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Matrix approach to Schrödinger-type equations initiated by perturbational invariants

Nenad Vesić[a,*], Ivana Djurišić[b], Nenad Milojević[c]

[a]*Mathematical Institute of Serbian Academy of Sciences and Arts, University of Belgrade, Serbia*
[b]*Institute for Multidisciplinary Research, University of Belgrade, Serbia*
[c]*Faculty of Sciences and Mathematics, University of Niš, Serbia*

Abstract. This research is motivated by results obtained by Vesić et al. about scalar perturbational invariants caused by scalar perturbations of a gravitational field. Five linearly independent scalar perturbational invariants are obtained there. After some algebraic computing, these five perturbational invariants are transformed to new five linearly independent scalar perturbational invariants. From the set of last mentioned five perturbational invariants, two pairs of them are selected such that the second invariant in this pair is equal to the partial derivative by conformal time of the first one. The pair invariant for the fifth one, analogous to the second invariants in the two pairs, was not obtained. In this manuscript, these existing results are expanded with respect to the basics of linear algebra and quantum mechanics. We obtained the corresponding Hermitian time-dependent matrices which transform three of obtained scalar perturbational invariants to their partial derivatives by conformal time in here. These matrices are the corresponding Hamiltonians. Their eigenvalues (energy levels) and eigenstates (energy functions) are determined. After that, the expectation values of Hamiltonians and their squares, together with the corresponding uncertainties of these Hamiltonians in the states of scalar perturbational invariants are obtained.

1. Introduction

The cosmological perturbation theory is important in modeling the Universe [11]. J. Bardeen [3] obtained two scalar perturbational invariants known as Bardeen's potentials. The third scalar perturbational invariant is the Mukhanov-Sasaki variable [11]. It is confirmed that only three of five scalar perturbational invariants obtained in [20] are functionally independent. Motivated with definitions and results from the theory of quantum mechanics [1, 5, 15], we will express the obtained three functionally

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* Corresponding author: Nenad Vesić
Email addresses: n.o.vesic@outlook.com (Nenad Vesić), ivanadjurisic@yahoo.com (Ivana Djurišić), nenad.milojevic@pmf.edu.rs (Nenad Milojević)
ORCID iDs: https://orcid.org/0000-0002-7598-9058 (Nenad Vesić), https://orcid.org/0000-0002-9414-0711 (Ivana Djurišić), https://orcid.org/0000-0002-5639-2192 (Nenad Milojević)

independent scalar perturbational invariants as column matrices and obtain the corresponding transformation matrices to their partial derivatives by conformal time η. These transformation matrices will be the most suitable for determining the eigenvalues and eigenvectors of them. With respect to these matrices, we will obtain the corresponding expectations of Hamiltonians, the expectations of squares of Hamiltonians, and the corresponding uncertainties.

1.1. Necessary details about cosmology and quantum mechanics

In this subsection, we will present the facts about cosmology and quantum mechanics necessary for our research. Based on these facts, we will use the knowledge about vector spaces and isomorphisms between them [19] to present some quantum mechanical properties of some of perturbational invariants obtained in [20].

1.1.1. Perturbations in cosmology

The theory of cosmological perturbations has a deep impact in cosmological inflation. With respect to the inflation, the Universe underwent a period of almost exponential expansion shortly after Big Bang, after which it transits into two next eras (the radiation and matter dominated ones). These eras are described by conventional Big Bang theory [16, 21].

Starting from spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = a^2(\eta)\Big\{-d\eta^2 + dx^{1^2} + dx^{2^2} + dx^{3^2}\Big\} \tag{1}$$

for spatial coordinates x^1, x^2, x^3, the conformal time η, $d\eta = dt/a(t)$, where t is the physical time, and the scale factor $a(\eta)$. The perturbed metric is

$$ds^2 = a^2(\eta)\Big\{-(1+2A)d\eta^2 + 2\big(\partial_i B\big)dx^i d\eta + \Big[\Big(1 - 2\Big(D + \tfrac{1}{3}\big(\partial_k\partial_k E\big)\Big)\Big)\delta_{ij} + 2\big(\partial_i\partial_j E\big)\Big]dx^i dx^j\Big\}, \tag{2}$$

for $x^0 = \eta$, and scalar functions A, B, D, E, the partial derivation by x^i denoted by ∂_i, and the Einstein's summation convention applied to any repeated indices. In other words, the term $\big(\partial_i\partial_i E\big)$ is the shortened form of $\delta^{ij}\big(\partial_i\partial_j E\big)$, where the Einstein's summation convention is repeated to the mute indices i and j which take the values of $\{1, 2, 3\}$.

Expressed in different reference frames $O'x'$ and $O''x''$, the values of functions A, B, D, E, satisfy the equalities

$$\begin{cases} A' = A'' - \frac{\partial\xi^0}{\partial\eta} - \xi^0\mathcal{H}, \\ B' = B'' - \Big(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\Big) + \xi^0, \\ D' = D'' + \frac{1}{3}\big(\partial_i\xi^i\big) + \xi^0\mathcal{H}, \\ E' = E'' - \big(\partial_i^{-1}\xi^i\big), \end{cases} \tag{3}$$

where $\mathcal{H} = \mathcal{H}(\eta) = a'(\eta)/a(\eta)$ is the Hubble scalar and $\xi = \big(\xi^0, \xi^1, \xi^2, \xi^3\big)$ is the coordinate transformation vector, $\xi = x' - x''$, $x' = (x'^0, x'^1, x'^2, x'^3)$, $x'' = (x''^0, x''^1, x''^2, x''^3)$. The mater with respect to FLRW metric is determined by a scalar field $\bar\varphi$ which describes the dominant cosmological fluid and its perturbation $\delta\varphi = \varphi - \bar\varphi$, where $\bar\varphi$ is the unperturbed field. The perturbations $\delta\varphi'$ and $\delta\varphi''$ of this field in different coordinate systems satisfy the equality

$$\delta\varphi' = \delta\varphi'' - \xi^0\frac{d\bar\varphi}{d\eta}. \tag{4}$$

Bardeen [3] pointed that only linear combinations of A, B, D, E, $\delta\varphi$ which do not depend on ξ have an inherent physical meaning. With respect to that, two Bardeen's potentials and the Mukhanov-Sasaki variable are obtained [3, 11]. They are expressed in the coordinate system Ox as

$$\Phi_B = A + \mathcal{H}\Big(B - \frac{\partial E}{\partial\eta}\Big) + \frac{\partial}{\partial\eta}\Big(B - \frac{\partial E}{\partial\eta}\Big), \tag{5}$$

$$\Psi_B = \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) - \mathcal{H}\Big(B - \frac{\partial E}{\partial \eta}\Big), \tag{6}$$

$$v_{MS} = a\mathcal{H}^{-1}\frac{d\bar{\varphi}}{d\eta}\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + a\delta\varphi. \tag{7}$$

New point of view on perturbational invariants is expressed in [20]. In this research, it was obtained how many linearly independent, and functionally independent as well, perturbational invariants may be obtained from the transformation rules (3, 4) and their partial derivatives by η. It is proved that there are three perturbational invariants which are generators for any of other perturbational invariant. In this manuscript, we are interested what are quantum mechanical characteristics of these three perturbational invariants.

The obtained perturbational invariants are

$$\mathcal{J}^0 = A - c_1^0\delta\varphi - c_2^0\frac{\partial\delta\varphi}{\partial\eta}, \tag{8}$$

$$\mathcal{J}^2 = \Big(B - \frac{\partial E}{\partial\eta}\Big) + c_2^0\delta\varphi, \tag{9}$$

$$\mathcal{J}^3 = \frac{\partial}{\partial\eta}\Big(B - \frac{\partial E}{\partial\eta}\Big) - c_1^3\delta\varphi + c_2^0\frac{\partial\delta\varphi}{\partial\eta}, \tag{10}$$

$$\mathcal{J}^4 = \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + c_2^0\mathcal{H}\delta\varphi, \tag{11}$$

$$\mathcal{J}^5 = \frac{\partial}{\partial\eta}\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + c_1^5\delta\varphi + c_2^0\mathcal{H}\frac{\partial\delta\varphi}{\partial\eta}, \tag{12}$$

where $c_2^0 = \big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}$, $c_1^0 = c_2^0\mathcal{H} - \frac{d^2\bar{\varphi}}{d\eta^2}{c_2^0}^2$, $c_1^3 = \frac{d^2\bar{\varphi}}{d\eta^2}{c_2^0}^2$, $c_1^5 = \frac{d\mathcal{H}}{d\eta}c_2^0 - \frac{d^2\bar{\varphi}}{d\eta^2}{c_2^0}^2$. It is proved [20] that the next equalities hold

$$\mathcal{J}^{2k+1} = \frac{\partial\mathcal{J}^{2k}}{\partial\eta}, \tag{13}$$

for $k = 1, 2$.

The Bardeen's potentials Φ_B and Ψ_B, and the Mukhanov-Sasaki variable v_{MS} are expressed as the linear combinations of perturbational invariants $\mathcal{J}^0$, $\mathcal{J}^2$, $\mathcal{J}^3$, $\mathcal{J}^4$, $\mathcal{J}^5$ in [20]. For this reason, and because $\mathcal{J}^0$, $\mathcal{J}^2$, $\mathcal{J}^4$ are functionally independent scalar perturbational invariants, we will consider the perturbational invariants

$$m_0\mathcal{J}^0, \quad m_2\mathcal{J}^2, \quad m_4\mathcal{J}^4, \tag{14}$$

for the scalar and perturbational invariant scalar functions $m_0 = m_0(\eta, x^1, x^2, x^3)$, $m_2 = m_2(\eta, x^1, x^2, x^3)$, $m_4 = m_4(\eta, x^1, x^2, x^3)$, where we will use $m'_{2k} = \frac{\partial m_{2k}}{\partial\eta}$, $k = 0, 1, 2$. We will obtain the aimed quantum mechanical characteristics of these three perturbational invariants in this research.

1.2. Matrix approach to quantum mechanics

Computations in quantum mechanics are based on physical time t. The purpose of this research is to obtain energy levels and wave functions of perturbational invariants in cosmology, the analogues to the corresponding ones which have been obtained in quantum mechanics. For this reason, we will review the corresponding basic definitions from quantum mechanics with respect to the physical time, but the main results of this research will be expressed in terms of conformal time.

By the first postulate of quantum mechanics [1, 5, 15], a wave function $\Psi = \Psi(x, t)$ (for one dimensional problems) may be approached to any quantum mechanical system. This function, defined on the interval $(-\infty, +\infty)$, is complex and integral of its squared module over the full space is $\int_{-\infty}^{+\infty}\big|\Psi(x,t)\big|^2 dx = 1$.

The time evolution of this system [1, 5, 15] is determined with the time dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{15}$$

for a Hermitian operator $H = H(t)$ which is a function of physical time. The time independent Schrödinger equation is

$$H\psi = E\psi, \tag{16}$$

for a Hermitian operator H which is not a function of t, and the scalar E which is the energy. The energy E is an eigenvalue of the operator H.

The time independent Schrödinger equation determined by the Hamiltonian H of the type $n \times n$ may be solved as the eigenvalue problem. Namely, if $E_0, E_1, \dots E_{n-1}$ are n eigenvalues of the matrix H, and if ψ^0, $\psi^1, \dots, \psi^{n-1}$ are the corresponding linearly independent eigenvectors of the matrix H, the general solution of Schrödinger equation $H\psi = E\psi$ is

$$\Psi(x,t) = c_0\psi^0 e^{-\frac{i}{\hbar}E_0 t} + c_1\psi^1 e^{-\frac{i}{\hbar}E_1 t} + \dots + c_{n-1}\psi^{n-1} e^{-\frac{i}{\hbar}E_{n-1} t}, \tag{17}$$

where $c_0, c_1, \dots, c_{n-1}$ are scalars.

The time dependent Schrödinger equation mostly has been solved approximatively. The methodology for solving time dependent Scrhödinger equations is presented in many articles and books [2, 4, 12, 13]. For applying this method, known as the finite differences method, we determine points (x, t^k). At these points, the time-dependent Hamiltonian becomes constant and we solve the corresponding time-independent Schrödinger equations. That means that if we are able to obtain time dependent eigenvalues and eigenvectors of the time dependent Hamiltonian $H = H(t)$ at any time, we directly obtain the corresponding solution of the time dependent Schrödinger equation given by (15). In other words, the analytically expressed time dependent eigenvalues and eigenvectors of the Hamiltonian H are enough for the analyzed time-dependent Schrödinger equation (15) to be solved.

It also should be pointed that if $\Psi(x,t)$ is a finite-dimensional vector, the Hamiltonian H is corresponded to a non-symmetric matrix H. Because H is a Hermitian operator, the doubled symmetric part $H + H^T$ of this matrix is real but the doubled anti-symmetric part $H - H^T$ of the matrix H is imaginary.

1.3. Motivation

Different physical systems are described by corresponding Schrödinger equations [2, 12, 13, 17]. Their solutions are mostly approximated ones. In cosmology and astronomy, Schrödinger equations have been used as well. I. V. Formin, S. V. Chervon, and S. D. Maharaj [7] considered the new representation of Schrödinger-like equation for scalar field Friedmann cosmology. In this model, the scalar field is the argument, but the Hubble parameter is the analogue to the wave function. V. Husain and O. Singh [9] studied the model of semiclassical cosmology. In that research, the semiclassical approximation with back reaction for the coupled evolution of a classical FLRW cosmology and quantum scalar field with respect to the corresponding equations presented in there is one of results. The other result are results of the corresponding numerical computing. V. Husain and S. Singh [9] proposed and studied the cosmological system in which the matter field evolution is determined by the corresponding Schrödinger equation. B. Gumjudpai [8] studied the cosmological model correlated to the non-linear Schrödinger-type formulation. In this case, the Schrödinger wave function is not normalizable. V. Husain and O. Winkler [10] presented the model for semiclassical matter-geometry states for homogeneous and isotropic cosmological models. C. J. Short and P. Coles [18] presented the method for studding large-scale structure formation. In this research, the coupled Schrödinger and Poisson equations were used for studding the dynamics of scalar fields which represent the corresponding self-gravitating cold dark matter. A. Feoli [6] analyzed the model with assumption that Dark Matter is composed of a quantum particle of very low mass. The cosmological Friedmann-Einstein dynamical system which corresponds to this case is reduced to a kind of Schrödinger equation.

The research presented in [14] is the closest to the focus of the research presented in this article. Namely, J. Martin quantized the Mukhanov-Sasaki variable. As we may see in [20], this perturbational invariant is one of three scalar perturbational invariants which generate the all other ones.

We will be focused on quantum mechanical aspects of three perturbational invariants obtained in cosmology [20]. The corresponding time dependent Schrödinger equations for some kinds of generalizations of these invariants will be presented. For the solutions of these Schrödinger equations will be recommended the method presented in [4].

1.4. Purposes

1. At the start of research, we need to express the perturbational invariants $m_{2(k)}\mathcal{J}^{2(k)}$, $k = 0, 1, 2$, given by (14), as elements of a finite-dimensional vector space. The brackets (k) note that the index k does not obey Einstein's summation convention.
2. After that, we will obtain the corresponding Hermitian transformation matrices of the vectors $m_{2(k)}\mathcal{J}^{2(k)}$, $k = 0, 1, 2$, to the partial derivatives of these vectors by conformal time. In this way, the corresponding Schrödinger-type equations will be obtained and they will correspond to the corresponding theoretical particles which describe time, space-time, and space.
3. At the last part of research, we will obtain the corresponding energy levels, wave functions, expected energies of the states $m_0\mathcal{J}^0$, $m_2\mathcal{J}^2$, $m_4\mathcal{J}^4$ and the corresponding uncertainties.

2. Review on scalar perturbational invariants

It is well known [19] that the space of n-dimensional vectors is isomorphic to the space of matrices of the type $n \times 1$. We are interested to express perturbational invariants obtained in [20] as finite dimensional matrices.

In [20], the following scalar perturbational invariants $\mathcal{J}^0$, $\mathcal{J}^2$, $\mathcal{J}^3$, $\mathcal{J}^4$, $\mathcal{J}^5$, reviewed by the Equations (8–12), are obtained. These five scalar perturbational invariants and the scalar perturbational invariant

$$\mathcal{J}^1 \equiv \frac{\partial \mathcal{J}^0}{\partial \eta} = \frac{\partial A}{\partial \eta} - \frac{\partial c_1^0}{\partial \eta}\delta\varphi - \Big(c_1^0 + \frac{\partial c_2^0}{\partial \eta}\Big)\frac{\partial \delta\varphi}{\partial \eta} - c_2^0\frac{\partial^2 \delta\varphi}{\partial \eta^2}, \tag{18}$$

are elements of the following family

$$\begin{aligned} \mathcal{J} &= \alpha_A \cdot A + \alpha_{A_0} \cdot \tfrac{\partial A}{\partial \eta} + \alpha_{BE} \cdot \Big(B - \tfrac{\partial E}{\partial \eta}\Big) + \alpha_{BE_0} \cdot \tfrac{\partial}{\partial \eta}\Big(B - \tfrac{\partial E}{\partial \eta}\Big) \\ &+\alpha_{DE} \cdot \Big(D + \tfrac{1}{3}\big(\partial_i\partial_i E\big)\Big) + \alpha_{DE_0}\tfrac{\partial}{\partial \eta}\Big(D + \tfrac{1}{3}\big(\partial_i\partial_i E\big)\Big) \\ &+\alpha_\varphi \cdot \delta\varphi + \alpha_{\varphi_0} \cdot \tfrac{\partial \delta\varphi}{\partial \eta} + \alpha_{\varphi_{00}} \cdot \tfrac{\partial^2 \delta\varphi}{\partial \eta^2}, \end{aligned} \tag{19}$$

for coefficients $\alpha_A, \alpha_{A_0}, \alpha_{BE}, \alpha_{BE_0}, \alpha_{DE}, \alpha_{DE_0}, \alpha_\varphi, \alpha_{\varphi_0}, \alpha_{\varphi_{00}}$.

This family may be transformed to the family of matrices

$$\mathcal{M}_{\mathcal{J}} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \end{bmatrix}^T, \tag{20}$$

for $\mu_1 = \alpha_A \cdot A$, $\mu_2 = \alpha_{A_0} \cdot \frac{\partial A}{\partial \eta}$, $\mu_3 = \alpha_{BE} \cdot \Big(B - \frac{\partial E}{\partial \eta}\Big)$, $\mu_4 = \alpha_{BE_0} \cdot \frac{\partial}{\partial \eta}\Big(B - \frac{\partial E}{\partial \eta}\Big)$, $\mu_5 = \alpha_{DE} \cdot \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)$, $\mu_6 = \alpha_{DE_0}\frac{\partial}{\partial \eta}\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)$, $\mu_7 = \alpha_\varphi \cdot \delta\varphi$, $\mu_8 = \alpha_{\varphi_0} \cdot \frac{\partial \delta\varphi}{\partial \eta}$, $\mu_9 = \alpha_{\varphi_{00}} \cdot \frac{\partial^2 \delta\varphi}{\partial \eta^2}$.

The transformation $\mathcal{J} \to \mathcal{M}_{\mathcal{J}}$ is injective. If $\mathcal{M}_{\bar{\mathcal{J}}} = \mathcal{M}_{\mathcal{J}}$ are two equal matrix expressions, where is

$$\begin{aligned} \mathcal{M}_{\bar{\mathcal{J}}} &= \begin{bmatrix} \bar\mu_1 & \bar\mu_2 & \bar\mu_3 & \bar\mu_4 & \bar\mu_5 & \bar\mu_6 & \bar\mu_7 & \bar\mu_8 & \bar\mu_9 \end{bmatrix}^T, \\ \mathcal{M}_{\mathcal{J}} &= \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \end{bmatrix}^T, \end{aligned}$$

than it holds $\bar\mu_1 = \mu_1$, $\bar\mu_2 = \mu_2$, $\bar\mu_3 = \mu_3$, $\bar\mu_4 = \mu_4$, $\bar\mu_5 = \mu_5$, $\bar\mu_6 = \mu_6$, $\bar\mu_7 = \mu_7$, $\bar\mu_8 = \mu_8$, $\bar\mu_9 = \mu_9$. Because $\bar{\mathcal{J}} = \bar\mu_1 + \ldots \bar\mu_9$ and $\mathcal{J} = \mu_1 + \ldots + \mu_9$, we get $\bar{\mathcal{J}} = \mathcal{J}$. For any matrix $\mathcal{M}_{\mathcal{J}} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \end{bmatrix}^T$, and the row matrix $\mathbf{1}_{1\times 9} = \left[\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$, the only element of the matrix $\mathbf{1}_{1\times 9} \cdot \mathcal{M}_{\mathcal{J}}$ is the original of this result under transformation, which completes the proof that the transformation $\mathcal{J} \to \mathcal{M}_{\mathcal{J}}$ is bijective. Moreover, if γ_1 and γ_2 are scalars, it evidently holds the equality $\mathcal{M}_{\gamma_1\bar{\mathcal{J}}+\gamma_2\mathcal{J}} = \gamma_1\mathcal{M}_{\bar{\mathcal{J}}} + \gamma_2\mathcal{M}_{\mathcal{J}}$ which proves the linearity of the analyzed transformation. Hence the function $\mathcal{J} \to \mathcal{M}_{\mathcal{J}}$ is an isomorphism.

In this way, the forthcoming theorem is proved.

Theorem 2.1. *Six scalar perturbational invariants $\mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3, \mathcal{J}^4, \mathcal{J}^5$, determine the nine-dimensional vector space. Five scalar perturbational invariants $\mathcal{J}^0$, $\mathcal{J}^2$, $\mathcal{J}^3$, $\mathcal{J}^4$, $\mathcal{J}^5$ determine the seven-dimensional vector space. These vector spaces are respectively isomorphic to the corresponding vector spaces of column-matrices of the types 9×1 and 7×1. The relations*

$$\mathcal{J}^{2k+1} = \frac{\partial \mathcal{J}^{2k}}{\partial \eta} \tag{21}$$

are satisfied for $k = 0, 1, 2$.

Three of scalar perturbational invariants $\mathcal{J}^0, \ldots, \mathcal{J}^5$ are functionally independent. These three independent scalar perturbational invariants are $\mathcal{J}^0$, $\mathcal{J}^2$, $\mathcal{J}^4$, and all other scalar perturbational invariants are obtained as the linear combinations of these three functionally independent scalar perturbational invariants and their partial derivatives by conformal time η or space coordinates x^i. $\square$

The matrix expressions of functionally independent scalar perturbational invariants $m_0\mathcal{J}^0$, $m_2\mathcal{J}^2$, and $m_4\mathcal{J}^4$ are

$$\mathcal{M}_{m_0\mathcal{J}^0} = \left[\begin{array}{ccccccccc} m_0 A & 0 & 0 & 0 & 0 & 0 & -m_0 c_1^0 \delta\varphi & -m_0 c_2^0 \frac{\partial \delta\varphi}{\partial\eta} & 0 \end{array}\right]^T, \tag{22}$$

$$\mathcal{M}_{m_2\mathcal{J}^2} = \left[\begin{array}{ccccccccc} 0 & 0 & m_2\Big(B - \frac{\partial E}{\partial \eta}\Big) & 0 & 0 & 0 & m_2 c_2^0 \delta\varphi & 0 & 0 \end{array}\right]^T, \tag{23}$$

$$\mathcal{M}_{m_4\mathcal{J}^4} = \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & m_4\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) & 0 & m_4 c_2^0 \mathcal{H}\delta\varphi & 0 & 0 \end{array}\right]^T, \tag{24}$$

for the above defined coefficients c_1^0 and c_2^0.

3. Quantum mechanical characteristics of scalar perturbational invariants

In this section, we will present the main results of our research. Because the form of a time dependent Schrödinger equation is given by (15), and because the perturbational invariants $m_0\mathcal{J}^0$, $m_2\mathcal{J}^2$, $m_4\mathcal{J}^4$, are states expressed in the 9-dimensional vector space, we are initially interested to obtain Hermitian matrices $H_{(k)} = H_{(k)}(\eta)$, $k = 0, 1, 2$, of the type 9×9 such that

$$i\hbar \frac{\partial \mathcal{M}_{m_{2(k)}\mathcal{J}^{2(k)}}}{\partial \eta} = H_{(k)} \mathcal{M}_{m_{2(k)}\mathcal{J}^{2(k)}}, \tag{25}$$

where brackets (k) about k mean that the index k does not obey Einstein's summation convention.

By the equations (25), it was emphasized that the matrix expressions $\mathcal{M}_{m_{2(k)}\mathcal{J}^{2(k)}}$ of the analyzed perturbational invariants are solutions of the Schrödinger-type equations

$$i\hbar \frac{\partial \mathcal{M}_\Psi}{\partial \eta} = H_{(k)} \mathcal{M}_\Psi, \tag{26}$$

for the Hermitian matrices $H_{(k)}$, $k = 0, 1, 2$. The operators $H_{(k)}$ are hamiltonians of the corresponding systems. The eigenvalues of the operators $H_{(k)}$ are the energy levels of the analyzed system. The eigenvectors of the operators $H_{(k)}$ are the wave functions of the corresponding systems.

For a time interval $(a, b) \subset \mathbb{R}$, and the known eigenvalues $\mathcal{E}_0, \ldots, \mathcal{E}_8$ and the eigenvectors $\psi^0, \ldots, \psi^8$ the Schrödinger equations (25) may be solved in the following manner:

(i) Divide the interval (a, b) into n subintervals (a_{p-1}, a_p), $a_0 = a < a_1 < \dots, a_n = b$, of equal with $\Delta\eta$. It will be satisfied the following equalities: $a_0 = a$, $a_p = a + p\Delta\eta$, $p = 1, \dots, n$.

(ii) Select the concrete conformal times $\eta^p \in (a_{p-1}, a_p)$, $p = 1, \dots, n$.

(iii) Determine the Hermitian matrices $H_{(k)}$ such that

$$i\hbar \frac{\partial \mathcal{M}_{m_{2(k)}\mathcal{J}^{2(k)}}}{\partial \eta} = H_{(k)}(\eta^{(k)})\mathcal{M}_{m_{2(k)}\mathcal{J}^{(2k)}}. \tag{27}$$

Determine the eigenvalues and the eigenvectors of the matrices $H_{(k)}$.

(iv) The Schrödinger-type equation which corresponds to the transformation of scalar perturbational invariant $m_{2(k)}\mathcal{J}^{2(k)}$ to its partial derivative by conformal time η is $i\hbar\frac{\partial\psi}{\partial\eta} = H_{(k)}\psi$.

Remark 3.1. *As we may see from the above presented way to solve the Scrhödinger-type equations, if eigenvalues and eigenvectors of a hamiltonian $H_{(k)}(\eta)$ are expressed in the corresponding analytical forms, we may treat the corresponding Scrhödinger equations as solved.*

The components of matrices $H_{(k)}$ are $\left[h_{(k).\mu\nu}\right] \equiv \left[h_{\mu\nu}\right]$, for

$$h_{\mu\nu} = \begin{cases} s_{\mu\nu}, & \mu = \nu, \\ s_{\mu\nu} + in_{\mu\nu}, & \mu < \nu, \\ s_{\mu\nu} - in_{\mu\nu}, & \mu > \nu, \end{cases} \tag{28}$$

for the real functions $s_{\mu\nu} = \frac{1}{2}\left(h_{\mu\nu} + h_{\nu\mu}\right) = s_{\nu\mu}$ and $n_{\mu\nu} = \frac{1}{2i}\left(h_{\mu\nu} - h_{\nu\mu}\right) = -n_{\nu\mu}$.

In the case of $k = 0$, the equation (25) by $H_{(k)}$ is reduced to the equality $i\hbar\frac{\partial\mathcal{M}_{m_0\mathcal{J}^0}}{\partial\eta} = H_0\mathcal{M}_{m_0\mathcal{J}^0}$ which is, with respect to the matrix expression $\mathcal{M}_{m_0\mathcal{J}^0}$ given by (22), equivalent to the following system of linear equations

$$S_1^0 : \begin{cases} -m_0(As_{00} - c_1^0\delta\varphi s_{06} - c_2^0\frac{\partial\delta\varphi}{\partial\eta}s_{07}) + i\Big(m_0' A\hbar + (c_1^0\delta\varphi n_{06} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{07})m_0\Big) = 0 \\ -m_0\Big(As_{01} - c_1^0\delta\varphi s_{16} - c_2^0\frac{\partial\delta\varphi}{\partial\eta}s_{17}\Big) + im_0\Big(\frac{\partial A}{\partial\eta}\hbar + An_{01} + c_1^0\delta\varphi n_{16} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{17}\Big) = 0 \\ -m_0\Big(As_{02} - c_1^0\delta\varphi s_{26} - c_2^0\delta\varphi s_{27}\Big) + im_0\Big(An_{02} + c_1^0\delta\varphi n_{26} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{27}\Big) = 0 \\ -m_0\Big(As_{03} - c_1^0\delta\varphi s_{36} - c_2^0\delta\varphi s_{37}\Big) + im_0\Big(An_{03} + c_1^0\delta\varphi n_{36} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{37}\Big) = 0 \\ -m_0\Big(As_{04} - c_1^0\delta\varphi s_{46} - c_2^0\delta\varphi s_{47}\Big) + im_0(An_{04} + c_1^0\delta\varphi n_{46} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{47}\Big) = 0 \\ -m_0\Big(As_{05} - c_1^0\delta\varphi s_{56} - c_2^0\delta\varphi s_{57}\Big) + im_0(An_{05} + c_1^0\delta\varphi n_{56} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{57}\Big) = 0 \\ -m_0\Big(As_{06} - c_1^0\delta\varphi s_{66} - c_2^0\frac{\partial\delta\varphi}{\partial\eta}s_{67}\Big) \\ +i\left(m_0\Big(An_{06} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{67} - \delta\varphi\frac{\partial c_1^0}{\partial\eta}\hbar\Big) - c_1^0 m_0'\delta\varphi\hbar\right) = 0 \\ -m_0\Big(As_{07} - c_1^0\delta\varphi s_{67} - c_2^0\delta\varphi s_{77}) \\ +i\left(m_0\Big(An_{07} - \Big(c_1^0 + \frac{\partial^2 c_2^0}{\partial\eta^2}\Big)\frac{\partial\delta\varphi}{\partial\eta}\hbar - c_1^0\delta\varpi n_{67}\Big) - c_2^0 m_0'\frac{\partial\delta\varphi}{\partial\eta}\hbar\right) = 0 \\ -m_0\Big(As_{08} - c_1^0\delta\varphi s_{68} - c_2^0\frac{\partial\delta\varphi}{\partial\eta}s_{78}\Big) \\ +im_0\Big(An_{08} - c_2^0\frac{\partial^2\delta\varphi}{\partial\eta^2}\hbar - c_1^0\delta\varphi n_{68} - c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{78}\Big) = 0. \end{cases}$$

For a nonconstant function m_0, this system has infinitely many solutions if it is solvable. For this system to be solvable, we need to take special values of values $s_{\mu\nu}$ and special values of some concrete $n_{\mu\nu}$. The system S_1^0, to be solvable, it is necessary to the imaginary parts of equations in this system be vanished. Because variables n_{06}, n_{07}, n_{67} are variables of the first, seventh, and eighth equation of this system, these three imaginary parts should be vanished by solving this subsystem by n_{06}, n_{07}, n_{67}.

One of these infinite many solutions of the system (the most suitable one for this research) is

$$
Sol_1^0 : \begin{cases}
s_{\mu\nu} = 0, \qquad \mu,\nu \in \{0,\dots,8\}, \\
n_{\mu\nu} = \frac{\hbar m_0'}{2m_0}\Big(\frac{c_1^0\delta\varphi}{A} - \frac{A^2 + c_2^{0\,2}\left(\frac{\partial\delta\varphi}{\partial\eta}\right)^2}{A c_1^0 \delta\varphi}\Big) + \frac{\hbar\Big(c_1^0 c_1^{0\prime}\delta\varphi^2 - (c_1^0 + c_2^{0\prime})c_2^0\left(\frac{\partial\delta\varphi}{\partial\eta}\right)^2\Big)}{2Ac_1^0\delta\varphi}, \quad \mu = 0, \nu = 6, \\
n_{\mu\nu} = \frac{\hbar m_0'}{2m_0}\Big(\frac{c_2^0\frac{\partial\delta\varphi}{\partial\eta}}{A} - \frac{m_0'\Big(A^2 + c_1^{0\,2}\delta\varphi^2\Big)}{2Ac_2^0\frac{\partial\delta\varphi}{\partial\eta}}\Big) + \frac{\hbar\Big(-c_1^0 c_1^{0\prime}\delta\varphi + (c_1^0 + c_2^{0\prime})c_2^0\left(\frac{\partial\delta\varphi}{\partial\eta}\right)^2\Big)}{2Ac_2^0\frac{\partial\delta\varphi}{\partial\eta}}, \quad \mu = 0, \nu = 7, \\
n_{\mu\nu} = -\frac{\hbar m_0'}{2m_0}\frac{A^2 + c_1^{0\,2}\delta\varphi^2 + c_2^{0\,2}\left(\frac{\partial\delta\varphi}{\partial\eta}\right)^2}{2c_1^0 c_2^0\delta\varphi\frac{\partial\delta\varphi}{\partial\eta}} - \frac{\hbar\Big(c_1^0 c_1^{0\prime}\delta\varphi^2 + (c_1^0 + c_2^{0\prime})c_2^0\left(\frac{\partial\delta\varphi}{\partial\eta}\right)^2\Big)}{2c_1^0 c_2^0\delta\varphi\frac{\partial\delta\varphi}{\partial\eta}}, \quad \mu = 6, \nu = 7, \\
n_{01} = -\hbar A^{-1}\frac{\partial A}{\partial\eta}, \quad n_{10} = \hbar A^{-1}\frac{\partial A}{\partial\eta}, \quad n_{08} = c_1^0 A^{-1}\frac{\partial^2\delta\varphi}{\partial\eta^2}, \quad n_{80} = -c_1^0 A^{-1}\frac{\partial^2\delta\varphi}{\partial\eta^2}, \\
n_{60} = -n_{06}, \quad n_{70} = -n_{07}, \quad n_{76} = -n_{67}, \\
n_{\mu\nu} = 0, \qquad \text{otherwise.}
\end{cases}
$$

The eigenvalues of matrix H_0 are

$$
\begin{aligned}
&\mathcal{E}_0^0 = \mathcal{E}_1^0 = \mathcal{E}_2^0 = \mathcal{E}_3^0 = \mathcal{E}_4^0 = 0, \\
&\mathcal{E}_5^0 = \sqrt{N_1^0 + \sqrt{N_1^{0\,2} - M_1^0}}, \qquad \mathcal{E}_6^0 = \sqrt{N_1^0 - \sqrt{N_1^{0\,2} - M_1^0}}, \\
&\mathcal{E}_7^0 = -\sqrt{N_1^0 - \sqrt{N_1^{0\,2} - M_1^0}}, \quad \mathcal{E}_8^0 = -\sqrt{N_1^0 + \sqrt{N_1^{0\,2} - M_1^0}},
\end{aligned} \tag{29}
$$

where $N_1^0 = \dfrac{1}{2}\Big((n_{01})^2 + (n_{06})^2 + (n_{07})^2 + (n_{08})^2 + (n_{67})^2\Big)$ and $M_1^0 = \Big((n_{01})^2 + (n_{08})^2\Big)(n_{67})^2$.

The corresponding eigenvectors are

$$
Evec_1^0 : \begin{cases}
\mathcal{V}_0^0 = \Big(0, -\frac{n_{08}}{n_{01}}, 0, 0, 0, 0, 0, 0, 1\Big), \\
\mathcal{V}_1^0 = \Big(0, 0, 0, 0, 0, 1, 0, 0, 0\Big), \\
\mathcal{V}_2^0 = \Big(0, 0, 0, 0, 1, 0, 0, 0, 0\Big), \\
\mathcal{V}_3^0 = \Big(0, 0, 0, 1, 0, 0, 0, 0, 0\Big), \\
\mathcal{V}_4^0 = \Big(0, 0, 1, 0, 0, 0, 0, 0, 0\Big), \\
\mathcal{V}_5^0 = \Big(i\frac{\mathcal{E}_5^0}{n_{08}}, \frac{n_{01}}{n_{08}}, 0, 0, 0, 0, \frac{i\mathcal{E}_5^0\Big(n_{06}\mathcal{E}_5^0 + in_{07}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_5^{0\,2} - (n_{67})^2\Big)}, i\frac{\mathcal{E}_5^0\Big(n_{07}\mathcal{E}_5^0 - n_{06}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_5^{0\,2} - (n_{67})^2\Big)}, 1\Big), \\
\mathcal{V}_6^0 = \Big(i\frac{\mathcal{E}_6^0}{n_{08}}, \frac{n_{01}}{n_{08}}, 0, 0, 0, 0, \frac{i\mathcal{E}_6^0\Big(n_{06}\mathcal{E}_6^0 + in_{07}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_6^{0\,2} - (n_{67})^2\Big)}, -i\frac{\mathcal{E}_6^0\Big(n_{07}\mathcal{E}_6^0 - n_{06}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_6^{0\,2} - (n_{67})^2\Big)}, 1\Big), \\
\mathcal{V}_7^0 = \Big(i\frac{\mathcal{E}_7^0}{n_{08}}, \frac{n_{01}}{n_{08}}, 0, 0, 0, 0, \frac{i\mathcal{E}_7^0\Big(n_{06}\mathcal{E}_7^0 + in_{07}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_7^{0\,2} - (n_{67})^2\Big)}, i\frac{\mathcal{E}_7^0\Big(n_{07}\mathcal{E}_7^0 - n_{06}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_7^{0\,2} - (n_{67})^2\Big)}, 1\Big), \\
\mathcal{V}_8^0 = \Big(i\frac{\mathcal{E}_8^0}{n_{08}}, \frac{n_{01}}{n_{08}}, 0, 0, 0, 0, \frac{i\mathcal{E}_8^0\Big(n_{06}\mathcal{E}_8^0 + in_{07}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_8^{0\,2} - (n_{67})^2\Big)}, -i\frac{\mathcal{E}_8^0\Big(n_{07}\mathcal{E}_8^0 - n_{06}n_{67}\Big)}{n_{08}\Big(\mathcal{E}_8^{0\,2} - (n_{67})^2\Big)}, 1\Big).
\end{cases}
$$

With respect to the matrix H_0, we obtain that the expectation of hamiltonian H_0 in the state $\mathcal{M}_{m_0\mathcal{J}^0}$ is

$$
\langle H_0 \rangle = \mathcal{M}_{m_0\mathcal{J}^0}^T H_0 \mathcal{M}_{m_0\mathcal{J}^0} = 0. \tag{30}
$$

Moreover, the next equality holds

$$
\Big(\Delta H_0\Big)^2 = \Big\langle {H_0}^2 \Big\rangle - \langle H_0 \rangle^2 = \Big\langle H_0 \Big\rangle^2,
$$

i.e. the square of uncertainty of energy $\left(\Delta H_0\right)^2$ is

$$\begin{aligned}\left(\Delta H_0\right)^2 &= (n_{01})^2 + (n_{07})^2 + (n_{08})^2 + \left(c_2^0\frac{\partial\delta\varphi}{\partial\eta}\right)^2(n_{07})^2 + 2c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{06}n_{67}\\ &+\left(n_{06} - c_1^0\delta\varphi n_{67}\right)^2 + \left(c_1^0\delta\varphi n_{06} + c_2^0\frac{\partial\delta\varphi}{\partial\eta}n_{67}\right)^2.\end{aligned} \tag{31}$$

The next theorem holds.

Theorem 3.2. *Let $m_0 = m_0(\eta, x)$ be a scalar function. Five of the energy levels caused by $m_0\mathcal{J}^0$ are*

$$\mathcal{E}_0^0 = \mathcal{E}_1^0 = \mathcal{E}_2^0 = \mathcal{E}_3^0 = \mathcal{E}_4^0 = 0. \tag{32}$$

The two positive of other four energy levels are

$$\mathcal{E}_5^0 = \sqrt{N_1^0 + \sqrt{N_1^{0^2} - M_1^0}}, \tag{33}$$

$$E_6^0 = \sqrt{N_1^0 - \sqrt{N_1^{0^2} - M_1^0}}, \tag{34}$$

for the corresponding values M_1^0 and N_1^0. The last two energy levels, the negative ones, are $\mathcal{E}_7^0 = -\mathcal{E}_6^0$ and $\mathcal{E}_8^0 = -\mathcal{E}_5^0$. These energy levels satisfy the next inequalities $\mathcal{E}_8^0 \leq \mathcal{E}_7^0 \leq \mathcal{E}_6^0 \leq \mathcal{E}_5^0$, where all the equalities hold if and only if $M_1^0 = N_1^{0^2}$.

The corresponding wave functions are $\mathcal{V}_p^0$, $p = 0, \ldots, 8$, given in the list $Evec_1^0$.

The expectation of hamiltonian H_0 in the state $\mathcal{M}_{m_0\mathcal{J}^0}$ is equal 0. The uncertainty of hamiltonian H_0 in the state $\mathcal{M}_{m_0\mathcal{J}^0}$ is given by the Equation (31). □

In the case of $k = 1$, the Schrödinger-type equation (26) is satisfied for $\mathcal{M}_{m_2\mathcal{J}^2}$, which is expressed in the form

$$i\hbar\frac{\partial\mathcal{M}_{m_2\mathcal{J}^2}}{\partial\eta} = H_1\mathcal{M}_{m_2\mathcal{J}^2}, \tag{35}$$

for the corresponding Hermitian matrix H_1 of the type 9×9. This relation is equivalent to the following system of linear equations

$$S_3^2 : \begin{cases} -\left(B - \frac{\partial E}{\partial\eta}\right)s_{02} - m_2c_2^0\delta\varphi s_{06} - i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{02} + m_2c_2^0\delta\varphi n_{06}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{12} - m_2c_2^0\delta\varphi s_{16} - i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{12} + m_2c_2^0\delta\varphi n_{16}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{22} - m_2c_2^0\delta\varphi s_{26} + i\left(\left(B - \frac{\partial E}{\partial\eta}\right)m_2'\hbar + m_2c_2^0\delta\varphi n_{26}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{23} - m_2c_2^0\delta\varphi s_{36} - i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{23} + \frac{\partial}{\partial\eta}\left(B - \frac{\partial E}{\partial\eta}\right) - m_2c_2^0\delta\varphi n_{36}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{24} - m_2c_2^0\delta\varphi s_{46} + i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{24} + m_2c_2^0\delta\varphi n_{46}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{25} - m_2c_2^0\delta\varphi s_{56} + i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{25} + m_2c_2^0\delta\varphi n_{56}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{26} - m_2c_2^0\delta\varphi s_{66} + i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{26} + \left(m_2\frac{\partial c_2^0}{\partial\eta} + m_2'c_2^0\right)\hbar\delta\varphi\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{27} - m_2c_2^0\delta\varphi s_{67} + i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{27} + m_2c_2^0\delta\varphi n_{67} + m_2c_2^0\hbar\frac{\partial\delta\varphi}{\partial\eta}\right) = 0\\ -\left(B - \frac{\partial E}{\partial\eta}\right)s_{28} - m_2c_2^0\delta\varphi s_{46} - i\left(\left(B - \frac{\partial E}{\partial\eta}\right)n_{24} + m_2c_2^0\delta\varphi n_{46}\right) = 0. \end{cases}$$

This system is solvable if and only if the imaginary parts of its equations may be vanished simultaneously. This is possible if and only if

$$m_2c_2^0\delta\varphi \neq 0, \quad n_{26} = \left(B - \frac{\partial E}{\partial\eta}\right)\frac{m_2'\hbar}{m_2c_2^0\delta\varphi}, \quad \left(B - \frac{\partial E}{\partial\eta}\right)^2 m_2' = -m_2c_2^0\frac{\partial m_2c_2^0}{\partial\eta}\delta\varphi^2.$$

Because $\left(B-\frac{\partial E}{\partial\eta}\right)$ and $\delta\varphi$ are real valued functions, the imaginary parts of the third and seventh equation of system S_3^2 are vanished if and only if $m_2m_2'c_2^0\frac{\partial m_2c_2^0}{\partial\eta}<0$. This inequality leads to the relation between $\left(B-\frac{\partial E}{\partial\eta}\right)$ and $\delta\varphi$ which is

$$\left(B-\frac{\partial E}{\partial\eta}\right)^2=-m_2c_2^0\left(c_2^0+\frac{m_2}{m_2'}\frac{\partial c_2^0}{\partial\eta}\right). \tag{36}$$

The most suitable solution of the system S_3^2 for our research is

$$Sol_3^2:\begin{cases} s_{\mu\nu}=0, & \mu,\nu\in\{0,\dots,8\},\\ n_{\mu\nu}=-\dfrac{\frac{\partial}{\partial\eta}\left(B-\frac{\partial E}{\partial\eta}\right)m_2\hbar}{\left(B-\frac{\partial E}{\partial\eta}\right)}, & \mu=2,\nu=3,\\ n_{\mu\nu}=\left(B-\dfrac{\partial E}{\partial\eta}\right)\dfrac{m_2'\hbar}{m_2c_2^0\delta\varphi}, & \mu=2,\nu=6,\\ n_{\mu\nu}=-\dfrac{m_2c_2^0\delta\varphi}{\left(B-\frac{\partial E}{\partial\eta}\right)}n_{67}-\dfrac{m_2c_2^0\hbar\frac{\partial\delta\varphi}{\partial\eta}}{\left(B-\frac{\partial E}{\partial\eta}\right)}, & \mu=2,\nu=7,\\ n_{32}=-n_{23},\quad n_{62}=-n_{26}, & n_{72}=-n_{27},\\ n_{\mu\nu}=0, & \text{otherwise.}\end{cases}$$

The eigenvalues of the hamiltonian whose components are determined by the solution Sol_3^2 are

$$\begin{aligned} &\mathcal{E}_0^2=\mathcal{E}_1^2=\mathcal{E}_2^2=\mathcal{E}_3^2=\mathcal{E}_4^2=0,\\ &\mathcal{E}_5^2=\sqrt{N_3^2+\sqrt{N_3^{2\,2}-M_3^2}},\qquad \mathcal{E}_6^2=\sqrt{N_3^2-\sqrt{N_3^{2\,2}-M_3^2}},\\ &\mathcal{E}_7^2=-\sqrt{N_3^2-\sqrt{N_3^{2\,2}-M_3^2}},\quad \mathcal{E}_8^2=-\sqrt{N_3^2+\sqrt{N_3^{2\,2}-M_3^2}}, \end{aligned} \tag{37}$$

where $N_3^2=\frac{1}{2}\left((n_{23})^2+(n_{26})^2+(n_{27})^2+(n_{67})^2\right)$ and $M_3^2=(n_{23})^2(n_{67})^2$.

Based on the solutions Sol_3^2, we form the hamiltonian H_1 whose matrix is H_1 and which corresponds to the analyzed transformation.

The eigenvectors of Hamiltonian H_1 are

$$Evec_3^2:\begin{cases} \mathcal{V}_0^2=(0,0,0,0,0,0,0,0,1),\\ \mathcal{V}_1^2=(0,0,0,0,0,1,0,0,0),\\ \mathcal{V}_2^2=(0,0,0,0,1,0,0,0,0),\\ \mathcal{V}_3^2=(0,1,0,0,0,0,0,0,0),\\ \mathcal{V}_4^2=(1,0,0,0,0,0,0,0,0),\\ \mathcal{V}_5^2=\left(0,0,-\frac{\mathcal{E}_5^{2\,2}-(n_{67})^2}{n_{26}n_{67}-in_{27}\mathcal{E}_5^2},-i\frac{\mathcal{E}_5^2\left(\mathcal{E}_5^{2\,2}-2(N_3^2-(n_{67})^2)\right)}{n_{23}(n_{26}n_{67}-in_{27}\mathcal{E}_5^2)},0,0,\frac{n_{27}n_{67}+in_{26}\mathcal{E}_5^2}{n_{26}n_{67}-in_{27}\mathcal{E}_5^2},1,0\right),\\ \mathcal{V}_6^2=\left(0,0,-\frac{\mathcal{E}_6^{2\,2}-(n_{67})^2}{n_{26}n_{67}-in_{27}\mathcal{E}_6^2},i\frac{\mathcal{E}_6^2\left(\mathcal{E}_6^{2\,2}-2(N_3^2-(n_{67})^2)\right)}{n_{23}(n_{26}n_{67}-in_{27}\mathcal{E}_6^2)},0,0,\frac{n_{27}n_{67}+in_{26}\mathcal{E}_6^2}{n_{26}n_{67}-in_{27}\mathcal{E}_6^2},1,0\right),\\ \mathcal{V}_7^2=\left(0,0,-\frac{\mathcal{E}_7^{2\,2}-(n_{67})^2}{n_{26}n_{67}-in_{27}\mathcal{E}_7^2},-i\frac{\mathcal{E}_7^2\left(\mathcal{E}_7^{2\,2}-2(N_3^2-(n_{67})^2)\right)}{n_{23}(n_{26}n_{67}-in_{27}\mathcal{E}_7^2)},0,0,\frac{n_{27}n_{67}+in_{26}\mathcal{E}_7^2}{n_{26}n_{67}-in_{27}\mathcal{E}_7^2},1,0\right),\\ \mathcal{V}_8^2=\left(0,0,-\frac{\mathcal{E}_8^{2\,2}-(n_{67})^2}{n_{26}n_{67}-in_{27}\mathcal{E}_8^2},i\frac{\mathcal{E}_8^2\left(\mathcal{E}_8^{2\,2}-2(N_3^2-(n_{67})^2)\right)}{n_{23}(n_{26}n_{67}-in_{27}\mathcal{E}_8^2)},0,0,\frac{n_{27}n_{67}+in_{26}\mathcal{E}_8^2}{n_{26}n_{67}-in_{27}\mathcal{E}_8^2},1,0\right). \end{cases}$$

The next equalities also hold

$$\langle H_1\rangle=0, \tag{38}$$

$$\left(\Delta H_1\right)^2=(n_{23})^2+\left(1+(c_2^0)^2\delta\varphi^2\right)(n_{26})^2+(n_{27}-c_2^0\delta\varphi n_{67})^2. \tag{39}$$

Theorem 3.3. *Let $m_2 = m_2(\eta, x)$ be a scalar function. Five of the energy levels caused by $m_2\mathcal{J}^2$ are*

$$\mathcal{E}_0^2 = \mathcal{E}_1^2 = \mathcal{E}_2^2 = \mathcal{E}_3^2 = \mathcal{E}_4^2 = 0. \tag{40}$$

The two positive of other four energy levels are

$$\mathcal{E}_5^2 = \sqrt{N_3^2 + \sqrt{N_3^{2^2} - M_3^2}}, \tag{41}$$

$$E_6^2 = \sqrt{N_3^2 - \sqrt{N_3^{2^2} - M_3^2}}, \tag{42}$$

for the corresponding values M_3^2 and N_3^2. The last two energy levels, the negative ones, are $\mathcal{E}_7^2 = -\mathcal{E}_6^2$ and $\mathcal{E}_8^2 = -\mathcal{E}_5^2$. These energy levels satisfy the next inequalities $\mathcal{E}_8^2 \leq \mathcal{E}_7^2 \leq \mathcal{E}_6^2 \leq \mathcal{E}_5^2$, where all the equalities hold if and only if $M_3^2 = N_3^{2^2}$.

The corresponding wave functions are $\mathcal{V}_p^2$, $p = 0, \dots, 8$, given in the list $Evec_3^2$.

The expectation of hamiltonian H_1 in the state $\mathcal{M}_{m_2\mathcal{J}^2}$ is equal 0. *The uncertainty of hamiltonian H_1 in the state $\mathcal{M}_{m_2\mathcal{J}^2}$ is given by the Equation* (39). □

In the case of $k = 2$, the Schrödinger-type equation (26) is satisfied for $m_4\mathcal{J}^4$, which is expressed as

$$i\hbar\frac{\partial \mathcal{M}_{m_4\mathcal{J}^4}}{\partial\eta} = H_2\mathcal{M}_{m_4\mathcal{J}^4}. \tag{43}$$

This equality is equivalent to the following system of linear equations

$$S_5^4 : \begin{cases} -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{04} - m_4c_2^0\delta\varphi s_{06} - i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{04} + m_4c_2^0\delta\varphi n_{06}\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{14} - m_4c_2^0\delta\varphi s_{16} - i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{14} + m_4c_2^0\delta\varphi n_{16}\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{24} - m_4c_2^0\delta\varphi s_{26} - i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{24} + m_4c_2^0\delta\varphi n_{26}\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{34} - m_4c_2^0\delta\varphi s_{36} - i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{34} + m_4c_2^0\delta\varphi n_{36}\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{44} - m_4c_2^0\delta\varphi s_{46} - i\Big(m_4c_2^0\mathcal{H}\delta\varphi n_{46} - \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)m_4'\hbar\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{45} - m_4c_2^0\delta\varphi s_{56} \\ +i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{45} + \frac{\partial}{\partial\eta}\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)m_4\hbar - m_4c_2^0\mathcal{H}\delta\varphi n_{56}\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{46} - m_4c_2^0\delta\varphi s_{66} + i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{46} + \frac{\partial m_4c_2^0\mathcal{H}}{\partial\eta}\hbar\delta\varphi\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{47} - m_4c_2^0\delta\varphi s_{67} \\ +i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{47} + m_4c_2^0\mathcal{H}\Big(\delta\varphi n_{67} + \frac{\partial\delta\varphi}{\partial\eta}\hbar\Big)\Big) = 0 \\ -\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)s_{48} - m_4c_2^0\delta\varphi s_{68} + i\Big(\Big(D + \big(\partial_i\partial_i E\big)\Big)n_{48} + m_4c_2^0\mathcal{H}\delta\varphi n_{68}\Big) = 0. \end{cases}$$

The variable n_{46} is a component of the imaginary parts in the fifth and seventh equations of the preceding system. These two imaginary parts may be simultaneously vanished if and only if

$$m_4m_4'c_2^0 \neq 0, \quad n_{46} = \Big(D + \tfrac{1}{3}\big(\partial_i\partial_i E\big)\Big)\frac{m_4'\hbar}{m_4c_2^0\mathcal{H}\delta\varphi}, \quad \Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big)^2 = -\frac{m_4c_2^0\mathcal{H}}{m_4'}\frac{\partial(m_4c_2^0\mathcal{H})}{\partial\eta}\delta\varphi^2.$$

In this case, the system S_5^4 has infinitely many solutions. The most suitable of these solutions for our research is

$$
Sol_5^4 : \begin{cases} s_{\mu\nu} = 0, & \mu,\nu \in \{0,\dots,8\}, \\ n_{\mu\nu} = -\dfrac{\frac{\partial(m_4 c_2^0 \mathcal{H})}{\partial\eta}}{\left(D+\frac{1}{3}\left(\partial_i\partial_i E\right)\right)} m_4\hbar, & \mu = 4, \nu = 5 \\ n_{\mu\nu} = -\dfrac{\frac{\partial(m_4 c_2^0 \mathcal{H})}{\partial\eta}}{\left(D+\frac{1}{3}\left(\partial_i\partial_i E\right)\right)} \hbar, & \mu = 4, \nu = 6 \\ n_{\mu\nu} = -\dfrac{\delta\varphi n_{67} + \frac{\partial\delta\varphi}{\partial\eta}\hbar}{\left(D+\frac{1}{3}\left(\partial_i\partial_i E\right)\right)} \hbar, & \mu = 4, \nu = 7 \\ n_{54} = -n_{54}, \quad n_{64} = -n_{46}, \quad n_{74} = -n_{47}, \quad n_{67}, n_{76} = -n_{67} \in \mathbb{R}[\eta, x] \\ n_{\mu\nu} = 0, & \text{otherwise.} \end{cases}
$$

The eigenvalues of matrix H_2 are

$$
\begin{aligned} &\mathcal{E}_0^4 = \mathcal{E}_1^4 = \mathcal{E}_2^4 = \mathcal{E}_3^4 = \mathcal{E}_4^4 = 0, \\ &\mathcal{E}_5^4 = \sqrt{N_5^4 + \sqrt{{N_5^4}^2 - M_5^4}}, \quad \mathcal{E}_6^4 = \sqrt{N_5^4 - \sqrt{{N_5^4}^2 - M_5^4}}, \\ &\mathcal{E}_7^4 = -\sqrt{N_5^4 - \sqrt{{N_5^4}^2 - M_5^4}}, \quad \mathcal{E}_8^4 = -\sqrt{N_5^4 + \sqrt{{N_5^4}^2 - M_5^4}}, \end{aligned} \tag{44}
$$

where $N_4^5 = \frac{1}{2}\left((n_{45})^2 + (n_{46})^2 + (n_{47})^2 + (n_{67})^2\right)$ and $M_5^4 = (n_{45})^2 (n_{67})^2$.

The eigenvectors of Hamiltonian H_2 are

$$
Evec_5^4 : \begin{cases} \mathcal{V}_0^4 = (0,0,0,0,0,0,0,0,1), \\ \mathcal{V}_1^4 = (0,0,0,0,0,1,0,0,0), \\ \mathcal{V}_2^4 = (0,0,0,0,1,0,0,0,0), \\ \mathcal{V}_3^4 = (0,1,0,0,0,0,0,0,0), \\ \mathcal{V}_4^4 = (1,0,0,0,0,0,0,0,0), \\ \mathcal{V}_5^4 = \left(0,0,0,0, \frac{{\mathcal{E}_5^4}^2 - (n_{67})^2}{n_{46}n_{67} - i n_{47}\mathcal{E}_5^4}, i\frac{\mathcal{E}_5^4\left({\mathcal{E}_5^4}^2 - 2\left(N_5^4 - 2(n_{45})^2\right)\right)}{n_{45}(n_{46}n_{67} + i n_{27}\mathcal{E}_5^4)}, -\frac{n_{47}n_{67} - i n_{46}\mathcal{E}_5^4}{n_{46}n_{67} + i n_{47}\mathcal{E}_5^4}, 1, 0\right), \\ \mathcal{V}_6^4 = \left(0,0,0,0, \frac{{\mathcal{E}_6^4}^2 - (n_{67})^2}{n_{46}n_{67} - i n_{47}\mathcal{E}_6^4}, i\frac{\mathcal{E}_6^4\left({\mathcal{E}_6^4}^2 - 2\left(N_5^4 - 2(n_{45})^2\right)\right)}{n_{45}(n_{46}n_{67} + i n_{27}\mathcal{E}_6^4)}, -\frac{n_{47}n_{67} - i n_{46}\mathcal{E}_6^4}{n_{46}n_{67} + i n_{47}\mathcal{E}_6^4}, 1, 0\right), \\ \mathcal{V}_7^4 = \left(0,0,0,0, \frac{{\mathcal{E}_7^4}^2 - (n_{67})^2}{n_{46}n_{67} - i n_{47}\mathcal{E}_7^4}, i\frac{\mathcal{E}_7^4\left({\mathcal{E}_7^4}^2 - 2\left(N_5^4 - 2(n_{45})^2\right)\right)}{n_{45}(n_{46}n_{67} + i n_{27}\mathcal{E}_7^4)}, -\frac{n_{47}n_{67} - i n_{46}\mathcal{E}_7^4}{n_{46}n_{67} + i n_{47}\mathcal{E}_7^4}, 1, 0\right), \\ \mathcal{V}_8^4 = \left(0,0,0,0, \frac{{\mathcal{E}_8^4}^2 - (n_{67})^2}{n_{46}n_{67} - i n_{47}\mathcal{E}_8^4}, i\frac{\mathcal{E}_8^4\left({\mathcal{E}_8^4}^2 - 2\left(N_5^4 - 2(n_{45})^2\right)\right)}{n_{45}(n_{46}n_{67} + i n_{27}\mathcal{E}_8^4)}, -\frac{n_{47}n_{67} - i n_{46}\mathcal{E}_8^4}{n_{46}n_{67} + i n_{47}\mathcal{E}_8^4}, 1, 0\right). \end{cases}
$$

The next equalities hold

$$
\langle H_2 \rangle = 0, \tag{45}
$$

$$
\left(\triangle H_2\right)^2 = (n_{45})^2 + (1 + H^2 {c_2^0}^2 \delta\varphi^2)(n_{46})^2 + (n_{47} + \mathcal{H} c_2^0 \delta\varphi)^2. \tag{46}
$$

In this way, we proved the next theorem.

Theorem 3.4. *Let $m_4 = m_4(\eta, x)$ be a scalar function. Five of the energy levels caused by $m_4\mathcal{J}^4$ are*

$$
\mathcal{E}_0^4 = \mathcal{E}_1^4 = \mathcal{E}_2^4 = \mathcal{E}_3^4 = \mathcal{E}_4^4 = 0. \tag{47}
$$

The two positive of other four energy levels are

$$
\mathcal{E}_5^4 = \sqrt{N_5^4 + \sqrt{{N_5^4}^2 - M_5^4}}, \tag{48}
$$

$$E_6^4 = \sqrt{N_5^4 - \sqrt{N_5^{4\,2} - M_5^4}}, \tag{49}$$

for the corresponding values M_5^4 and N_5^4. The last two energy levels, the negative ones, are $\mathcal{E}_7^4 = -\mathcal{E}_6^4$ and $\mathcal{E}_8^4 = -\mathcal{E}_5^4$. These energy levels satisfy the next inequalities $\mathcal{E}_8^4 \leq \mathcal{E}_7^4 \leq \mathcal{E}_6^4 \leq \mathcal{E}_5^4$, where all the equalities hold if and only if $M_5^4 = N_5^{4\,2}$.

The corresponding wave functions are $\mathcal{V}_p^4$, $p = 0, \ldots, 8$, given in the list $Evec_5^4$.

The expectation of hamiltonian H_2 in the state $\mathcal{M}_{m_4\mathcal{J}^4}$ is equal 0. The uncertainty of hamiltonian H_2 in the state $\mathcal{M}_{m_4\mathcal{J}^4}$ is given by the Equation (46). $\square$

Thus, for any of the three cases analyzed above, we obtained five eigenvalues equal to 0, two positive eigenvalues, and two negative eigenvalues whose absolute values are equal to those of the corresponding positive ones. For a clearer illustration, the obtained results are presented in Figure 1. The vertical axis represents the energy axis. The black spheres correspond to the five eigenvalues of the Hamiltonian equal to 0. The blue and red spheres above them correspond to the positive eigenvalues, while the spheres below them correspond to the negative eigenvalues.

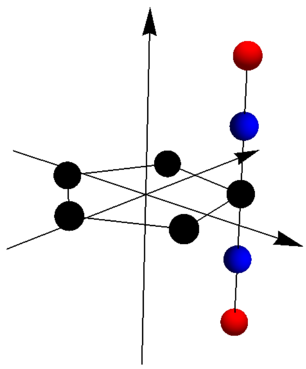

Figure 1: Graphical presentation of energies in studied model: black pentagon is analogy to the Brillouin zone, four cones correspond to the rank four of the hamiltonians, black lines over the pentagon are LUMO and LUMO- energies, but the black lines under the pentagon are HOMO and HOMO+ energies.

4. Discussion and conclusion

In Section 2, we reviewed scalar perturbational invariants obtained in [20]. Following the well-known knowledge from linear algebra, and taken from [19], we expressed these perturbational invariants as matrices. In this section, we proved the Theorem 2.1 which says that three scalar perturbational invariants and their conformal time derivatives are elements of a nine-dimensional vector space. Because in [20] two scalar perturbational invariants are obtained together with their conformal time derivatives but the fifth perturbational invariant is obtained without its conformal time derivative, we proved (Theorem 2.1) that these perturbational invariants form a vector space of dimension seven.

In the Section 3, we multiplied scalar perturbational invariants by scalar functions m_0, m_2, m_4. For analyzing of these three scalar perturbational invariants quantum-mechanically, the Schrödinger-type equations given by the Equation (26) were necessary. With respect to these equations, we obtained the quantum mechanical characteristics (energy levels and wave functions) of scalar perturbational invariants $m_0\mathcal{J}^0$, $m_2\mathcal{J}^2$, $m_4\mathcal{J}^4$.

In the future research, we will try to obtain common hamiltonian for two different scalar perturbational invariants. The second question which is opened does exist a hamiltonian which correspond to three scalar perturbational invariants analyzed here as states.

5. Conclusion

We reviewed three functionally independent scalar perturbational invariants under scalar perturbations in cosmology. These invariants are completed with their partial derivatives by conformal time. It is proved that all of these six scalar perturbational invariants are elements of the nine-dimensional vector space of column matrices. The three functionally independent scalar perturbations multiplied by scalar functions are presented as matrices which was given in the Equations (22, 23, 24).

The three functionally independent scalar perturbational invariants expressed as the column matrices are correlated with their partial derivatives by conformal time η by hermitian matrices of the type 9×9.

The eigenvectors and eigenvalues of the last transformation matrices are obtained. With that knowledge about them, we are able to solve the corresponding Schrödinger equations. For any of scalar perturbational invariants $m_0\mathcal{J}^0$, $m_2\mathcal{J}^2$, $m_4\mathcal{J}^4$ as quantum mechanical states, we obtained the corresponding expectations and uncertainties.

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