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# On $q$-Berezin number inequalities in reproducing kernel Hilbert space

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**Abstract.** This work focuses on the classical Berezin number and the $q$-Berezin number of bounded linear operators on a reproducing kernel Hilbert space. In this study, we also present the $q$-Berezin transform, the $q$-Berezin interval, and the $q$-Berezin number of the reproducing kernel Hilbert space, as well as show various $q$-Berezin number inequalities that generalize prior inequalities provided with the standard Berezin number. Some other connected questions are also addressed.

## 1. Introduction

This work introduces the $\mathfrak{q}$-Berezin number, a more extended variant of the Berezin number in reproducing kernel Hilbert spaces, and proves several new $\mathfrak{q}$-Berezin number inequalities for operators operating on kernel Hilbert spaces. Let $\mathfrak{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators acting on a non-trivial complex Hilbert space $\mathcal{H}$ with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. For $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$, $\mathrm{T}^*$ denotes the adjoint of T and $|\mathrm{T}| = \sqrt{\mathrm{T}^*\mathrm{T}}$. Recall that, the numerical range of $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ is defined by

$$W(\mathrm{T}) = \{\langle \mathrm{T}x, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\},$$

while the numerical radius is defined as

$$w(\mathrm{T}) = \sup\{\langle \mathrm{T}x, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}.$$

It is well-known that the norm $\|.\|$ and the numerical radius $w(.)$ are equivalent, where the following sharp two sided inequality holds:

$$\frac{1}{2}\|\mathrm{T}\| \leq w(\mathrm{T}) \leq \|\mathrm{T}\|,$$

for any $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$.

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For some results about the numerical radius inequalities and application, we refer to see [6, 7, 9, 16, 27, 36]. There exist several generalizations of the classical numerical range in the literature. Our focus will be on the $\mathfrak{q}$-numerical range and its radius of an operator. Let $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ and $\mathfrak{q} \in [0,1]$. The $\mathfrak{q}$-numerical range $W_{\mathfrak{q}}(\mathrm{T})$ and $\mathfrak{q}$-numerical radius $w_{\mathfrak{q}}(\mathrm{T})$ of T are defined respectively as

$$\begin{aligned} W_{\mathfrak{q}}(\mathrm{T}) &= \{\langle \mathrm{T}x, y\rangle : x, y \in H, \|x\| = \left\|y\right\| = 1, \langle x, y\rangle = \mathfrak{q}\}, \\ w_{\mathfrak{q}}(\mathrm{T}) &= \sup_{z \in W_{\mathfrak{q}}(\mathrm{T})} |z|. \end{aligned}$$

It is easy to verify that if $\mathfrak{q} = 1$ then $W_{\mathfrak{q}}(\mathrm{T})$ reduces to the classical numerical range $W(\mathrm{T})$.

Let us introduce the reproducing kernel Hilbert space. Let $\Omega$ be a subset of a topological space $\mathcal{X}$ such that the boundary $\partial\Omega$ is nonempty. We say that an infinite dimensional Hilbert space $\mathcal{H}$ of functions defined on $\Omega$ is a reproducing kernel Hilbert space (RKHS) if the following conditions are satisfied (see Aronszajn [1], and also [8, 23, 24]):

(I) For any $\rho \in \Omega$, the (evaluation) functionals $f \to f(\rho)$ are continuous on $\mathcal{H}$;

(II) For any $\rho \in \Omega$, there exists $f_\rho \in \mathcal{H}$ such that $f_\rho(\rho) \neq 0$.

According to the classical Riesz representation theorem, assumption (I) implies that for any $\rho \in \Omega$, there exists a unique $k_\rho \in \mathcal{H}$ with the reproducing property that

$$f(\rho) = \left\langle f, k_\rho \right\rangle_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

The function $k_\rho$ is called the reproducing kernel of $\mathcal{H}$ at point $\rho$. Note that by (II), we surely have $k_\rho \neq 0$ and denote by $\mathfrak{K}_\rho$ the normalized reproducing kernel, that is $\mathfrak{K}_\rho = k_\rho / \left\|k_\rho\right\|_{\mathcal{H}}$.

Following Nordgren and Rosenthal [30], we say that a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is standard if $\mathfrak{K}_\rho \to 0$ (weakly) as $\rho \to \zeta$ for any point $\zeta \in \partial\Omega$. It is easy to see that all finite dimensional spaces are nonstandard since in such spaces weak and strong convergences coincide. The common RKHSs of analytic functions, including Hardy, Bergman, Dirichlet, and Fock spaces, are standard in this sense. This concept is useful in many questions. For example, Nordgren and Rosenthal [30] established a characterization of compact operators acting on such spaces in terms of the so-called Berezin symbols of their unitary orbits.

Let T be a bounded linear operator on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of T, which firstly have been shown by Berezin [5] is the function of $\widetilde{\mathrm{T}}$ on $\Omega$ defined by

$$\widetilde{\mathrm{T}}(\rho) = \left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle.$$

The Berezin set and Berezin number of the operator T are defined restively by:

$$\mathrm{Ber}(\mathrm{T}) = \left\{\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle : \rho \in \Omega\right\}$$

and

$$\mathrm{ber}(\mathrm{T}) = \sup\left\{\left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle\right| : \rho \in \Omega\right\}.$$

It is clear that Berezin symbol $\widetilde{\mathrm{T}}$ is the bounded function on $\Omega$ whose value lies in the numerical range of T and hence for any $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$,

$$\mathrm{Ber}(\mathrm{T}) \subset W(\mathrm{T}) \text{ and } \mathrm{ber}(\mathrm{T}) \leq w(\mathrm{T}).$$

Furthermore, the Berezin number of an operator T obtain the following properties:

(*i*) $\mathrm{ber}(\mathrm{T}) = \mathrm{ber}(\mathrm{T}^*)$.

(*ii*) $\frac{1}{2}\|\mathrm{T}\| \leq \mathrm{ber}(\mathrm{T}) \leq \|\mathrm{T}\|$.

(*iii*) $\mathrm{ber}(\alpha\mathrm{T}) = |\alpha|\,\mathrm{ber}(\mathrm{T})$ for all $\alpha \in \mathbb{C}$.

(*iv*) $\mathrm{ber}(\mathrm{T} + K) \leq \mathrm{ber}(\mathrm{T}) + \mathrm{ber}(K)$ for all $\mathrm{T}, K \in \mathfrak{B}(\mathcal{H})$.

Notice that, in general, the Berezin number does not define a norm $\mathfrak{B}(\mathcal{H})$ but if the RKHS $\mathcal{H}$ has the "Ber" property (i.e., for any operators $T, K \in \mathfrak{B}(\mathcal{H})$, $\widetilde{T}(\rho) = \widetilde{K}(\rho)$ for all $\rho \in \Omega$ implies $T = K$) then it defines a norm $\mathfrak{B}(\mathcal{H})$ (see [22]). The Berezin symbol of an operator provides important information about the operator and it has wide application in operator theory. It has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. Many researchers have explored the Berezin symbol, Berezin set, and Berezin number throughout the years, including [2–4, 13–15, 17, 18, 21, 25, 28, 31–35].

In this paper, we prove $\mathfrak{q}$-Berezin symbols inequalities involving the $\mathfrak{q}$-Berezin number of operators.

## 2. Preliminary

In this section, we present some useful lemmas that we need for improving and generalizing some inequalities. The first lemma is known in the literature as the generalized mixed Schwarz inequality.

**Lemma 2.1 ([10, 26]).** *Let* $T \in \mathfrak{B}(\mathcal{H})$ *and for any* $x, y \in \mathcal{H}$. *If* $0 \leq \alpha \leq 1$, *then*

$$\left|\langle Tx, y\rangle\right| \leq \left\langle |T|^{2\alpha} x, x\right\rangle^{\frac{1}{2}} \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle^{\frac{1}{2}}. \tag{1}$$

**Lemma 2.2.** *For* $a, b \geq 0$ *and* $0 \leq \alpha \leq 1$ *and* $\frac{1}{p} + \frac{1}{\mathfrak{q}} = 1$

$$\text{(i) } a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{1/r} \text{ for } r \geq 1, \tag{2}$$

$$\text{(ii) } (ab)^{1/2} \leq \frac{a+b}{2}, \tag{3}$$

$$\text{(iii) } ab \leq \frac{a^p}{p} + \frac{b^{\mathfrak{q}}}{\mathfrak{q}} \leq \left(\frac{a^{pr}}{p} + \frac{b^{\mathfrak{q}r}}{\mathfrak{q}}\right)^{1/r} \text{ for } r \geq 1. \tag{4}$$

**Lemma 2.3 ([11, 27]).** *If* $T, S \in \mathfrak{B}(\mathcal{H})$ *are positive operators, then*

$$\left\|T^{1/2} S^{1/2}\right\| \leq \|TS\|^{1/2}. \tag{5}$$

**Lemma 2.4 ([27]).** *If* $T, S \in \mathfrak{B}(\mathcal{H})$ *are positive operators, then*

$$\|T + S\| \leq \frac{1}{2}\left(\|T\| + \|S\| + \sqrt{(\|T\| - \|S\|)^2 + 4\left\|T^{1/2} S^{1/2}\right\|^2}\right). \tag{6}$$

**Lemma 2.5 ([29]).** *Suppose* $0 \leq \mathfrak{q} \leq 1$ *and* $T \in M_2(\mathbb{C})$. *Then* $T$ *is unitary similar to* $e^{it}\begin{pmatrix} \gamma & \alpha \\ \beta & \gamma \end{pmatrix}$ *for some* $0 \leq t \leq 2\pi$ *and* $0 \leq \beta \leq \alpha$. *Also*

$$W_{\mathfrak{q}}(T) = e^{it}\left\{\gamma \mathfrak{q} + r\left((c + pd)\cos(s) + i(d + ps)\sin(s)\right) : 0 \leq r \leq 1, 0 \leq s \leq 2\pi\right\} \tag{7}$$

*with* $c = \frac{a+b}{2}$, $d = \frac{a-b}{2}$ *and* $p = \sqrt{1-\mathfrak{q}^2}$.

The following well-known result follows from the spectral theorem for positive operators and Jensen's inequality (see [26]).

**Lemma 2.6 (Hölder McCarthy inequality).** *Let* $T \in \mathfrak{B}(\mathcal{H})$, $T \geq 0$ *and let* $x \in \mathcal{H}$ *be any unit vector. Then*
(a) $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle$ *for* $r \geq 1$,
(b) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r$ *for* $0 < r \leq 1$.

## 3. $\mathfrak{q}$-Berezin number inequality

**Definition 3.1.** *Let $\mathcal{H}$ be an RKHS and* T *be a bounded linear operator on $\mathcal{H}$.*
($i$) *For $\rho, \tau \in \Omega$, the $\mathfrak{q}$-Berezin transform of* T *(or $\mathfrak{q}$-Berezin symbol of* T*) is defined as:*

$$\widetilde{\mathrm{T}}_{\mathfrak{q}}(\rho, \tau) = \left\{ \left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle : \left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q} \right\}.$$

($ii$) *The $\mathfrak{q}$-Berezin range of* T *(or $\mathfrak{q}$-Berezin set of* T*) is defined as:*

$$\mathrm{Ber}_{\mathfrak{q}}(\mathrm{T}) = \left\{ \left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle : \rho, \tau \in \Omega, \left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q} \right\}.$$

($iii$) *The $\mathfrak{q}$-Berezin number of* T *(or $\mathfrak{q}$-Berezin radius of* T*) is defined as:*

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) = \left\{ \sup_{\rho,\tau\in\Omega} |\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle|, \left( \left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q} \right) \right\}.$$

If $\rho = \tau$ and $\mathfrak{q} = 1$, we get the Berezin number. So, this new concept generalizes the Berezin number of reproducing kernel Hilbert space operators. It is clear that $\mathrm{Ber}_{\mathfrak{q}}(\mathrm{T}) \subset W_{\mathfrak{q}}(\mathrm{T})$ and $\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \omega_{\mathfrak{q}}(\mathrm{T})$. For $\mathrm{T}, \mathrm{S} \in \mathfrak{B}(\mathcal{H})$, as Berezin symbol, it is clear from the definition of $\mathrm{ber}_q(\mathrm{T})$ to obtain the following properties:
($i$) $\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) = \mathrm{ber}_{\mathfrak{q}}(\mathrm{T}^*)$.
($ii$) $\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \mathrm{ber}_{\mathfrak{q}}(U^*\mathrm{T}U)$, where $U$ is unitary operator on $\mathcal{H}$.
($iii$) $\mathrm{ber}_{\mathfrak{q}}(\alpha\mathrm{T}) = |\alpha| \, \mathrm{ber}_{\mathfrak{q}}(\mathrm{T})$ for all $\alpha \in \mathbb{C}$.
($iv$) $\mathrm{ber}_{\mathfrak{q}}(\mathrm{T} + \mathrm{S}) \leq \mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) + \mathrm{ber}_{\mathfrak{q}}(\mathrm{S})$.

**Theorem 3.2.** *Let $H$ be a standard RKHS on a connected domain $\Omega$. Then for any $q \in (0, 1)$, there exist $\rho_1, \rho_2 \in \Omega$ such that $\|K_{\rho_1}\|_H = \|K_{\rho_2}\|_H = 1$ and $\langle K_{\rho_1}, K_{\rho_2} \rangle_H = q$.*

*Proof.* Let $\rho_0 \in \Omega$ be arbitrary but fixed. Define the function

$$F : \Omega \to \mathbb{C}, \quad F(\rho) := \langle K_\rho, K_{\rho_0} \rangle_H.$$

We have

$$F(\rho) = \frac{k_\rho(\rho_0)}{\|k_\rho\|_H \, \|k_{\rho_0}\|_H}.$$

If the scalar kernel function

$$K(\rho, \sigma) := \langle k_\sigma, k_\rho \rangle_H$$

is jointly continuous on $\Omega \times \Omega$ (a standard assumption in RKHS theory when $\Omega$ has a topology), then both the numerator $\rho \mapsto k_\rho(\rho_0)$ and the denominator $\rho \mapsto \|k_\rho\|_H$ are continuous. Moreover, $\|k_\rho\|_H > 0$ for all $\rho$ by non-degeneracy. Therefore $F$ is continuous on $\Omega$. By definition of normalized kernel,

$$F(\rho_0) = \langle K_{\rho_0}, K_{\rho_0} \rangle_H = 1.$$

If $\rho \to \zeta \in \partial\Omega$, the standard RKHS property ensures that $K_\rho \rightharpoonup 0$ weakly in $H$. Since inner products are continuous with respect to weak convergence in the first argument,

$$\lim_{\rho\to\zeta} F(\rho) = \langle 0, K_{\rho_0} \rangle_H = 0.$$

Let $\gamma : [0, 1) \to \Omega$ be a continuous path such that $\gamma(0) = \rho_0$ and $\gamma(t) \to \zeta \in \partial\Omega$ as $t \to 1^-$. The existence of such a path is guaranteed by the connectedness of $\Omega$. The composition $t \mapsto F(\gamma(t))$ is continuous, with

$$F(\gamma(0)) = 1, \quad \lim_{t\to1^-} F(\gamma(t)) = 0.$$

By the real intermediate value theorem, for any $q \in (0,1)$ there exists $t_q \in (0,1)$ such that

$$F(\gamma(t_q)) = q.$$

Setting $\rho_1 := \rho_0$ and $\rho_2 := \gamma(t_q)$ yields

$$\|K_{\rho_1}\|_H = \|K_{\rho_2}\|_H = 1, \quad \langle K_{\rho_1}, K_{\rho_2}\rangle_H = q.$$

This completes the proof. □

**Remark 3.3.** *(i) Joint continuity of $K(\rho,\sigma)$ holds for many classical RKHS examples, including Hardy and Bergman spaces on smoothly bounded domains.*

*(ii) The connectedness of $\Omega$ is essential; without it, the intermediate value theorem may fail to connect the value 1 at $\rho_0$ with 0 near the boundary.*

*(iii) The use of weak convergence to zero at the boundary is sufficient because we only need convergence of the scalar product $\langle K_\rho, K_{\rho_0}\rangle_H$, not convergence in norm.*

**Theorem 3.4.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an RKHS. Let $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ and $\mathfrak{q} \in (0,1)$. Then*

$$\frac{\mathfrak{q}}{4}\|\mathrm{T}\| \le \mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \le \|\mathrm{T}\| \tag{8}$$

*and for any normal operator* T,

$$\frac{\mathfrak{q}}{2}\|\mathrm{T}\| \le \mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \le \|\mathrm{T}\|. \tag{9}$$

*Proof.* Let $\mathfrak{K}_\rho$ and $\mathfrak{K}_\tau$ be normalized reproducing kernels and let $0 \le \mathfrak{q} \le 1$. For proving (8), note first that, if $\mathfrak{K}_\rho$, $\mathfrak{K}_\tau$ in $\mathcal{H}$ are such that $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle = \mathfrak{q}$, then $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \neq -\mathbf{y}$. Put $\mathbf{x} = \frac{\mathfrak{K}_\rho + \mathfrak{K}_\tau}{\left\|\mathfrak{K}_\rho + \mathfrak{K}_\tau\right\|}$, $\mathbf{z} = \frac{\mathfrak{K}_\rho - \mathfrak{K}_\tau}{\left\|\mathfrak{K}_\rho - \mathfrak{K}_\tau\right\|}$ and $\mathbf{y} = \mathfrak{q}\mathbf{x} + \sqrt{1-\mathfrak{q}^2}\mathbf{z}$. $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ is easily obtained. Also $\langle \mathbf{x}, \mathbf{y}\rangle = \mathfrak{q}$ and $\left\|\mathbf{y}\right\| = 1$. Putting $\mathbf{y}' = \mathfrak{q}\mathbf{z} + \sqrt{1-\mathfrak{q}^2}\mathbf{x}$, we have

$$\mathrm{Re}\left(\langle \mathrm{T}\mathbf{x}, \mathbf{y}\rangle + \langle \mathrm{T}\mathbf{z}, \mathbf{y}'\rangle\right) \le \left|\langle \mathrm{T}\mathbf{x}, \mathbf{y}\rangle + \langle \mathrm{T}\mathbf{z}, \mathbf{y}'\rangle\right| \le 2\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}). \tag{10}$$

It is known that $\left\|\mathfrak{K}_\rho + \mathfrak{K}_\tau\right\|^2 = 2 + 2\mathfrak{q}$ and $\left\|\mathfrak{K}_\rho - \mathfrak{K}_\tau\right\|^2 = 2 - 2\mathfrak{q}$. On the other hand,

$$\begin{aligned}
&\mathrm{Re}\left(\langle \mathrm{T}\mathbf{x}, \mathbf{y}\rangle + \langle \mathrm{T}\mathbf{z}, \mathbf{y}'\rangle\right)\\
&= \mathrm{Re}\Bigg[\frac{\mathfrak{q}}{2+2\mathfrak{q}}\left\langle \mathrm{T}\left(\mathfrak{K}_\rho + \mathfrak{K}_\tau\right), \left(\mathfrak{K}_\rho + \mathfrak{K}_\tau\right)\right\rangle + \frac{\sqrt{1-\mathfrak{q}^2}}{\sqrt{2+2\mathfrak{q}}\sqrt{2-2\mathfrak{q}}}\left\langle \mathrm{T}\left(\mathfrak{K}_\rho + \mathfrak{K}_\tau\right), \left(\mathfrak{K}_\rho - \mathfrak{K}_\tau\right)\right\rangle\\
&-\frac{\mathfrak{q}}{2-2\mathfrak{q}}\left\langle \mathrm{T}\left(\mathfrak{K}_\rho - \mathfrak{K}_\tau\right), \left(\mathfrak{K}_\rho - \mathfrak{K}_\tau\right)\right\rangle + \frac{\sqrt{1-\mathfrak{q}^2}}{\sqrt{2+2\mathfrak{q}}\sqrt{2-2\mathfrak{q}}}\left\langle \mathrm{T}\left(\mathfrak{K}_\rho - \mathfrak{K}_\tau\right), \left(\mathfrak{K}_\rho + \mathfrak{K}_\tau\right)\right\rangle\Bigg] \qquad (11)\\
&= \mathrm{Re}\Bigg[\frac{\mathfrak{q}}{2+2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho\right\rangle + \frac{\mathfrak{q}}{2+2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle + \frac{\mathfrak{q}}{2+2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\rho\right\rangle + \frac{\mathfrak{q}}{2+2\mathfrak{q}}\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\tau\rangle\\
&+\frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho\right\rangle - \frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle + \frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\rho\right\rangle - \frac{1}{2}\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\tau\rangle\\
&+\frac{\mathfrak{q}}{2-2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho\right\rangle - \frac{\mathfrak{q}}{2-2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle - \frac{\mathfrak{q}}{2-2\mathfrak{q}}\left\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\rho\right\rangle + \frac{\mathfrak{q}}{2-2\mathfrak{q}}\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\tau\rangle\\
&+\frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho\right\rangle + \frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle - \frac{1}{2}\left\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\rho\right\rangle - \frac{1}{2}\langle \mathrm{T}\mathfrak{K}_\tau, \mathfrak{K}_\tau\rangle\Bigg]. \qquad (12)
\end{aligned}$$

By (10) and (12), we get

$$\begin{aligned}
&\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}+1\right)\operatorname{Re}\left(\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\rho}\right\rangle\right)+\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}-1\right)\operatorname{Re}\left(\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\tau}\rangle\right)\\
&\leq 2\operatorname{ber}_{\mathfrak{q}}(T)-\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}-\frac{\mathfrak{q}}{2-2\mathfrak{q}}\right)\operatorname{Re}\left(\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\tau}\right\rangle+\left\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\rho}\right\rangle\right)\\
&\leq 2\operatorname{ber}_{\mathfrak{q}}(T)+\left(\frac{\mathfrak{q}}{2-2\mathfrak{q}}-\frac{\mathfrak{q}}{2+2\mathfrak{q}}\right)\left(\left|\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\tau}\right\rangle+\left\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\rho}\right\rangle\right|\right)\\
&\leq 2\operatorname{ber}_{\mathfrak{q}}(T)+\frac{4\mathfrak{q}^2}{4-4\mathfrak{q}^2}\left(\left|\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\tau}\right\rangle+\left\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\rho}\right\rangle\right|\right)\\
&\leq 2\operatorname{ber}_{\mathfrak{q}}(T)+\frac{4\mathfrak{q}^2}{4-4\mathfrak{q}^2}\left(\left|\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\tau}\right\rangle\right|+\left|\left\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\rho}\right\rangle\right|\right)\\
&\leq\left(1+\frac{4\mathfrak{q}^2}{4-4\mathfrak{q}^2}\right)2\operatorname{ber}_{\mathfrak{q}}(T)\\
&=\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}+1\right)\operatorname{Re}\left(\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\rho}\right\rangle\right)&\leq\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+\left(1-\frac{\mathfrak{q}}{2+2\mathfrak{q}}-\frac{\mathfrak{q}}{2-2\mathfrak{q}}\right)|\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\tau}\rangle|\\
&\leq\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+|\langle T\mathfrak{K}_{\tau},\mathfrak{K}_{\tau}\rangle|\\
&\leq\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+\operatorname{ber}(T). \qquad (13)
\end{aligned}$$

Replacing T by $e^{i\theta}T$ with $\theta\in\mathbb{R}$ in (13), we have

$$\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}+1\right)\operatorname{Re}\left(\left\langle e^{i\theta}T\mathfrak{K}_{\rho},\mathfrak{K}_{\rho}\right\rangle\right)\leq\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+\operatorname{ber}(T).$$

By taking the supremum over $\theta\in\mathbb{R}$ in the above inequality, we reach

$$\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}+1\right)\left|\left\langle T\mathfrak{K}_{\rho},\mathfrak{K}_{\rho}\right\rangle\right|\leq\frac{2-\mathfrak{q}^2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+\operatorname{ber}(T).$$

Taking the supremum over $\rho\in\Omega$ in the above inequality, we have

$$\left(\frac{\mathfrak{q}}{2+2\mathfrak{q}}+\frac{\mathfrak{q}}{2-2\mathfrak{q}}+1\right)\operatorname{ber}(T)\leq\frac{2-\mathfrak{q}^2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T)+\operatorname{ber}(T).$$

Hence

$$\frac{\mathfrak{q}}{1-\mathfrak{q}^2}\operatorname{ber}(T)\leq\frac{2}{1-\mathfrak{q}^2}\operatorname{ber}_{\mathfrak{q}}(T).$$

It follows that

$$\frac{\mathfrak{q}}{2}\operatorname{ber}(T)\leq\operatorname{ber}_{\mathfrak{q}}(T). \qquad (14)$$

From $\frac{1}{2}\|T\|\leq\operatorname{ber}(T)$, we have

$$\frac{\mathfrak{q}}{4}\|T\|\leq\operatorname{ber}_{\mathfrak{q}}(T). \qquad (15)$$

For proving the other side of the inequality in (10), we proceed with the following inequality

$$\left|\left\langle T\mathfrak{K}_\rho,\mathfrak{K}_\tau\right\rangle\right| \leq \left\|T\mathfrak{K}_\rho\right\| \|\mathfrak{K}_\tau\| \leq \|T\| \left\|\mathfrak{K}_\rho\right\| \|\mathfrak{K}_\tau\| = \|T\|.$$

Taking the supremum over all $\rho, \tau \in \Omega$ in the above inequality with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q}$, we reach

$$\mathrm{ber}_{\mathfrak{q}}(T) \leq \|T\|. \tag{16}$$

Combining the inequality (15) and inequality (16), we have

$$\frac{\mathfrak{q}}{4}\|T\| \leq \mathrm{ber}_{\mathfrak{q}}(T) \leq \|T\|.$$

To prove (9), we note that for a normal operator T, $\mathrm{ber}(T) = \|T\|$ and so by (14)

$$\frac{\mathfrak{q}}{2}\|T\| \leq \mathrm{ber}_{\mathfrak{q}}(T) \leq \|T\|.$$

It is clear that, putting $\mathfrak{q} \to 1$ in (8), we have the well-known inequalities

$$\frac{1}{4}\|T\| \leq \mathrm{ber}(T) \leq \|T\|.$$

This completes the proof. □

**Corollary 3.5.** *Let* $T \in \mathfrak{B}(\mathcal{H})$ *and* $\mathfrak{q} \in (0,1)$. *Then we deduce that*

(*i*) $\frac{\mathfrak{q}}{2}\mathrm{ber}(T) \leq \mathrm{ber}_{\mathfrak{q}}(T)$,

(*ii*) $\frac{\mathfrak{q}}{4}\|T\| \leq \mathrm{ber}_{\mathfrak{q}}(T) \leq \|T\|$,

(*iii*) *If* $T \in \mathfrak{B}(\mathcal{H})$ *is normal operator, then* $\frac{\mathfrak{q}}{2}\|T\| \leq \mathrm{ber}_{\mathfrak{q}}(T) \leq \|T\|$.

**Lemma 3.6.** *Suppose* $0 \leq \mathfrak{q} \leq 1$ *and* $T \in M_2(\mathbb{C})$. *Then* T *is unitary similar to* $e^{i\theta}\begin{pmatrix}\gamma & \alpha\\ \beta & \gamma\end{pmatrix}$ *for some* $0 \leq t \leq 2\pi$ *and* $0 \leq \beta \leq \alpha$. *Also*

$$\mathrm{Ber}_{\mathfrak{q}}(T) = \{\gamma\mathfrak{q} + r\left((c+pd)\cos(s) + i(d+pc)\sin(s)\right) : 0 \leq r \leq 1, 0 \leq s \leq 2\pi\},$$

*with* $c = \frac{\alpha+\beta}{2}$, $d = \frac{\alpha-\beta}{2}$ *and* $p = \sqrt{1-\mathfrak{q}^2}$.

*Proof.* From $\mathrm{Ber}_{\mathfrak{q}}(T) \subseteq W_{\mathfrak{q}}(T)$, proof is the similar to the one in the case of $W_{\mathfrak{q}}(T)$. □

**Theorem 3.7.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be an RKHS. Let* T *be a bounded linear operator on* $\mathcal{H}$ *with* $T^2 = 0$. *Then for any* $\mathfrak{q} \in [0,1)$,

$$\mathrm{ber}_{\mathfrak{q}}(T) \leq \left(1 - \frac{3\mathfrak{q}^2}{4} + \mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)^{1/2}\|T\|. \tag{17}$$

*Proof.* For any $\mathfrak{K}_\rho$ in $\mathcal{H}$, by the equality $\mathcal{H} = ran(T) \oplus \ker(T^*)$, one can uniquely write $\mathfrak{K}_\rho = u + v$, for some $u$ in $\ker(T^*)$ and $v$ in $ran(T)$. The inequality $T^2 = 0$ stands for $ran(T) \perp ran(T^*)$, since $\left\langle T^*\mathfrak{K}_\rho, T\mathfrak{K}_\rho\right\rangle = 0$, for any

$\rho \in \Omega$. Thus, $\left\langle T\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle = \langle Tu, v\rangle$. Now for each $\rho, \tau \in \Omega$ with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q}$, we reach

$$\begin{aligned}
\left|\left\langle T\mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle\right|^2 &= \left|\left\langle T\mathfrak{K}_\rho, \mathfrak{q}\mathfrak{K}_\rho + \sqrt{1-\mathfrak{q}^2}\widehat{k_\xi} \right\rangle\right|^2 \\
&\le \left(\left|\mathfrak{q}\left\langle T\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle\right| + \left|\sqrt{1-\mathfrak{q}^2}\left\langle T\mathfrak{K}_\rho, \widehat{k_\xi}\right\rangle\right|\right)^2 \\
&\le \mathfrak{q}^2\left|\left\langle T\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle\right|^2 + \left(1-\mathfrak{q}^2\right)\left|\left\langle T\mathfrak{K}_\rho, \widehat{k_\xi}\right\rangle\right|^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left|\left\langle T\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle\right|\left|\left\langle T\mathfrak{K}_\rho, \widehat{k_\xi}\right\rangle\right| \\
&\le \mathfrak{q}^2\left|\langle Tu, v\rangle\right|^2 + \left(1-\mathfrak{q}^2\right)\left|\left\langle T\mathfrak{K}_\rho, \widehat{k_\xi}\right\rangle\right|^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left|\langle Tu, v\rangle\right|\left|\left\langle T\mathfrak{K}_\rho, \widehat{k_\xi}\right\rangle\right| \\
&\le \mathfrak{q}^2\|T\|^2\|u\|^2\|v\|^2 + \left(1-\mathfrak{q}^2\right)\|T\|^2\left\|\mathfrak{K}_\rho\right\|^2\left\|\widehat{k_\xi}\right\|^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\|u\|\|v\|\left\|\mathfrak{K}_\rho\right\|\left\|\widehat{k_\xi}\right\|\|T\|^2 \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
&\le \mathfrak{q}^2\|T\|^2\left(\frac{\|u\|^2+\|v\|^2}{2}\right)^2 + \left(1-\mathfrak{q}^2+2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left(\frac{\|u\|+\|v\|}{2}\right)^2\right)\|T\|^2 \\
&\quad \text{(by the inequality (3))} \\
&\le \mathfrak{q}^2\|T\|^2\left(\frac{\langle u+v, u+v\rangle}{2}\right)^2 + \left(1-\mathfrak{q}^2+2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left(\frac{\|u\|^2+\|v\|^2}{2}\right)\right)\|T\|^2 \\
&\quad \left(\text{by } f\left(\frac{u+v}{2}\right) \le \frac{f(u)+f(v)}{2}\right) \\
&= \frac{\mathfrak{q}^2}{4}\|T\|^2\left\|\mathfrak{K}_\rho\right\|^2 + \left(1-\mathfrak{q}^2+\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left\|\mathfrak{K}_\rho\right\|^2\right)\|T\|^2 \\
&= \frac{\mathfrak{q}^2}{4}\|T\|^2 + \left(1-\mathfrak{q}^2+\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)\|T\|^2 \\
&= \left(1-\frac{3\mathfrak{q}^2}{4}+\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)\|T\|^2.
\end{aligned}$$

Taking the supremum over $\rho, \tau \in \Omega$ in the above inequality with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q}$, we have

$$\mathrm{ber}_{\mathfrak{q}}(T) \le \left(1-\frac{3\mathfrak{q}^2}{4}+\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)^{1/2}\|T\|.$$

This completes the proof. □

**Corollary 3.8.** *When* $\mathfrak{q}$ *tends to* 1 *in (17), we have*

$$\mathrm{ber}(T) = \frac{1}{2}\|T\|,$$

*for any* $T \in \mathfrak{B}(\mathcal{H})$ *with* $T^2 = 0$.

**Theorem 3.9.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be an RKHS. If* $T \in \mathfrak{B}(\mathcal{H})$ *and* $\mathfrak{q} \in (0,1)$, *then*

$$\mathrm{ber}_{\mathfrak{q}}(T) \le \left(\frac{\mathfrak{q}^2}{4}\left(\|T\| + \left\|T^2\right\|^{1/2}\right)^2 + \left(1-\mathfrak{q}^2+2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)\|T\|^2\right)^{1/2}. \tag{18}$$

*Proof.* Let $\Re_\rho$, $\Re_\tau$ and $\widehat{k_\xi}$ be normalized reproducing kernels and let $0 \le \mathfrak{q} \le 1$. Then

$$\begin{aligned}
\left|\left\langle T\Re_\rho,\Re_\tau\right\rangle\right|^2 &\le \left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \langle |T^*|\,\Re_\tau,\Re_\tau\rangle \\
&\quad \text{(by the inequality (1)}\\
&= \left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \left\langle |T^*| \left(\mathfrak{q}\Re_\rho + \sqrt{1-\mathfrak{q}^2}\widehat{k_\xi}\right), \left(\mathfrak{q}\Re_\rho + \sqrt{1-\mathfrak{q}^2}\widehat{k_\xi}\right)\right\rangle \\
&\le \mathfrak{q}^2 \left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \left\langle |T^*|\,\Re_\rho,\Re_\rho\right\rangle + \left(1-\mathfrak{q}^2\right)\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \left\langle |T^*|\,\widehat{k_\xi},\widehat{k_\xi}\right\rangle \\
&\quad + 2\,\mathrm{Re}\left(\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \left\langle |T^*|\,\Re_\rho,\widehat{k_\xi}\right\rangle\right) \\
&\le \mathfrak{q}^2\left(\frac{\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle + \left\langle |T^*|\,\Re_\rho,\Re_\rho\right\rangle}{2}\right)^2 + \left(1-\mathfrak{q}^2\right)\frac{\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle^2 + \left\langle |T^*|\,\widehat{k_\xi},\widehat{k_\xi}\right\rangle^2}{2} \\
&\quad + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle \left|\left\langle |T^*|\,\Re_\rho,\widehat{k_\xi}\right\rangle\right| \\
&\quad \text{(by inequality (3) and } f\left(\frac{u+v}{2}\right) \le \frac{f(u)+f(v)}{2}) \\
&\le \frac{\mathfrak{q}^2}{4}\left(\left\langle \left(|T|+|T^*|\right)\Re_\rho,\Re_\rho\right\rangle\right)^2 + \frac{1-\mathfrak{q}^2}{2}\left(\left\langle |T|\,\Re_\rho,\Re_\rho\right\rangle^2 + \left\langle |T^*|\,\widehat{k_\xi},\widehat{k_\xi}\right\rangle^2\right) \\
&\quad + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\,|||T|||\,|||T|^*||\,\left\|\Re_\rho\right\|^3\left\|\widehat{k_\xi}\right\| \\
&\quad \text{(By Cauchy-Schwarz inequality)} \\
&\le \frac{\mathfrak{q}^2}{4}\mathrm{ber}^2\left(|T|+|T^*|\right) + \frac{1-\mathfrak{q}^2}{2}\left(\mathrm{ber}^2(|T|) + \mathrm{ber}^2(|T^*|)\right) + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\,|||T|||\,|||T|^*|| \\
&\le \frac{\mathfrak{q}^2}{4}\left(|||T|+|T^*|||^2\right) + \frac{1-\mathfrak{q}^2}{2}\left(|||T|||^2 + |||T|^*||^2\right) + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\,|||T|||\,|||T|^*||\,.
\end{aligned}$$

Here, $|||T||| = |||T^*||| = \|T\|$, $|||T|\,|T^*||| = \left\|T^2\right\|$ and applying Lemma 2.4 and Lemma 2.3 for the pair $|T|$ , $|T^*|$, we reach

$$\begin{aligned}
|||T|+|T^*||| &\le \frac{1}{2}\left(|||T|||+|||T^*||| + \sqrt{(|||T|||-|||T^*|||)^2 + 4\left\||T|^{1/2}\,|T^*|^{1/2}\right\|^2}\right) \\
&\le \frac{1}{2}\left(2\,\|T\| + \sqrt{4\,|||T|\,|T^*|||}\right) \\
&= \|T\| + \left\|T^2\right\|^{1/2}.
\end{aligned}$$

Thus

$$\left|\left\langle T\Re_\rho,\Re_\tau\right\rangle\right|^2 \le \frac{\mathfrak{q}^2}{4}\left(\|T\| + \left\|T^2\right\|^{1/2}\right)^2 + \left(1-\mathfrak{q}^2\right)|||T|||^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\,|||T|||^2\,.$$

In the above inequality, by taking the supremum over all $\rho, \tau \in \Omega$ with $\left\langle \Re_\rho, \Re_\tau\right\rangle = \mathfrak{q}$, we get

$$\mathrm{ber}_{\mathfrak{q}}^2(T) \le \frac{\mathfrak{q}^2}{4}\left(\|T\| + \left\|T^2\right\|^{1/2}\right)^2 + \left(1-\mathfrak{q}^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)|||T|||^2\,.$$

Here, we reach

$$\mathrm{ber}_{\mathfrak{q}}(T) \le \left(\frac{\mathfrak{q}^2}{4}\left(\|T\| + \left\|T^2\right\|^{1/2}\right)^2 + \left(1-\mathfrak{q}^2 + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\right)\|T\|^2\right)^{1/2}.$$

Which completes the proof. □

**Corollary 3.10.** *If* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *and if we let q tend to 1, then we have*

$$\mathrm{ber}(\mathrm{T}) \leq \frac{1}{2}\left(\|\mathrm{T}\| + \left\|\mathrm{T}^2\right\|^{1/2}\right)$$

*(see, [19, 20]).*

**Theorem 3.11.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be an RKHS. For any* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *and* $\mathfrak{q} \in (0,1)$*, the following inequalities hold:*

$$\frac{\mathfrak{q}^2}{16}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\| \leq \mathrm{ber}_{\mathfrak{q}}^2(\mathrm{T}) \leq \frac{\mathfrak{q}^2}{2\left(1-\sqrt{1-\mathfrak{q}^2}\right)^2}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\|. \tag{19}$$

*Proof.* Let $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$. For each $\rho, \tau, \xi \in \Omega$ with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q}$, by using the $\mathfrak{K}_\tau = \mathfrak{q}\mathfrak{K}_\rho + \sqrt{1-\mathfrak{q}^2}\widehat{k_\xi}$ with $\left\langle \mathfrak{K}_\rho, \widehat{k_\xi} \right\rangle = 0$, we get

$$\begin{aligned}
\left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle\right|^2 &\leq \left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{q}\mathfrak{K}_\rho + \sqrt{1-\mathfrak{q}^2}\widehat{k_\xi} \right\rangle\right|^2 \\
&\leq \left(\mathfrak{q}\left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\rho \right\rangle\right| + \sqrt{1-\mathfrak{q}^2}\left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \widehat{k_\xi} \right\rangle\right|\right)^2 \\
&\leq \left(\mathfrak{q}\,\mathrm{ber}(\mathrm{T}) + \sqrt{1-\mathfrak{q}^2}\,\mathrm{ber}_{\mathfrak{q}}(\mathrm{T})\right)^2.
\end{aligned}$$

By taking the supremum over all $\rho, \tau \in \Omega$ with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau \right\rangle = \mathfrak{q}$, we have

$$\mathrm{ber}_{\mathfrak{q}}^2(\mathrm{T}) \leq \left(\mathfrak{q}\,\mathrm{ber}(\mathrm{T}) + \sqrt{1-\mathfrak{q}^2}\,\mathrm{ber}_{\mathfrak{q}}(\mathrm{T})\right)^2.$$

Therefore

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \mathfrak{q}\,\mathrm{ber}(\mathrm{T}) + \sqrt{1-\mathfrak{q}^2}\,\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}).$$

Thus

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \frac{\mathfrak{q}}{1-\sqrt{1-\mathfrak{q}^2}}\mathrm{ber}(\mathrm{T}). \tag{20}$$

From the inequality $\mathrm{ber}^2(\mathrm{T}) \leq \frac{1}{2}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\|$ and (20), we reach

$$\mathrm{ber}_{\mathfrak{q}}^2(\mathrm{T}) \leq \frac{\mathfrak{q}^2}{\left(1-\sqrt{1-\mathfrak{q}^2}\right)^2}\mathrm{ber}^2(\mathrm{T}) \leq \frac{\mathfrak{q}^2}{2\left(1-\sqrt{1-\mathfrak{q}^2}\right)^2}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\|.$$

This proves the right-hand side of (19). For the other side of the inequality , we will need to use the inequalities $\frac{1}{4}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\| \leq \mathrm{ber}^2(\mathrm{T})$ and (14). Then we reach

$$\frac{\mathfrak{q}^2}{16}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\| \leq \mathrm{ber}_{\mathfrak{q}}^2(\mathrm{T}).$$

This completes the proof. □

**Corollary 3.12.** *If we tend* $\mathfrak{q}$ *to* 1 *in Theorem 3.11, then we have*

$$\frac{1}{16}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\|_{\mathrm{ber}} \leq \mathrm{ber}^2(\mathrm{T}) \leq \frac{1}{2}\|\mathrm{T}^*\mathrm{T} + \mathrm{T}\mathrm{T}^*\|_{\mathrm{ber}}.$$

**Corollary 3.13.** *Applying inequalities (14) and (20), we obtain the following inequality*

$$\frac{\mathfrak{q}}{2}\mathrm{ber}(\mathrm{T}) \leq \mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \frac{\mathfrak{q}}{1-\sqrt{1-\mathfrak{q}^2}}\mathrm{ber}(\mathrm{T}),$$

*for all* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *and* $\mathfrak{q} \in (0,1)$.

**Theorem 3.14.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be an RKHS. If* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *is a positive operator and* $\mathfrak{q} \in (0,1)$, *then*

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}^m) \leq (\mathrm{ber}(\mathrm{T}))^m \text{ for any } m \in [0,1], \tag{21}$$

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}^m) \geq (\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}))^m \text{ for any } m > 1. \tag{22}$$

*Proof.* Assume that the inequality (21) holds for some $\alpha, \beta \in [0,1]$. Then we only have to prove (21) holds for $\frac{\alpha+\beta}{2} \in [0,1]$ by continuity of an operator. In fact, we have for any $\rho, \tau \in \Omega$ that

$$\begin{aligned}
\left|\left\langle \mathrm{T}^{\frac{\alpha+\beta}{2}}\mathfrak{K}_\rho,\mathfrak{K}_\tau\right\rangle\right|^2 &\leq \left|\left\langle \mathrm{T}^{\frac{\alpha}{2}}\mathfrak{K}_\rho,\mathrm{T}^{\frac{\beta}{2}}\mathfrak{K}_\tau\right\rangle\right|^2 \\
&\leq \left\langle \mathrm{T}^{\alpha}\mathfrak{K}_\rho,\mathfrak{K}_\rho\right\rangle \left\langle \mathrm{T}^{\beta}\mathfrak{K}_\tau,\mathrm{T}\mathfrak{K}_\tau\right\rangle \\
&\quad \text{(by Cauchy-Schwarz inequality)} \\
&\leq \left\langle \mathrm{T}\mathfrak{K}_\rho,\mathfrak{K}_\rho\right\rangle^{\alpha} \langle \mathrm{T}\mathfrak{K}_\tau,\mathfrak{K}_\tau\rangle^{\beta} \\
&\quad \text{(by Lemma 2.6 (a))} \\
&\leq (\mathrm{ber}(\mathrm{T}))^{\alpha}(\mathrm{ber}(\mathrm{T}))^{\beta} \\
&= (\mathrm{ber}(\mathrm{T}))^{\alpha+\beta}.
\end{aligned}$$

By taking the supremum over $\rho, \tau \in \Omega$ in the above inequality with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle = \mathfrak{q}$, we have

$$\left(\mathrm{ber}_{\mathfrak{q}}\left(\mathrm{T}^{\frac{\alpha+\beta}{2}}\right)\right) \leq (\mathrm{ber}(\mathrm{T}))^{\frac{\alpha+\beta}{2}}.$$

This implies the desired inequality $\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}^m) \leq (\mathrm{ber}(\mathrm{T}))^m$ for any $m \in [0,1]$.

Let $m > 1$. Then $\frac{1}{m} \in (0,1)$. For any $\rho, \tau \in \Omega$,

$$\begin{aligned}
\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle &= \left\langle \mathrm{T}^{m\frac{1}{m}}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle \\
&\leq \left\langle \mathrm{T}^{m}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle^{\frac{1}{m}} \text{ (by the inequality (21))}
\end{aligned}$$

for any $m > 1$. If we take the power of $m$ and then take absolute value from both sides of the inequality, then we obtain the following inequality:

$$\left|\left\langle \mathrm{T}^m\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle\right| \geq \left|\left\langle \mathrm{T}\mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle\right|^m.$$

By taking the supremum over $\rho, \tau \in \Omega$ in the above inequality with $\left\langle \mathfrak{K}_\rho, \mathfrak{K}_\tau\right\rangle = \mathfrak{q}$, we get

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}^m) \geq (\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}))^m \text{ for any } m > 1.$$

This completes the proof. □

**Corollary 3.15.** *Let* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *be a positive operator. Then we get*

$$\mathrm{ber}_{\mathfrak{q}}(\mathrm{T}) \leq \mathrm{ber}(\mathrm{T}).$$

*Proof.* If we set $m = 1$ in the inequality (21), then we have the desired inequality. □

**Corollary 3.16.** *Let* $\mathrm{T} \in \mathfrak{B}(\mathcal{H})$ *be a positive operator and let* $m > 1$. *Then we get*

$$\mathrm{ber}(\mathrm{T}^m) \geq (\mathrm{ber}(\mathrm{T}))^m,$$

*(see [12, 21]).*

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