-\tau} < 1$, since $\tau > 2$ is a real number.
Thus:

$$\max_{x\in[0,1]} |Tu(x) - Tv(x)|^2 \le e^{-\tau} M_d(x,y),$$

and

$$d(Tx,Ty) \le \psi(M_d(x,y)),$$

where

$$M_d(x,y) = \max\{\alpha_1|u-v|^2, \alpha_2(|u-Tu|^2 + |v-Tv|^2), \alpha_3(|u-Tv|^2 + |v-Tu|^2)\},$$

and $\psi(t) = e^{-\tau}t$, for $t \in [0,\infty)$. Clearly, T is a contraction.
We conclude that the operator T satisfies all the conditions of Theorem 2.3. Hence, T has a unique fixed point in the given $(\frac{1}{2}, 2)$ interpolative metric space, which will be the solution of (31). □

Remark 3.3. *:- Theorem 3.2 also holds for b− metric spaces with* $s = 2$.

Conclusion

This study examines fixed point theory in interpolative metric spaces, extending the classical theory related to standard metric spaces. We illustrated the applicability of generalized contractions, such as Banach, Kannan, and Chatterjea contractions, in interpolative metric spaces, emphasizing their ability to resolve intricate nonlinear systems. The completeness of interpolative metric spaces and the flexibility offered by the interpolative inequality establishes a strong basis for the progression of fixed-point theory. Our findings resulted in an application that illustrates solutions to non-linear integral and fractional differential equations, confirming the practical significance and adaptability of the suggested framework. These findings connect theoretical progress with practical applications, enhancing the broader domain of nonlinear analysis.

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