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Some extensions of the Pinelis-Stolarsky's inequality for $q-$integrals

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Abstract. The main results presented are inequalities similar to the Pinelis extension of Stolarsky's type inequality for $q-$integrals. Also, the applications for $q-$gamma and $q-$beta functions are given.

1. Introduction

Let us recall Stolarsky's inequality given in [6] in 1991.
If a function $f : [0,1] \to [0,1]$ is decreasing, then for any $a, b > 0$ it holds

$$\int_0^1 f\left(x^{\frac{1}{a+b}}\right) dx \geq \int_0^1 f\left(x^{\frac{1}{a}}\right) dx \cdot \int_0^1 f\left(x^{\frac{1}{b}}\right) dx. \tag{1}$$

Putting substitution $t = \frac{1}{x^a}$ in (1), we obtain

$$\int_0^1 f\left(x^{\frac{1}{a}}\right) dx = a \int_0^1 t^{a-1} f(t)\, dt.$$

Then, putting $Q(f,a) = \int_0^1 \frac{d}{dt}(t^a) f(t)\, dt$, the Stolarsky inequality (1) can be written in the form

$$Q(f, a+b) \geq Q(f,a) \cdot Q(f,b). \tag{2}$$

Moreover (see [3] and [4]), by defining $Q(f,0) = \lim_{a\to 0} Q(f,a)$, the following more precisely formulation of (2) is obtained: if $f : [0,1] \to [0,1]$ is decreasing, then

$$Q(f,0) \cdot Q(f, a+b) \geq Q(f,a) \cdot Q(f,b). \tag{3}$$

Also, Pečarić ([3]) proved reversed Stolarsky's inequality: if $f : [0,1] \to [0,1]$ is increasing, then

$$Q(f,0) \cdot Q(f, a+b) \leq Q(f,a) \cdot Q(f,b). \tag{4}$$

In [4], the following generalizations of the Stolarsky inequality are given by I. Pinelis:

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Theorem A. *Let α, β and γ be non-negative numbers.*

(i) If f and g are non-negative, decreasing functions on $[0,1]$ such that they are left-continuous on $\langle 0,1]$ and right-continuous at 0, then

$$\begin{aligned}&Q(f,\gamma)\cdot Q(g,\alpha+\beta+\gamma)+Q(g,\gamma)\cdot Q(f,\alpha+\beta+\gamma)\\ \geq\ &Q(f,\alpha+\gamma)\cdot Q(g,\beta+\gamma)+Q(g,\alpha+\gamma)\cdot Q(f,\beta+\gamma),\end{aligned}\tag{5}$$

where

$$Q(f,a)=\int_0^1 f\left(x^{\frac{1}{a}}\right)dx=a\int_0^1 x^{a-1}f(x)\,dx.$$

(ii) If f and g are non-negative, increasing function on $[0,1]$ such that they are right-continuous on $[0,1\rangle$ and left-continuous at 1, then the reverse sign in (5) holds.

Results from Theorem A are called the Pinelis-Stolarsky inequalities. Further investigation of the Pinelis results is done in papers [8] and [9].

In this paper we will establish some results related to this inequality, similar to the Pinelis extension of the Stolarsky inequality, given in [4] (see also [8]), but for $q-$integrals. Moreover, some applications for $q-$gamma and $q-$beta functions will be given.

First, let us introduce the definitions of $q-$derivative and the definite $q-$integral as well as some basic properties (see [2]).

The $q-$derivative of a function at point x ($q\in\langle 0,1\rangle$) is defined as a quotient

$$D_q f(x)=\frac{f(x)-f(qx)}{(1-q)x}.$$

Then, for each $i=0,1,2,...,n$, it follows that

$$f(q^i x)-f(q^{i+1}x)=(1-q)q^i x D_q f(q^i x).$$

By adding the above $n+1$ identities and then letting $n\to\infty$, we obtain

$$f(x)-f(0)=(1-q)x\sum_{i=0}^{\infty} q^i D_q f(q^i x).$$

The definite $q-$integral of a function $f:[0,c\rangle\to\mathbb{R}$ is defined as a

$$\int_0^b f(x)d_qx=(1-q)b\sum_{k=0}^{\infty} q^k f(bq^k).$$

for $b\in\langle 0,c\rangle$, provided that the right-hand side converges.

As shown above, we have an analogue of the Newton-Leibniz formula

$$\int_0^b D_q f(x)d_qx=f(b)-f(0).\tag{6}$$

Moreover, it's easy to prove $q-$product rule for $q-$derivatives:

$$D_q(f(x)g(x))=f(x)D_q g(x)+g(qx)D_q f(x),$$

(see [2]) from which the formula for integration by parts for $q-$integrals follows:

$$\int_0^b f(x)D_q g(x)d_qx=f(b)g(b)-f(0)g(0)-\int_0^b g(qx)D_q f(x)d_qx.$$

2. Main results

Through the article, we will use the following notation:

$$Q_q(f,r,a) = \int_0^a D_q(t^r)f(t)d_qt.$$

But for simplicity, in Theorems 2.1 and 2.2 we will use $Q_q(f,r)$ for $Q_q(f,r,a)$ since there will be no difference.

Now we are going to state and prove our results:

Theorem 2.1. *Let α, β, γ, δ be positive real numbers. Let f_1 and f_2 be non-negative and decreasing integrable functions on $[0,a]$. Let $Q_q(f,r) = \int_0^a D_q(t^r)f(t)d_qt$ for $r > 0$. Then the following inequality holds:*

$$\begin{aligned} &Q_q(f_1,\delta)\cdot Q_q(f_2,\alpha+\beta+\gamma) + Q_q(f_1,\alpha+\beta+\delta)\cdot Q_q(f_2,\gamma) \\ &\geq Q_q(f_1,\alpha+\delta)\cdot Q_q(f_2,\beta+\gamma) + Q_q(f_1,\beta+\delta)\cdot Q_q(f_2,\alpha+\gamma). \end{aligned} \tag{7}$$

Proof. Since f_1 and f_2 are decreasing functions on $[0,a]$ and $q \in \langle 0,1\rangle$, we have

$$D_qf_i(t) = \frac{f_i(t) - f_i(qt)}{(1-q)t} \leq 0 \tag{8}$$

for $i = 1,2$ and for each $t \in [0,a]$.

The integration by parts gives us

$$Q_q(f,r) = \int_0^a D_q(t^r)f(t)d_qt = a^rf(a) - \int_0^a (qt)^rD_qf(t)d_qt. \tag{9}$$

Before the main calculation, let us transform the product of two integrals which will appear in the proof. Using the definition of the integral and the multiplication of two series, we have the following:

$$\begin{aligned} &\int_0^a (qt)^AD_qf_1(t)d_qt \cdot \int_0^a (qt)^BD_qf_2(t)d_qt \\ &\quad = (1-q)^2a^2\Big(\sum_{k=0}^{\infty} q^k(aq^{k+1})^AD_qf_1(aq^k)\Big)\Big(\sum_{k=0}^{\infty} q^k(aq^{k+1})^BD_qf_2(aq^k)\Big) \\ &\quad = (1-q)^2a^2\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} q^k(aq^{k+1})^Aqf_1(aq^k)q^{n-k}(aq^{n-k+1})^BD_qf_2(aq^{n-k})\Big) \\ &\quad = (1-q)^2a^{2+A+B}q^{A+B}\sum_{n=0}^{\infty} q^n\Big(\sum_{k=0}^{n}(q^k)^A(q^{n-k})^BD_qf_1(aq^k)D_qf_2(aq^{n-k})\Big). \end{aligned} \tag{10}$$

Using (9), multiplying given expressions and applying formula (10) four times, we get

$$\begin{aligned} &Q_q(f_1,\delta)\cdot Q_q(f_2,\alpha+\beta+\gamma) + Q_q(f_1,\alpha+\beta+\delta)\cdot Q_q(f_2,\gamma) \\ &- Q_q(f_1,\alpha+\delta)\cdot Q_q(f_2,\beta+\gamma) - Q_q(f_1,\beta+\delta)\cdot Q_q(f_2,\alpha+\gamma) \\ &\quad = f_1(a)f_2(a)a^{\alpha+\beta+\gamma+\delta} - a^{\delta}f_1(a)\int_0^a (qt)^{\alpha+\beta+\gamma}D_qf_2(t)d_qt \\ &\quad - a^{\alpha+\beta+\gamma}f_2(a)\int_0^a (qt)^{\delta}D_qf_1(t)d_qt + \int_0^a (qt)^{\delta}D_qf_1(t)d_qt \cdot \int_0^a (qt)^{\alpha+\beta+\gamma}D_qf_2(t)d_qt \\ &\quad + f_1(a)f_2(a)a^{\alpha+\beta+\gamma+\delta} - a^{\alpha+\beta+\delta}f_1(a)\int_0^a (qt)^{\gamma}D_qf_2(t)d_qt \end{aligned}$$

$$
\begin{aligned}
&- a^{\gamma} f_2(a) \int_0^a (qt)^{\alpha+\beta+\delta} D_q f_1(t) d_q t + \int_0^a (qt)^{\alpha+\beta+\delta} D_q f_1(t) d_q t \cdot \int_0^a (qt)^{\gamma} D_q f_2(t) d_q t \\
&- f_1(a) f_2(a) a^{\alpha+\beta+\gamma+\delta} + a^{\alpha+\delta} f_1(a) \int_0^a (qt)^{\beta+\gamma} D_q f_2(t) d_q t \\
&+ a^{\beta+\gamma} f_2(a) \int_0^a (qt)^{\alpha+\delta} D_q f_1(t) d_q t - \int_0^a (qt)^{\alpha+\delta} D_q f_1(t) d_q t \cdot \int_0^a (qt)^{\beta+\gamma} D_q f_2(t) d_q t \\
&- f_1(a) f_2(a) a^{\alpha+\beta+\gamma+\delta} + a^{\beta+\delta} f_1(a) \int_0^a (qt)^{\alpha+\gamma} D_q f_2(t) d_q t \\
&+ a^{\alpha+\gamma} f_2(a) \int_0^a (qt)^{\beta+\delta} D_q f_1(t) d_q t - \int_0^a (qt)^{\beta+\delta} D_q f_1(t) d_q t \cdot \int_0^a (qt)^{\alpha+\gamma} D_q f_2(t) d_q t \\
&= a^{\delta} f_1(a) \int_0^a (qt)^{\beta+\gamma} (a^{\alpha} - (qt)^{\alpha}) D_q f_2(t) d_q t + a^{\beta+\delta} f_1(a) \int_0^a (qt)^{\gamma} ((qt)^{\alpha} - a^{\alpha}) D_q f_2(t) d_q t \\
&+ a^{\beta+\gamma} f_2(a) \int_0^a (qt)^{\delta} ((qt)^{\alpha} - a^{\alpha}) D_q f_1(t) d_q t + a^{\gamma} f_2(a) \int_0^a (qt)^{\beta+\delta} (a^{\alpha} - (qt)^{\alpha}) D_q f_1(t) d_q t \\
&+ (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} q^{\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \Big(\sum_{k=0}^{n} (q^k)^{\delta} (q^{n-k})^{\alpha+\beta+\gamma} D_q f_1(aq^k) D_q f_2(aq^{n-k}) \Big) \\
&+ (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} q^{\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \Big(\sum_{k=0}^{n} (q^k)^{\alpha+\beta+\delta} (q^{n-k})^{\gamma} D_q f_1(aq^k) D_q f_2(aq^{n-k}) \Big) \\
&- (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} q^{\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \Big(\sum_{k=0}^{n} (q^k)^{\alpha+\delta} (q^{n-k})^{\beta+\gamma} D_q f_1(aq^k) D_q f_2(aq^{n-k}) \Big) \\
&- (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} q^{\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \Big(\sum_{k=0}^{n} (q^k)^{\beta+\delta} (q^{n-k})^{\alpha+\gamma} D_q f_1(aq^k) D_q f_2(aq^{n-k}) \Big) \\
&= a^{\delta} f_1(a) \int_0^a (qt)^{\gamma} (a^{\alpha} - (qt)^{\alpha}) ((qt)^{\beta} - a^{\beta}) D_q f_2(t) d_q t \\
&+ a^{\gamma} f_2(a) \int_0^a (qt)^{\delta} (a^{\alpha} - (qt)^{\alpha}) ((qt)^{\beta} - a^{\beta}) D_q f_1(t) d_q t + (1-q)^2 a^2 (aq)^{\alpha+\beta+\gamma+\delta} \\
&\times \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (q^k)^{\delta} (q^{n-k})^{\gamma} ((q^{n-k})^{\alpha} - (q^k)^{\alpha}) ((q^{n-k})^{\beta} - (q^k)^{\beta}) D_q f_1(aq^k) D_q f_2(aq^{n-k}).
\end{aligned}
$$

Now we can conclude that

$$(a^{\alpha} - (qt)^{\alpha})((qt)^{\beta} - a^{\beta}) \leq 0,$$

for each $t \in [0, a]$ since $qt < a$ for $q \in \langle 0, 1 \rangle$

and also that

$$(\,(q^{n-k})^{\alpha} - (q^k)^{\alpha}\,) \cdot (\,(q^{n-k})^{\beta} - (q^k)^{\beta}\,) \geq 0,$$

since both factors are either non-positive or non-negative.

Then, using (8), we conclude that the last expression is non-negative, so inequality (7) is proven. □

Theorem 2.2. *Let α, β, γ, δ be positive real numbers. Let f_1 and f_2 be non-negative and increasing integrable functions on $[0, a]$. Then the following inequality holds:*

$$
\begin{aligned}
&Q_q(f_1, \delta) \cdot Q_q(f_2, \alpha+\beta+\gamma) + Q_q(f_1, \alpha+\beta+\delta) \cdot Q_q(f_2, \gamma) \\
&\leq Q_q(f_1, \alpha+\delta) \cdot Q_q(f_2, \beta+\gamma) + Q_q(f_1, \beta+\delta) \cdot Q_q(f_2, \alpha+\gamma),
\end{aligned} \tag{11}
$$

where $Q_q(f, r)$ is defined as in Theorem 2.1.

Proof. Since f_1 and f_2 are increasing functions on $[0,a]$ and $q \in \langle 0,1\rangle$, we have

$$D_q f_i(t) = \frac{f_i(t) - f_i(qt)}{(1-q)t} \geq 0 \tag{12}$$

for $i = 1,2$ and for each $t \in [0,a]$.

The integration by parts gives us

$$Q_q(f,r) = \int_0^a D_q(t^r) f(t) d_q t = a^r f(a) - \int_0^a (qt)^r D_q f(t) d_q t.$$

Moreover, by using the analogue of the Newton-Leibniz formula (6), we obtain

$$Q_q(f,r) = a^r \cdot \frac{\int_0^a D_q(f(t)) d_q t}{\int_0^a D_q(f(t)) d_q t} f(a) - \int_0^a (qt)^r D_q f(t) d_q t = \int_0^a \Big(\frac{a^r f(a)}{f(a) - f(0)} - (qt)^r \Big) D_q f(t) d_q t. \tag{13}$$

Let us transform the product $Q_q(f_1,A) \cdot Q_q(f_2,B)$ according to (13) and using the formula for the product of two series. For easier reading, we introduce the following abbreviations:

$$F_i = \frac{f_i(a)}{f_i(a) - f_i(0)},\ i = 1,2,\ \ G_1 = q^{k+1},\ G_2 = q^{n-k+1}.$$

Now, we have

$$\begin{aligned}
Q_q(f_1,A) \cdot Q_q(f_2,B) &= \int_0^a (a^A F_1 - (qt)^A) D_q f_1(t) d_q t \cdot \int_0^a (a^B F_1 - (qt)^B) D_q f_2(t) d_q t \\
&= (1-q)^2 a^2 \Big(\sum_{k=0}^{\infty} q^k (a^A F_1 - (aq^{k+1})^A) D_q f_1(aq^k) \Big) \Big(\sum_{k=0}^{\infty} q^k (a^B F_2 - (aq^{k+1})^B) D_q f_2(aq^k) \Big) \\
&= (1-q)^2 a^{2+A+B} \sum_{n=0}^{\infty} q^n \Big(\sum_{k=0}^{n} (F_1 - G_1^A)(F_2 - G_2^B) D_q f_1(aq^k) D_q f_2(aq^{n-k}) \Big).
\end{aligned} \tag{14}$$

Using (14), we obtain

$$\begin{aligned}
&Q_q(f_1,\delta) \cdot Q_q(f_2,\alpha+\beta+\gamma) + Q_q(f_1,\alpha+\beta+\delta) \cdot Q_q(f_2,\gamma) \\
&- Q_q(f_1,\alpha+\delta) \cdot Q_q(f_2,\beta+\gamma) - Q_q(f_1,\beta+\delta) \cdot Q_q(f_2,\alpha+\gamma) \\
&\quad = \Big(\int_0^a \Big(\frac{a^{\delta} f_1(a)}{f_1(a) - f_1(0)} - (qt)^{\delta} \Big) D_q f_1(t) d_q t \Big) \cdot \Big(\int_0^a \Big(\frac{a^{\alpha+\beta+\gamma} f_2(a)}{f_2(a) - f_2(0)} - (qt)^{\alpha+\beta+\gamma} \Big) D_q f_2(t) d_q t \Big) \\
&\quad + \Big(\int_0^a \Big(\frac{a^{\alpha+\beta+\delta} f_1(a)}{f_1(a) - f_1(0)} - (qt)^{\alpha+\beta+\delta} \Big) D_q f_1(t) d_q t \Big) \cdot \Big(\int_0^a \Big(\frac{a^{\gamma} f_2(a)}{f_2(a) - f_2(0)} - (qt)^{\gamma} \Big) D_q f_2(t) d_q t \Big) \\
&\quad - \Big(\int_0^a \Big(\frac{a^{\alpha+\delta} f_1(a)}{f_1(a) - f_1(0)} - (qt)^{\alpha+\delta} \Big) D_q f_1(t) d_q t \Big) \cdot \Big(\int_0^a \Big(\frac{a^{\beta+\gamma} f_2(a)}{f_2(a) - f_2(0)} - (qt)^{\beta+\gamma} \Big) D_q f_2(t) d_q t \Big) \\
&\quad - \Big(\int_0^a \Big(\frac{a^{\beta+\delta} f_1(a)}{f_1(a) - f_1(0)} - (qt)^{\beta+\delta} \Big) D_q f_1(t) d_q t \Big) \cdot \Big(\int_0^a \Big(\frac{a^{\alpha+\gamma} f_2(a)}{f_2(a) - f_2(0)} - (qt)^{\alpha+\gamma} \Big) D_q f_2(t) d_q t \Big) \\
&\quad = (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (F_1 - G_1^{\delta})(F_2 - G_2^{\alpha+\beta+\gamma}) D_q f_1(aq^k) D_q f_2(aq^{n-k}) \\
&\quad + (1-q)^2 a^{2+\alpha+\beta+\gamma+\delta} \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (F_1 - G_1^{\alpha+\beta+\delta})(F_2 - G_2^{\gamma}) D_q f_1(aq^k) D_q f_2(aq^{n-k})
\end{aligned}$$

$$
\begin{aligned}
&-(1-q)^2a^{2+\alpha+\beta+\gamma+\delta}\sum_{n=0}^{\infty}q^n\sum_{k=0}^{n}(F_1-G_1^{\alpha+\delta})(F_2-G_2^{\beta+\gamma})D_qf_1(aq^k)D_qf_2(aq^{n-k})\\
&-(1-q)^2a^{2+\alpha+\beta+\gamma+\delta}\sum_{n=0}^{\infty}q^n\sum_{k=0}^{n}(F_1-G_1^{\beta+\delta})(F_2-G_2^{\alpha+\gamma})D_qf_1(aq^k)D_qf_2(aq^{n-k})\\
&=(1-q)^2a^{2+\alpha+\beta+\gamma+\delta}\sum_{n=0}^{\infty}q^n\Big(\sum_{k=0}^{n}(F_1\cdot F_2-G_1^{\delta}F_2-G_2^{\alpha+\beta+\gamma}F_1+G_1^{\delta}G_2^{\alpha+\beta+\gamma}+F_1F_2\\
&-G_1^{\alpha+\beta+\delta}F_2-G_2^{\gamma}F_1+G_1^{\alpha+\beta+\delta}G_2^{\gamma}-F_1F_2+G_1^{\alpha+\delta}F_2+G_2^{\beta+\gamma}F_1-G_1^{\alpha+\delta}G_2^{\beta+\gamma}\\
&-F_1F_2+G_1^{\beta+\delta}F_2+G_2^{\alpha+\gamma}F_1-G_1^{\beta+\delta}G_2^{\alpha+\gamma}\Big)D_qf_1(aq^k)D_qf_2(aq^{n-k})\Big)\\
&=(1-q)^2a^{2+\alpha+\beta+\gamma+\delta}\sum_{n=0}^{\infty}q^n\Big(\sum_{k=0}^{n}\Big(-F_1G_2^{\gamma}(G_2^{\alpha}-1)(G_2^{\beta}-1)-F_2G_1^{\delta}(G_1^{\alpha}-1)(G_1^{\beta}-1)\\
&+G_1^{\delta}G_2^{\gamma}(G_2^{\alpha}-G_1^{\alpha})(G_2^{\beta}-G_1^{\beta})\Big)D_qf_1(aq^k)D_qf_2(aq^{n-k})\Big).
\end{aligned}
$$

Since f_1 and f_2 are non-negative and increasing on $[0,a]$, we have $F_i \geq 1$ for $i = 1,2$ and by using (12), we obtain

$$
\begin{aligned}
&Q_q(f_1,\delta)\cdot Q_q(f_2,\alpha+\beta+\gamma)+Q_q(f_1,\alpha+\beta+\delta)\cdot Q_q(f_2,\gamma)\\
&-Q_q(f_1,\alpha+\delta)\cdot Q_q(f_2,\beta+\gamma)-Q_q(f_1,\beta+\delta)\cdot Q_q(f_2,\alpha+\gamma)\\
&\quad\leq(1-q)^2a^{2+\alpha+\beta+\gamma+\delta}\sum_{n=0}^{\infty}q^n\Big(\sum_{k=0}^{n}\Big(-G_2^{\gamma}(G_2^{\alpha}-1)(G_2^{\beta}-1)-G_1^{\delta}(G_1^{\alpha}-1)(G_1^{\beta}-1)\\
&\quad+G_1^{\delta}G_2^{\gamma}(G_2^{\alpha}-G_1^{\alpha})(G_2^{\beta}-G_1^{\beta})\Big)D_qf_1(aq^k)D_qf_2(aq^{n-k})\Big)\leq 0,
\end{aligned}
$$

since, using $0 < q < 1$, we have

$$
\begin{aligned}
&-G_2^{\gamma}(G_2^{\alpha}-1)(G_2^{\beta}-1)-G_1^{\delta}(G_1^{\alpha}-1)(G_1^{\beta}-1)+G_1^{\delta}G_2^{\gamma}(G_2^{\alpha}-G_1^{\alpha})(G_2^{\beta}-G_1^{\beta})\\
&\quad=G_2^{\gamma}(G_2^{\alpha}-1)(G_2^{\beta}-1)(G_1^{\delta}-1)+G_1^{\delta}(G_1^{\alpha}-1)(G_1^{\beta}-1)(G_2^{\gamma}-1)\\
&\quad-G_1^{\delta}G_2^{\gamma}(G_2^{\alpha}-1)(G_1^{\beta}-1)-G_1^{\delta}G_2^{\gamma}(G_2^{\beta}-1)(G_1^{\alpha}-1)\leq 0,
\end{aligned}
$$

which proves inequality (11). □

Remark 2.3. *Putting $\gamma = \delta$ in Theorems 2.1 and 2.2, we get the $q-$versions of Theorems 2.1 and 2.2 from [8].*

Let us point out special results for Theorems 2.1 and 2.2 obtained for $f_1 = f_2 = f$ and for $\gamma = \delta$:

Corollary 2.4. *Let α, β and γ be positive real numbers. Let f be non-negative and integrable function on $[0,a]$ and let $Q_q(f,r) = \int_0^a D_q(t^r)f(t)d_qt$ for each $r > 0$.*

(i) If f is decreasing, then the following inequality holds:

$$Q_q(f,\gamma)\cdot Q_q(f,\alpha+\beta+\gamma)\geq Q_q(f,\alpha+\gamma)\cdot Q_q(f,\beta+\gamma). \tag{15}$$

(ii) If f is increasing, then the following inequality holds:

$$Q_q(f,\gamma)\cdot Q_q(f,\alpha+\beta+\gamma)\leq Q_q(f,\alpha+\gamma)\cdot Q_q(f,\beta+\gamma). \tag{16}$$

Remark 2.5. *For $\gamma = 0$ in (15) and (16), we especially obtain results of the same type as inequalities (3) and (4), that is, we obtain the Stolarsky inequality and its reversed inequality for $q-$integrals.*

Remark 2.6. *Instead of the monotonicity conditions of functions f_i, $i = 1,2$ in Theorems 2.1 and 2.2 and Corollary 2.4, it is enough to assume that $D_qf_i(t) \leq 0$ and $D_qf_i(t) \geq 0$ respectively for $i = 1,2$ and for each $t \in [0,a]$.*

3. Application to q-gamma and q-beta functions

Now we apply the main results to obtain inequalities for the $q-$gamma and $q-$beta functions.
First, we recall some notions and notations used in $q-$theory (see [1], [2]):

$$(1+a)_q^\infty = \prod_{j=0}^{\infty}(1+q^j a).$$

$$(1+a)_q^t = \frac{(1+a)_q^\infty}{(1+aq^t)_q^\infty}.$$

A $q-$analogue Γ_q of the gamma function is given as

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-qx} d_q x,\ t > 0, \tag{17}$$

where E_q^x is $q-$analogue of the exponential function:

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^\infty,$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Moreover, equation

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t),\ t > 0$$

is valid (see [1] for instance).

So, we can rewrite (17) in the following way

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) = \int_0^{\frac{1}{1-q}} D_q(x^t) E_q^{-qx} d_q x,\ t > 0.$$

The $q-$beta function $B_q(t,s)$ is defined as

$$B_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(t+s)},\ t,s > 0$$

and has the following $q-$integral representation

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x.$$

It is clear that:

$$[t]_q B_q(t,s) = \int_0^1 D_q(x^t)(1-qx)_q^{s-1} d_q x.$$

Some further results about the $q-$gamma and $q-$beta functions can be found in [1],[2],[5],[7].

Here, we will point out that in [7] it is proven that for $f(x) = (1-qx)_q^{s-1}$ we have $D_q f(x) \le 0$ for $s \ge 1$ and $D_q f(x) \ge 0$ for $0 < s < 1$.

Now, taking into account that both $\Gamma_q(t+1)$ and $[t]_q B_q(t,s)$ are of the form $\int_0^a D_q(x^t) f(x) d_q x$ for monotone function f, we obtain interesting inequalities for $q-$gamma and $q-$beta functions directly from the results of Theorems 2.1 and 2.2.

Corollary 3.1. *Let α, β, γ, δ be positive real numbers. Then the following inequality holds:*

$$\Gamma_q(\delta+1)\cdot\Gamma_q(\alpha+\beta+\gamma+1)+\Gamma_q(\alpha+\beta+\delta+1)\cdot\Gamma_q(\gamma+1)$$
$$\geq\Gamma_q(\alpha+\delta+1)\cdot\Gamma_q(\beta+\gamma+1)+\Gamma_q(\beta+\delta+1)\cdot\Gamma_q(\alpha+\gamma+1).$$

Corollary 3.2. *Let α, β, γ, δ be positive real numbers. Then the following inequalities hold:*
(i)

$$[\delta]_q[\alpha+\beta+\gamma]_qB_q(\delta,s)\cdot B_q(\alpha+\beta+\gamma,s)+[\gamma]_q[\alpha+\beta+\delta]_qB_q(\alpha+\beta+\delta,s)\cdot B_q(\gamma,s)$$
$$\geq[\alpha+\delta]_q[\beta+\gamma]_qB_q(\alpha+\delta,s)\cdot B_q(\beta+\gamma,s)+[\beta+\delta]_q[\alpha+\gamma]_qB_q(\beta+\delta,s)\cdot B_q(\alpha+\gamma,s)$$

for each $s\geq 1$,
(ii)

$$[\delta]_q[\alpha+\beta+\gamma]_qB_q(\delta,s)\cdot B_q(\alpha+\beta+\gamma,s)+[\gamma]_q[\alpha+\beta+\delta]_qB_q(\alpha+\beta+\delta,s)\cdot B_q(\gamma,s)$$
$$\leq[\alpha+\delta]_q[\beta+\gamma]_qB_q(\alpha+\delta,s)\cdot B_q(\beta+\gamma,s)+[\beta+\delta]_q[\alpha+\gamma]_qB_q(\beta+\delta,s)\cdot B_q(\alpha+\gamma,s)$$

for each $0<s<1$.

Remark 3.3. *Especially, for positive α, β and γ the following Stolarsky type inequality holds*

$$\Gamma_q(\gamma+1)\cdot\Gamma_q(\alpha+\beta+\gamma+1)\geq\Gamma_q(\alpha+\gamma+1)\cdot\Gamma_q(\beta+\gamma+1).$$

Moreover, for positive α, β and γ and for $s\geq 1$ the following inequality holds

$$[\gamma]_q[\alpha+\beta+\gamma]_qB_q(\gamma,s)\cdot B_q(\alpha+\beta+\gamma,s)\geq[\alpha+\gamma]_q[\beta+\gamma]_qB_q(\alpha+\gamma,s)\cdot B_q(\beta+\gamma,s)$$

and for each $0<s<1$ it holds

$$[\gamma]_q[\alpha+\beta+\gamma]_qB_q(\gamma,s)\cdot B_q(\alpha+\beta+\gamma,s)\leq[\alpha+\gamma]_q[\beta+\gamma]_qB_q(\alpha+\gamma,s)\cdot B_q(\beta+\gamma,s).$$

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