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On the modification of Picard operators on the semi-real axis

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Abstract. In this paper, we define and study the modified Picard singular integral operators on the half plane. We demonstrate uniform convergence of P_n^* operators, determine the degree of this uniform convergence as well and examine the asymptotic behavior of the operators by proving a quantitative Voronovskaya theorem. Moreover, we study uniform weighted approximation formula for the operators of P_n^* by using a weighted Korovkin type theorem. Finally, we give a result regarding global smoothness preservation properties for the operators of P_n^*.

1. Introduction

One of the most important branches of approximation theory is the use of linear positive operators. These operators play a key role in approximating functions and signals, particularly due to their stability and preserving properties. They have been widely studied and applied in various fields of analysis and numerical methods. One of these operators are the Picard integral operators.

Let f be real valued function in $\mathbb{R}$. For $n \in \mathbb{N}$ and $x \in (-\infty, \infty)$, the classical Picard integral operators are defined as

$$P_n(f;x) = \frac{n}{2}\int_{-\infty}^{\infty} f(x+t)\, e^{-n|t|}dt. \tag{1}$$

Over time, researchers have developed these types of operators by introducing various modifications, and their approximation behaviors have been extensively studied in the literature (see [1],[2],[5],[6], [15] and [16]). Therewithal, in [3], Anasstasiou's book contains numerous results on the different properties of such operators. Furthermore, in [8], Picard singular integral operators modified by means of non-isotropic distance and their pointwise approximation in different normed spaces are analyzed and in ([10], [14]) these type of integral operators were considered in exponential weighted spaces for function of one or two variables. Also, in [9], Bardaro et al. defined a generalized form of these operators which fix e^{ax} and e^{2ax} with $a > 0$ and proved some approximation theorems.

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In [5], Aral et al. studied modified Picard singular integral operators defined for functions with domain on the real line. The main motivation to introduce these operators with the domain restricted to the half-line was to prove the different theorems that were not given in [5].

We consider the following modification of the classical operators defined by (1), where it is shown that the corresponding operators fix the functions 1 and e^{2ax}. Moreover, we introduce these classes intended to be associated with functions defined on the semi-real axis. Specifically, the proposed generalizations of the above operators are defined as follows:

Let $\sigma_n := \frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)$, which is positive for every fixed $a > 0$ and for sufficiently large $n \in \mathbb{N}$ such that $n > 2a$. For $f \in C[0,\infty)$, we consider

$$\mathcal{P}_n^*(f;x) = n\int_0^\infty f(\gamma_n(x)+t)\,e^{-nt}dt,\quad n \geq 1 \tag{2}$$

and where

$$\gamma_n(x) = x - \sigma_n,\ x > \sigma_n,\quad n > 2a. \tag{3}$$

Moreover, we define

$$P_n^*(f;x) = \begin{cases} \mathcal{P}_n^*(f;x) & if\ x \in [\sigma_n,\infty), \\ f(x) & if\ x \in [0,\sigma_n). \end{cases}$$

In the present paper, in connection with the operators defined in (2), we first present some lemmas that are essential for proving our main theorems. In the third section, we establish the uniform convergence of the operators and also determine the rate of this convergence. In the fourth section, we investigate the asymptotic behavior of the operators P_n^* by proving a quantitative Voronovskaya theorem. Later on, we study a uniform weighted approximation formula for the operators P_n^* by applying a weighted Korovkin-type theorem. Subsequently, we show that similar results can also be obtained without relying on this theorem. Finally, in the last section, we present a result concerning the global smoothness preservation properties of the operators P_n^*.

2. Auxiliary Results

In this section, we begin by establishing some general results concerning these operators, as they will be required in the subsequent parts of the paper.

Lemma 2.1. *For the operators defined by (2), with $n \in \mathbb{N}$, $n > 2a$, $x \in [\sigma_n,\infty)$ and $e_r(t) = t^r, r = 0,1,2,\ldots,$ the moments given as follows:*

$$\begin{aligned} \mathcal{P}_n^*(e_0;x) &= 1, \\ \mathcal{P}_n^*(e_1;x) &= \gamma_n(x) + \frac{1}{n}, \\ \mathcal{P}_n^*(e_2;x) &= \left(\gamma_n(x) + \frac{1}{n}\right)^2 + \frac{1}{n^2}. \end{aligned}$$

Proof. From (2), we obtain

$$\mathcal{P}_n^*(e_0;x) = n\int_0^\infty e^{-nt}dt.$$

By changing the variable $t = \frac{u}{n}$,we have

$$\mathcal{P}_n^*(e_0; x) = \int_0^\infty e^{-u} du = \Gamma(1) = 0! = 1,$$

where $\Gamma(.)$ is Gamma function. Similarly, we can write

$$\mathcal{P}_n^*(e_1; x) = \gamma_n(x)\Gamma(1) + \frac{\Gamma(2)}{n} = \gamma_n(x) + \frac{1}{n}$$

and

$$\begin{aligned} \mathcal{P}_n^*(e_2; x) &= \gamma_n^2(x)\Gamma(1) + 2\gamma_n(x)\frac{\Gamma(2)}{n} + \frac{\Gamma(3)}{n^2} \\ &= \left(\gamma_n(x) + \frac{1}{n}\right)^2 + \frac{1}{n^2}, \end{aligned}$$

which proves the lemma. □

Lemma 2.2. *If we denote $T_{n,r}(x) = \mathcal{P}_n^*\left((t-x)^r; x\right)$, $\quad r = 0, 1, 2, \ldots$, and for each fixed x, then we have*

$$\begin{aligned} T_{n,0}(x) &= 1, \\ T_{n,1}(x) &= \gamma_n(x) - x + \frac{1}{n}, \\ T_{n,2}(x) &= (\gamma_n(x) - x)^2 + \frac{2}{n}(\gamma_n(x) - x) + \frac{2}{n^2}. \end{aligned}$$

Also, from (3), we obtain

$$\begin{aligned} \lim_{n\to\infty} n\left(\gamma_n(x) - x + \frac{1}{n}\right) &= 2 \\ \lim_{n\to\infty} n\left((\gamma_n(x) - x)^2 + \frac{2}{n}(\gamma_n(x) - x) + \frac{2}{n^2}\right) &= 0. \end{aligned}$$

Lemma 2.3. *For the operators $\mathcal{P}_n^*$, we have*

$$\begin{aligned} \mathcal{P}_n^*\left(e^{at}; x\right) &= e^{a\gamma_n(x)}\frac{n}{n-a} = \frac{\sqrt{n(n-2a)}}{n-a}e^{ax}, \\ \mathcal{P}_n^*\left(e^{2at}; x\right) &= e^{2a\gamma_n(x)}\frac{n}{n-2a} = e^{2ax}, \\ \mathcal{P}_n^*\left(e^{3at}; x\right) &= e^{3a\gamma_n(x)}\frac{n}{n-3a} = \frac{n-2a}{n-3a}\sqrt{\frac{n-2a}{n}}e^{3ax}, \\ \mathcal{P}_n^*\left(e^{4at}; x\right) &= e^{3a\gamma_n(x)}\frac{n}{n-4a} = \frac{(n-2a)^2}{n(n-4a)}e^{4ax}. \end{aligned}$$

3. Uniform convergence of P_n^*

In this section, we obtain the rate of convergence for P_n^* operators. Let us denote by $C[0,\infty)$, all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite. For this estimation, we use the following modulus of continuity:

$$\omega^*(f;\delta) = \sup_{\substack{x,t\in[0,\infty) \\ |e^{-x}-e^{-t}|\le\delta}} \left|f(t) - f(x)\right|,$$

defined for every $\delta \geq 0$ and every $f \in C[0,\infty)$. Also we use the following property of modulus of continuity.

$$\left|f(t) - f(x)\right| \leq \left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\right)\omega^*(f;\delta), \quad \delta > 0. \tag{4}$$

Further details on $\omega^*(.;\delta)$ are provided in [13], where the following statement is also presented.

Theorem 3.1. *If a sequence of positive linear operators $A_n : C[0,\infty) \to C[0,\infty)$ satisfy the following equalities*

$$\begin{aligned} \|A_n(1;x) - 1\|_{[0,\infty)} &= a_n, \\ \left\|A_n\left(e^{-t};x\right) - e^{-x}\right\|_{[0,\infty)} &= b_n, \\ \left\|A_n\left(e^{-2t};x\right) - e^{-2x}\right\|_{[0,\infty)} &= c_n, \end{aligned}$$

then

$$\left\|A_n f - f\right\|_{[\sigma_n,\infty)} \leq 2\omega^*\left(f; \sqrt{a_n + 2b_n + c_n}\right), \quad f \in C[0,\infty),$$

where $\left\|f\right\|_{[\sigma_n,\infty)} = \sup\limits_{x\in[\sigma_n,\infty)} \left|f(x)\right|$.

Theorem 3.2. *For each function $f \in C[0,\infty)$, the following inequality holds:*

$$\left\|P_n^* f - f\right\|_{[0,\infty)} \leq 2\omega^*\left(f; \sqrt{2b_n + c_n}\right),$$

where

$$\begin{aligned} b_n &= e^{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+1} - 1, \\ c_n &= e^{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+2} - 1, \end{aligned}$$

and both b_n and c_n tend to zero as $n \to \infty$.

Proof. By simple calculation, we obtain

$$\mathcal{P}_n^*\left(e^{-t};x\right) = e^{-\gamma_n(x)}\frac{n}{n+1} = e^{-x+\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+1},$$

$$\mathcal{P}_n^*\left(e^{-2t};x\right) = e^{-2\gamma_n(x)}\frac{n}{n+2} = e^{-2x+\frac{1}{a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+2}.$$

Therefore, by using Theorem 1, we have

$$\begin{aligned} \left\|\mathcal{P}_n^*(1;x) - 1\right\|_{[\sigma_n,\infty)} &= 0, \\ \left\|\mathcal{P}_n^*\left(e^{-t};x\right) - e^{-x}\right\|_{[\sigma_n,\infty)} &= b_n, \\ \left\|\mathcal{P}_n^*\left(e^{-2t};x\right) - e^{-2x}\right\|_{[\sigma_n,\infty)} &= c_n, \end{aligned}$$

where

$$\begin{aligned} b_n &= e^{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+1} - 1 \\ c_n &= e^{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)}\frac{n}{n+2} - 1. \end{aligned}$$

and both b_n and c_n tend to zero as $n \to \infty$. When $x \in [\sigma_n,\infty)$, we have

$$\left\|P_n^* f - f\right\|_{[0,\infty)} = \left\|\mathcal{P}_n^* f - f\right\|_{[\sigma_n,\infty)},$$

thus the proof is complete. □

4. Quantitative Type Voronovskaya Theorem

In this section, we give the quantitative type Voronovskaya theorem for P_n^* operators.

Theorem 4.1. *Let $f, f'' \in C[0,\infty)$. Then we have*

$$\left|n\left[P_n^*(f;x)-f(x)\right]-2f'(x)\right| \leq \left|nT_{n,1}(x)-2\right|\left|f'(x)\right|+\frac{1}{2}\left|nT_{n,2}(x)\right|\left|f''(x)\right|+D\omega^*\left(f'';\frac{1}{\sqrt{n}}\right),$$

where D is a positive constant.

Proof. By using the Taylor's formula, there exist η lying between x and t such that

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+h(t,x)(t-x)^2$$

where

$$h(t,x):=\frac{f''(\eta)-f''(x)}{2}$$

and h is a continuous function which vanishes at 0. In view of the definition of P_n^* operator, we obtain the following equality for $x\in[\sigma_n,\infty)$ and if we apply the above formula to the operator P_n^*, then

$$\begin{aligned} P_n^*(f;x)-f(x) &= \mathcal{P}_n^*(f;x)-f(x) \\ &= f'(x)T_{n,1}(x)+\frac{f''(x)}{2}T_{n,2}(x)+\mathcal{P}_n^*\left(h(t,x)(t-x)^2;x\right). \end{aligned}$$

Also we can write that

$$\left|n\left[\mathcal{P}_n^*(f;x)-f(x)\right]-2f'(x)\right| \leq f'(x)\left[nT_{n,1}(x)-2\right]+\frac{f''(x)}{2}\left[nT_{n,2}(x)\right]+n\mathcal{P}_n^*\left(|h(t,x)|(t-x)^2;x\right).$$

To estimate the last inequality by using the inequality (4), we get

$$|h(t,x)|\leq\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*\left(f'';\delta\right).$$

Since

$$|h(t,x)|\leq\begin{cases} 2\omega^*\left(f'';\delta\right), \left|e^{-x}-e^{-t}\right|<\delta \\ 2\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\omega^*\left(f'';\delta\right), \left|e^{-x}-e^{-t}\right|\geq\delta \end{cases},$$

we have

$$|h(t,x)|\leq 2\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*\left(f'';\delta\right).$$

We deduce that

$$n\mathcal{P}_n^*\left(|h(t,x)|(t-x)^2;x\right) \leq 2n\omega^*\left(f'';\delta\right)T_{n,2}(x)+\frac{2n}{\delta^2}\omega^*\left(f'';\delta\right)\mathcal{P}_n^*\left(\left(e^{-x}-e^{-t}\right)^2(t-x)^2;x\right).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$n\mathcal{P}_n^*\left(|h(t,x)|(t-x)^2;x\right) \leq 2n\omega^*\left(f'';\delta\right)T_{n,2}(x)+\frac{2n}{\delta^2}\omega^*\left(f'';\delta\right)\sqrt{\mathcal{P}_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{T_{n,4}(x)}.$$

Choosing $\delta = \frac{1}{\sqrt{n}}$,we get

$$n\mathcal{P}_n^*\left(|h(t,x)|(t-x)^2;x\right) \leq 2n\omega^*\left(f'';\frac{1}{\sqrt{n}}\right)T_{n,2}(x)+2\omega^*\left(f'';\frac{1}{\sqrt{n}}\right)\sqrt{n^2\mathcal{P}_n^*\left((e^{-x}-e^{-t})^4;x\right)}\sqrt{n^2T_{n,4}(x)}.$$

By using a computer program, the followings were obtained

$$\lim_{n\to\infty} n^2T_{n,4}(x) = 0,$$
$$\lim_{n\to\infty} n^2\mathcal{P}_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right) = 0.$$

Thus, we arrive at

$$\left|n\left[\mathcal{P}_n^*(f;x)-f(x)\right]-2f'(x)\right| \leq \left|nT_{n,1}(x)-2\right|\left|f'(x)\right|+\frac{1}{2}\left|nT_{n,2}(x)\right|\left|f''(x)\right|+D\omega^*\left(f'';\frac{1}{\sqrt{n}}\right),$$

which was our claim. □

Let $f\in C^r[0,\infty)$ with $\omega^*(f^{(r)};\delta)<\infty$, $\delta>0$ $r\in\mathbb{N}$, since for $x\in[\sigma_n,\infty)$ the operators P_n^* we set

$$\Delta_n(f;x) := \mathcal{P}_n^*(f;x)-n\sum_{k=0}^{m}\frac{f^{(k)}(\gamma(x))}{k!}\int_0^\infty t^ke^{-nt}dt.$$

We now study the convergence of P_n^* operators to the unit operator I with rates (*see* [3]).

Theorem 4.2. *The following inequality holds*

$$\left|\Delta_n(f;x)\right| \leq \frac{\omega^*\left(f;\sqrt{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\right)}{m!}A, \tag{5}$$

where

$$A := \left(\begin{array}{c} \Gamma(m+1)\left[\frac{1}{n^m}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\frac{n}{(n+2)^{m+1}}\right] \\ -\Gamma(m)\left[m\frac{1}{n^{m-1}}\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)+(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})m\frac{n}{(n+2)^{m-1}}\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^{1-m}\right] \\ \dots \\ +(-1)^m\left(\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m+(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})\frac{n}{n+2}\right). \end{array}\right)$$

Proof. Let $f\in C^m[0,\infty)$, $m\in\mathbb{N}$. By Taylor's formula (*see Lemma*2(2), *p*.2 *of* [4]), we have

$$f(\gamma(x)+t)=\sum_{k=0}^{m}\frac{f^{(k)}(\gamma(x))}{k!}t^k+R_m(f;x,\gamma(x)+t) \tag{6}$$

where

$$R_m(f;x,\gamma(x)+t) := \frac{1}{(m-1)!}\int_x^{\gamma(x)+t}\left[\left(f^{(m)}(s)-f^{(m)}(t)\right)\right](\gamma(x)+t-s)^{m-1}ds$$

for all $x,t\in\mathbb{R}$. Applying Theorem 6, (14) p. 4 of [4] , we can write

$$\left|R_m(f;x,\gamma(x)+t)\right| \leq \frac{1}{(m-1)!}\int_x^{\gamma(x)+t}\left|\left[f^{(m)}(s)-f^{(m)}(t)\right]\right|\left|(\gamma(x)+t-s)\right|^{m-1}ds.$$

From (4), we have

$$
\begin{aligned}
&\left|R_m\left(f;x,\gamma(x)+t\right)\right| \\
\leq\ & \frac{1}{(m-1)!}\int_x^{\gamma(x)+t}\left(1+\frac{\left(e^{-t}-e^{-s}\right)^2}{\delta^2}\right)\omega^*\left(f;\delta\right)\left|\left(\gamma(x)+t-s\right)\right|^{m-1}ds \\
\leq\ & \frac{1}{(m-1)!}\left(1+\frac{\left(e^{-t}-e^{-(\gamma(x)+t)}\right)^2}{\delta^2}\right)\omega^*\left(f;\delta\right)\int_x^{\gamma(x)+t}\left|\left(\gamma(x)+t-s\right)\right|^{m-1}ds \\
\leq & \left(1+\frac{\left(e^{-t}-e^{-(\gamma(x)+t)}\right)^2}{\delta^2}\right)\omega^*\left(f;\delta\right)\frac{\left(\gamma(x)+t-x\right)^m}{m!}
\end{aligned}
\tag{7}
$$

From (6), we find

$$f\left(\gamma(x)+t\right)-\sum_{k=0}^{m}\frac{f^{(k)}\left(\gamma(x)\right)}{k!}t^k=R_m\left(f;x,\gamma(x)+t\right)$$

and

$$n\int_0^\infty f\left(\gamma(x)+t\right)e^{-nt}dt-n\sum_{k=0}^{m}\frac{f^{(k)}\left(\gamma(x)\right)}{k!}\int_0^\infty t^k e^{-nt}dt=n\int_0^\infty R_m\left(f;x,\gamma(x)+t\right)e^{-nt}dt.$$

That is

$$\mathcal{P}_n^*\left(f;x\right)-f(\gamma(x))-\sum_{k=1}^{m}\frac{f^{(k)}\left(\gamma(x)\right)}{k!}\frac{1}{n^{k+1}}\Gamma(k+1)=n\int_0^\infty R_m\left(f;x,\gamma(x)+t\right)e^{-nt}dt.$$

We obtain

$$\Delta_n(f;x)=n\int_0^\infty R_m\left(f;x,\gamma(x)+t\right)e^{-nt}dt$$

for all $x\in[\sigma_n,\infty)$. Furthermore we have

$$\left|\Delta_n(f;x)\right|\leq n\int_0^\infty\left|R_m\left(f;x,\gamma(x)+t\right)e^{-nt}\right|dt$$

from (7),

$$
\begin{aligned}
\left|\Delta_n(f;x)\right| &\leq \frac{n\omega^*\left(f;\delta\right)}{m!}\int_0^\infty\left[\left(1+\frac{\left(e^{-t}-e^{-(\gamma(x)+t)}\right)^2}{\delta^2}\right)\left(\gamma(x)+t-x\right)^m\right]e^{-nt}dt \\
&\leq \frac{n\omega^*\left(f;\delta\right)}{m!}\int_0^\infty\left[\left(1+\frac{e^{-2t}-2e^{-\gamma(x)-2t}+e^{-2\gamma(x)-2t}}{\delta^2}\right)\left(t-\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m\right]e^{-nt}dt \\
&\leq \frac{n\omega^*\left(f;\delta\right)}{m!}\int_0^\infty\left[\left(e^{-nt}+e^{-2t-nt}\frac{\left(1-2e^{-\gamma(x)}+e^{-2\gamma(x)}\right)}{\delta^2}\right)\left(t-\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m\right]dt \\
&\leq \frac{n\omega^*\left(f;\delta\right)}{m!}\int_0^\infty\left[\begin{array}{c}\left(e^{-nt}+e^{-t(n+2)}\frac{\left(1-2e^{-\gamma(x)}+e^{-2\gamma(x)}\right)}{\delta^2}\right)\times \\ \left(\binom{m}{0}t^m-\binom{m}{1}t^{m-1}\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\ldots+(-1)^m\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m\right)\end{array}\right]dt
\end{aligned}
$$

$$\leq \frac{n\omega^*(f;\delta)}{m!}\left\{\begin{array}{c} \frac{1}{n^{m+1}}\Gamma(m+1)+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}\frac{1}{(n+2)^{m+1}}\Gamma(m+1) \\ -m\frac{1}{n^m}\Gamma(m)\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)-\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}m\frac{1}{(n+2)^m}\Gamma(m)\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right) \\ \cdots \\ +(-1)^m\left(\frac{1}{n}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}\frac{1}{n+2}\right)\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m \end{array}\right\}$$

$$\leq \frac{\omega^*(f;\delta)}{m!}\left\{\begin{array}{c} \Gamma(m+1)\left[\frac{1}{n^m}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}\frac{n}{(n+2)^{m+1}}\right] \\ -\Gamma(m)\left[m\frac{1}{n^{m-1}}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}m\frac{n}{(n+2)^{m-1}}\right]\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right) \\ \cdots \\ +(-1)^m\left(1+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\delta^2}\frac{n}{n+2}\right)\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m \end{array}\right\}.$$

If we choose $\delta=\sqrt{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}$,

$$\left|\Delta_n(f;x)\right| \leq \frac{\omega^*\left(f;\sqrt{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\right)}{m!}\left\{\begin{array}{c} \Gamma(m+1)\left[\frac{1}{n^m}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\frac{n}{(n+2)^{m+1}}\right] \\ -\Gamma(m)\left[\frac{m}{n^{m-1}}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\frac{mn}{(n+2)^{m-1}}\right]\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right) \\ \cdots \\ +(-1)^m\left(1+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\frac{n}{n+2}\right)\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m \end{array}\right\}$$

$$\leq \frac{\omega^*\left(f;\sqrt{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\right)}{m!}\left\{\begin{array}{c} \Gamma(m+1)\left[\frac{1}{n^m}+\frac{(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})}{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\frac{n}{(n+2)^{m+1}}\right] \\ -\Gamma(m)\left[\frac{m\ln\left(\frac{n}{n-2a}\right)}{2an^{m-1}}+(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})\frac{mn\ln\left(\frac{n}{n-2a}\right)^{1-m}}{2a(n+2)^{m-1}}\right] \\ \cdots \\ +(-1)^m\left(\left(\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)\right)^m+(1-2e^{-\gamma(x)}+e^{-2\gamma(x)})\frac{n}{n+2}\right) \end{array}\right\}.$$

Thus we obtained (5). □

Theorem 4.3. *Let $f\in C^{m-1}[0,\infty)$, such that $f^{(m)}$ exists, $m\in\mathbb{N}$. Furthermore suppose that $f^{(j)}(t)e^{-nt}\in L_1(\mathbb{R})$ for all $j=1,2,...,m-1$, $n>0$. Suppose that there exist $g_{i,n}\geq 0$ $j=1,2,...,m$, $g_{i,n}\in L_1(\mathbb{R})$ such that for each $x\in\mathbb{R}$, we have*

$$\left|f^{(j)}(x+t)\right|e^{-nt}\leq g_{i,n}(t),$$

for almost all $t\in\mathbb{R}$, all $t\in\mathbb{R}$, all $j=1,2,...,m$. Then $f^{(j)}(x+t)e^{-nt}$ defines a Lebesque integrable function with respect to t for each $x\in\mathbb{R}$, all $j=1,2,...,m$, and

$$\left(\mathcal{P}_n^*(f;x)\right)^j=\mathcal{P}_n^*\left(f^{(j)};x\right),$$

for all $x\in\mathbb{R}$, all $j=1,2,...,m$.

Proof. *It can be obtained as in Theorem 11,3 pp.141 of in [3].* □

Theorem 4.4. *Let $f \in C^{m+k}[0,\infty)$ $m \in \mathbb{N}$, $k \in \mathbb{Z}^+$ and $\omega^*\left(f^{(m+i)};\delta\right) < \infty$, $\delta > 0$, for $i = 0,1,2,...k$. We consider the assumptions of Theorem 5 as valid for $m = k$ there. Then*

$$\left|(\Delta_n(f;x))^{(i)}\right| \leq \frac{\omega^*\left(f^{(m+i)};\sqrt{\frac{1}{2a}\ln\left(\frac{n}{n-2a}\right)^m}\right)}{m!}A$$

$\delta > 0$, $i = 1,2,...,k.$, $m \in \mathbb{R}$.

Proof. We have this inequality by using Theorem 5, Theorem 6 and Theorem 11, 16 pp. 149 in [3]. □

5. Weighted Approximation

In this case we consider the exponential weighed space B_{ρ_a} with a fixed $a > 0$, which is the set of all real valued functions defined on $\mathbb{R}^+$ satisfying the condition $\left|f(x)\right| \leq M\rho_a(x)$, $\rho_a(x) = e^{ax}$, where M is a positive constant. Also C_{ρ_a} a denotes the subspace of all continuous functions belonging to B_{ρ_a}. This space is a normed space with the norm

$$\left\|f\right\|_{\rho_a} = \sup_{x\in\mathbb{R}^+}\frac{\left|f(x)\right|}{\rho_a(x)}.$$

Let $C^k_{\rho_a}(\mathbb{R}^+)$ be the subspace of all functions $f \in C_{\rho_a}(\mathbb{R}^+)$ such that $\lim_{x\to\infty}\frac{|f(x)|}{\rho_a(x)} = k$, where k is a positive constant.

Now we shows the conditions under which both classes of operators maps $C_{\rho_a}(\mathbb{R}^+)$ to $B_{\rho_b}(\mathbb{R}^+)$ with the uniformly bounded norm.

Lemma 5.1. *Let $0 < a < b$. P^*_n is a positive linear operator from the space C_{ρ_a} into C_{ρ_b} such that the norms $\left\|P^*_n\right\|_{C_{\rho_a}\to C_{\rho_b}}$ are uniformly bounded.*

Proof.

$$\begin{aligned}\left\|\mathcal{P}^*_n(\rho_a;x)\right\|_{\rho_b} &= \sup_{x\in[\sigma_n,\infty)}\frac{\left|e^{a\gamma_n(x)}\frac{n}{n-a}\right|}{\rho_b(x)} \\ &= \sup_{x\in[\sigma_n,\infty)}\frac{\sqrt{n(n-2a)}}{n-a}e^{(a-b)x} = 1,\end{aligned} \tag{8}$$

the operators $\mathcal{P}^*_n$ maps the space C_{ρ_a} into C_{ρ_b}. Also, for $x \in [\sigma_n,\infty)$ we obtain,

$$\begin{aligned}\frac{P^*_n(\rho_a;x)-\rho_a(x)}{\rho_b(x)} &= \frac{\mathcal{P}^*_n(\rho_a;x)-\rho_a(x)}{\rho_b(x)} = \frac{\left|\frac{\sqrt{n(n-2a)}}{n-a}e^{ax}-e^{ax}\right|}{e^{bx}} \\ &= e^{(a-b)x}\left|\frac{\sqrt{n(n-2a)}}{n-a}-1\right| \\ &= e^{(a-b)x}\left(1-\frac{\sqrt{n(n-2a)}}{n-a}\right)\end{aligned}$$

and

$$\left\|P^*_n(\rho_a;x)-\rho_a\right\|_{\rho_b} = \left(1-\frac{\sqrt{n(n-2a)}}{n-a}\right)\sup_{x\in[\sigma_n,\infty)}e^{(a-b)x}$$

$$\leq \quad 1-\frac{\sqrt{n\,(n-2a)}}{n-a},$$

which tends to zero when $n\to\infty$. This shows that the norms $\left\|P_n^*\right\|_{C_{\rho_a}\to C_{\rho_b}}$ are uniformly bounded. □

Theorem 5.2. *Let* $0<a<b$. *For* $f\in C_{\rho_a}(\mathbb{R}^+)$, *we have*

$$\lim_{n\to\infty}\left\|P_n^*(f;x)-f(x)\right\|_{\rho_b}=0.$$

Proof. Using the theorem in [11] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|P_n^*(\rho_a^{\nu};x)-\rho_a^{\nu}\right\|_{\rho_b}=0,\qquad \nu=0,1,2.$$

Since $\mathcal{P}_n^*(1;x)=1$ and $\mathcal{P}_n^*\left(e^{2at};x\right)=e^{2ax}$, for $\nu=0$ and $\nu=2$, the above equality is true. For $\nu=0$, from (8), the proof is obvious. □

6. Global Smoothness Preservation Results

Modulus of continuity with exponential weight is a good instrument to estimate the rate of convergence of sequences of linear positive operators in the exponential weighted spaces. For this purpose we use the weighted modulus of continuity defined by

$$\widetilde{\omega}(f;\delta)=\sup_{\substack{|t-x|\leq\delta\\ x\geq 0}}\frac{\left|f(t)-f(x)\right|}{e^{at}+e^{ax}},\qquad \delta\geq 0 \tag{9}$$

for $f\in C_{\rho_a}^k(\mathbb{R}^+)$ (See [7]).

By using the above weighted modulus of continuity, the result regarding global smoothness preservation properties for the operators of P_n^* will be given as follows:

Theorem 6.1. *Let* $\delta>0$ *and* $\widetilde{\omega}(f;\delta)<\infty$ *for* $f\in C[0,\infty)$, *then we have*

$$\widetilde{\omega}(P_n^*(f);\delta)\leq C\widetilde{\omega}(f;\delta).$$

This inequality is obtained with sharpness, i.e., $f(x)=e^{ax}$, $a>0$.

Proof.

$$P_n^*(f;x)-P_n^*(f;y)=n\int_0^{\infty}\left(f(\gamma_n(x)+t)-f(\gamma_n(y)+t)\right)e^{-nt}dt.$$

Thus, we have for $n>2a$

$$\begin{aligned}
&\left|P_n^*(f;x)-P_n^*(f;y)\right|\\
&\leq n\int_0^{\infty}\left|\frac{f(\gamma_n(x)+t)-f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)}+e^{a(\gamma_n(y)+t)}}\right|\left|\left(e^{a(\gamma_n(x)+t)}+e^{a(\gamma_n(y)+t)}\right)\right|e^{-nt}dt\\
&=n\int_0^{\infty}\left|\frac{f(\gamma_n(x)+t)-f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)}+e^{a(\gamma_n(y)+t)}}\right|e^{a(\gamma_n(x)+t)}e^{-nt}dt+n\int_0^{\infty}\left|\frac{f(\gamma_n(x)+t)-f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)}+e^{a(\gamma_n(y)+t)}}\right|e^{a(\gamma_n(y)+t)}e^{-nt}dt
\end{aligned}$$

$$\leq \sup_{\substack{|x-y|\leq\delta \\ x\geq 0}} \left| \frac{f(\gamma_n(x)+t) - f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)} + e^{a(\gamma_n(y)+t)}} \right| P_n^*\left(e^{at};x\right) + \sup_{\substack{|x-y|\leq\delta \\ x\geq 0}} \left| \frac{f(\gamma_n(x)+t) - f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)} + e^{a(\gamma_n(y)+t)}} \right| P_n^*\left(e^{at};y\right)$$

$$= \sup_{\substack{|x-y|\leq\delta \\ x\geq 0}} \left| \frac{f(\gamma_n(x)+t) - f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)} + e^{a(\gamma_n(y)+t)}} \right| \left(P_n^*\left(e^{at};x\right) + P_n^*\left(e^{at};y\right)\right)$$

$$= \sup_{\substack{|x-y|\leq\delta \\ x\geq 0}} \left| \frac{f(\gamma_n(x)+t) - f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)} + e^{a(\gamma_n(y)+t)}} \right| \left(\frac{\sqrt{n(n-2a)}}{n-a} e^{ax} + \frac{\sqrt{n(n-2a)}}{n-a} e^{ay} \right)$$

$$= \sup_{\substack{|x-y|\leq\delta \\ x\geq 0}} \left| \frac{f(\gamma_n(x)+t) - f(\gamma_n(y)+t)}{e^{a(\gamma_n(x)+t)} + e^{a(\gamma_n(y)+t)}} \right| \frac{\sqrt{n(n-2a)}}{n-a} \left(e^{ax} + e^{ay}\right).$$

$$\frac{\left|\mathcal{P}_n^*(f;x) - \mathcal{P}_n^*(f;y)\right|}{e^{ax} + e^{ay}} \leq \widetilde{\omega}\left(f; \left|a\left(\gamma_n(x) - \gamma_n(y)\right)\right|\right) \frac{\sqrt{n(n-2a)}}{n-a},$$

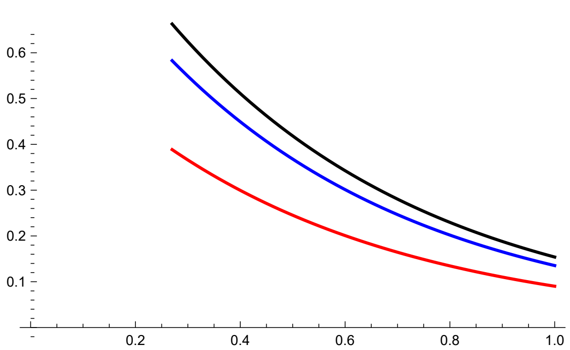
Thus for a sufficiently large n; we get

$$\widetilde{\omega}\left(P_n^*(f);\delta\right) \leq C\widetilde{\omega}(f;\delta)$$

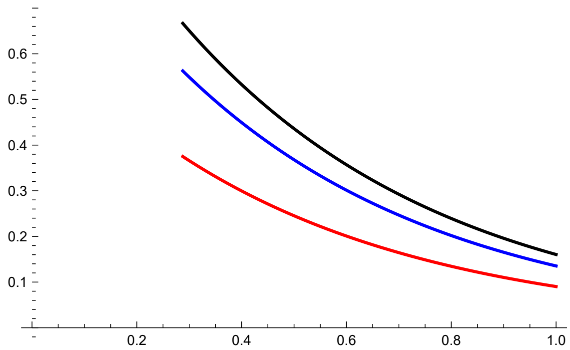
where C is a constant. Thus the proof is completed. □

7. Graphical Applications

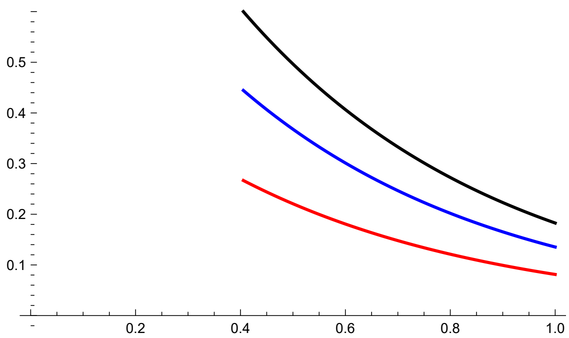
Figures illustrates the behavior of the approximation process for different choices of parameters. Each curve corresponds to a distinct parameter setting, as described in the related subsections. These visual representations clearly demonstrate how sensitive the operators are to parameter changes, supporting the theoretical findings discussed earlier.



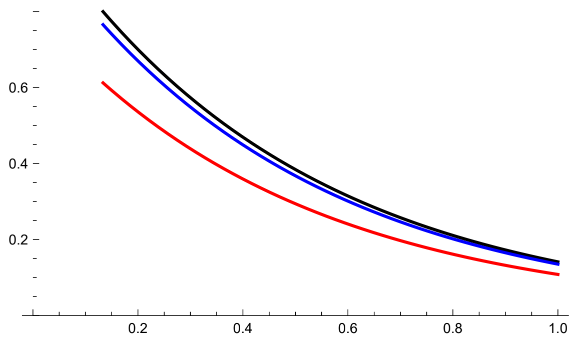
(a) $\mathcal{P}_n^*(f)$ (black), $\mathcal{P}_n(f)$ (red) and f (blue) for $f(x) = e^{-2x}$, $n = 4$ and $a = \frac{1}{4}$



(b) $\mathcal{P}_n^*(f)$ (black), $\mathcal{P}_n(f)$ (red) and f (blue) for $f(x) = e^{-2x}$, $n = 4$ and $a = \frac{1}{2}$



(a) $\mathcal{P}_n^*(f)$ (black), $\mathcal{P}_n(f)$ (red) and f (blue) for $f(x) = e^{-2x}$, $n = 3$ and $a = \frac{1}{2}$



(b) $\mathcal{P}_n^*(f)$ (black), $\mathcal{P}_n(f)$ (red) and f (blue) for $f(x) = e^{-2x}$, $n = 8$ and $a = \frac{1}{2}$

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