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Integral inequalities of Hermite–Hadamard type for (α, m)-GA-F-convex functions

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Abstract. In the paper, the authors introduce a notion of (α, m)-geometric-arithmetic-F-convex functions and, via an integral identity and other analytic techniques, establish several integral inequalities of the Hermite–Hadamard type for (α, m)-GA-F-convex functions.

1. Introduction

The following definitions are well known in the literature.

Definition 1 ([8, 9]). Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} = (-\infty, \infty)$. If for any $x, y \in I$ and $t \in [0, 1]$, the inequality

$$f(x^t y^{1-t}) \le t f(x) + (1 - t) f(y)$$

is valid, then we call f a geometric-arithmetic convex function on I. For simplicity, we call it a GA-convex function.

In the papers [1, 2], as a generalization of strongly convex functions, the notion of F-convex functions was defined as follows.

Definition 2 ([1, Definition 1] and [2, Definition 1]). Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $f : I \to \mathbb{R}$ is called to be F-convex if

$$f(tx + (1 - t)y) \le t f(x) + (1 - t) f(y) - t(1 - t)F(x - y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

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Definition 3 ([7]). For $f : [0, b] \to \mathbb{R}$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1 - t)y) \le t^\alpha f(x) + m(1 - t^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

In the paper [14], the notion of GA-F-convex functions was defined.

Definition 4 ([14, Definition 2.2]). Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is called GA-F-convex if

$$f(x^t y^{1-t}) \le tf(x) + (1 - t)f(y) - t(1 - t)F(x - y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Now we recall some Hermite–Hadamard type integral inequalities for GA-convex functions and F-convex functions in the form of theorems.

Theorem 1 ([6, Theorem 1]). *Let $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be GA-convex. Then*

$$f\left(\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \left(\frac{1}{\ln b - \ln a} - \frac{a}{b-a}\right)f(a) + \left(\frac{b}{b-a} - \frac{1}{\ln b - \ln a}\right)f(b).$$

Theorem 2 ([22, Theorem 3.1]). *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on the interior I° of an interval I, $a, b \in I$ with $a < b$, and $f' \in L([a, b])$. If $|f'|^q$ is GA-convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{bf(b) - af(a)}{b-a} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{[A(a, b)]^{1-1/q}}{[2(b-a)]^{1/q}}\{[L(a^2, b^2) - a^2]|f'(a)|^q + [b^2 - L(a^2, b^2)]|f'(b)|^q\}^{1/q},$$

where

$$A(u, v) = \frac{u+v}{2} \quad \text{and} \quad L(u, v) = \begin{cases} \dfrac{u - v}{\ln u - \ln v}, & u \ne v \\ u, & u = v \end{cases} \tag{1.2}$$

for $u, v > 0$ are the arithmetic mean and logarithmic mean, respectively.

Theorem 3 ([2, Theorem 5]). *Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function which is integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = \frac{\sup I - \inf I}{2}$. If an F-convex function $f : I \to \mathbb{R}$ is one-sided differentiable and $f_- \le f_+$, then*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b F\left(x - \frac{a+b}{2}\right)\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a) + f(b)}{2} - \frac{1}{6}F(a - b)$$

for all $a, b \in I$ with $a \ne b$.

Theorem 4 ([2, Theorem 6]). *Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function which is integrable on each compact subinterval of $(-\alpha, \alpha)$, where $\alpha = \sup I - \inf I$. If $f : I \to \mathbb{R}$ is an F-convex function, then*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{4(b-a)}\int_a^b F(2x - a - b)\,\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a) + f(b)}{2} - \frac{1}{6}F(a - b)$$

for all $a, b \in I$ with $a \ne b$.

For recent developments related to integral inequalities of the Hermite–Hadamard type for various convex functions, please refer to [3–5, 10–13, 15–21] and closely related references therein.

In this paper, we will define a new notion of (α, m)-GA-F-convex functions and establish several integral inequalities of the Hermite–Hadamard type for (α, m)-GA-F-convex functions.

2. A new definition and lemmas

Motivated by Definitions 2 to 4, we now define a new notion of (α, m)-GA-F-convex functions as follows.

Definition 5. Let $F : \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ be a fixed function and some constant $\alpha, m \in (0, 1]$. A function $f : (0, b] \to \mathbb{R}$ with $b > 0$ is called to be (α, m)-GA-F-convex if

$$f(x^t y^{m(1-t)}) \le t^\alpha f(x) + m(1 - t^\alpha) f(y) - t(1 - t)F(x - y) \tag{2.1}$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$.

Remark 1. If $f : (0, b] \to \mathbb{R}$ is an $(1, 1)$-GA-F-convex function, then f is a GA-F-convex function on $(0, b]$; see Definition 4.

Example 1. Let

$$f(x) = -(\ln x)^{1/100} \quad \text{and} \quad F(x - y) = \frac{|x - y|^{1/100}}{100\,\mathrm{e}^{1/100}}$$

for $x, y \in [1, \mathrm{e}]$. For $x, y \in [1, \mathrm{e}]$, assume $x < y$ without loss of generality, using Lagrange's mean value theorem, we have

$$\ln x - \ln y = \frac{x - y}{\xi}$$

for at least one interior point $\xi \in (x, y)$. Therefore, we acquire

$$\mathrm{e}\,|\ln x - \ln y| \ge \xi |\ln x - \ln y| = |x - y|.$$

For $\alpha = m = \frac{1}{2}$, we have

$$\begin{aligned}
& t^\alpha f(x) + m(1 - t^\alpha) f(v) - t(1 - t)F(x - y) - f(x^t y^{m(1-t)}) \\
&= -t^{1/2}(\ln x)^{1/100} - \frac{1}{2}(1 - t^{1/2})(\ln y)^{1/100} + [\ln(x^t y^{(1-t)/2})]^{1/100} - \frac{t(1-t)}{100\,\mathrm{e}^{1/100}}|x - y|^{1/100} \\
&\ge -t^{1/2}(\ln x)^{1/100} - \frac{1}{2}(1 - t^{1/2})(\ln y)^{1/100} + \left[t \ln x + \frac{1}{2}(1 - t)\ln y\right]^{1/100} - \frac{t(1-t)}{100}|\ln x - \ln y|^{1/100} \\
&= \left[\left(tu + \frac{1}{2}(1 - t)\right)^{1/100} - t^{1/2}u^{1/100} - \frac{1}{2}(1 - t^{1/2}) - \frac{t(1-t)}{100}(1 - u)^{1/100}\right](\ln y)^{1/100} \\
&\ge 0,
\end{aligned}$$

where $u = \frac{\ln x}{\ln y}$. As a result, the inequality (2.1) is valid. Consequently, the function $f(x) = -(\ln x)^{1/100}$ is $(\frac{1}{2}, \frac{1}{2})$-GA-$F$-convex on the interval $[1, \mathrm{e}]$.

This simple example implies that the (α, m)-GA-F-convex function surely exists and that Definition 5 is really significant.

For establishing integral inequalities of the Hermite–Hadamard type for (α, m)-GA-F-convex functions, we need the following lemmas.

Lemma 1 ([17, Lemma 2.1]). *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on the interior I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}\,\mathrm{d}t = \frac{\ln b - \ln a}{2}\int_0^1 t[a^{1-t}b^t f'(a^{1-t}b^t) - a^t b^{1-t} f'(a^t b^{1-t})]\,\mathrm{d}t. \tag{2.2}$$

Via an integral transformation, Lemma 1 is equivalently reduced to

Lemma 2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on the interior I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}\,\mathrm{d}t = \frac{\ln b - \ln a}{2}\int_0^1 (1-2t)a^t b^{1-t} f'(a^t b^{1-t})\,\mathrm{d}t.$$

Lemma 3. *Let $x, y \in \mathbb{R}_+$ with $x < y$ and $r > -1$. Then*

$$Q(x,y) \triangleq \int_0^1 t x^{1-t} y^t\,\mathrm{d}t = \frac{y - L(x,y)}{\ln y - \ln x},$$
$$R(x,y) \triangleq \int_0^1 t^2(1-t) x^t y^{1-t}\,\mathrm{d}t = \frac{2y + 4x + x(\ln y - \ln x) - 6L(x,y)}{(\ln y - \ln x)^3},$$
$$S(x,y,r) \triangleq \int_0^1 t^{r+1} x^t y^{1-t}\,\mathrm{d}t = \frac{[\Gamma(r+2) - \Gamma(r+2, \ln y - \ln x)]y}{(\ln y - \ln x)^{r+2}},$$
$$T(x,y,r) \triangleq \int_0^1 t(1-t^r) x^t y^{1-t}\,\mathrm{d}t = \frac{L(x,y) + x}{\ln y - \ln x} - S(x,y,r),$$

where $L(u, v)$ is defined by (1.2) *and the incomplete gamma function $\Gamma(x, z)$ is given by*

$$\Gamma(x,z) = \int_z^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \quad \Re(x) > 0, \quad z \ge 0$$

with $\Gamma(x) = \Gamma(x, 0)$.

Proof. This follows from a straightforward computation. □

3. Main Results

In this section, we establish several integral inequalities of the Hermite–Hadamard type for (α, m)-GA-F-convex functions.

Theorem 5. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $(\alpha, m) \in (0, 1]^2$. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ is an (α, m)-GA-F-convex function on $\mathbb{R}_+$ such that $f \in L_1([0, \max\{b, b^{1/m}\}])$ for $a, b \in \mathbb{R}_+$ with $a < b$. Then*

$$f(\sqrt{ab}) + \frac{1}{4(\ln b - \ln a)}\int_0^1 \frac{F(x - \frac{ab}{x})}{x}\,\mathrm{d}x \le \frac{1}{2^\alpha(\ln b - \ln a)}\left[\int_a^b \frac{f(x)}{x}\,\mathrm{d}x + m(2^\alpha - 1)\int_a^b \frac{f(x^{1/m})}{x}\,\mathrm{d}x\right] \quad (3.1)$$

and

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \le \frac{f(a) + m\alpha f(b^{1/m})}{\alpha + 1} - \frac{1}{6}F(a - b).$$

Proof. Using the (α, m)-GA-F-convexity of $f(x)$ on $\mathbb{R}_+$, we have

$$f(\sqrt{ab}) = f(\sqrt{a^t b^{1-t} a^{1-t} b^t}) \le \frac{1}{2^\alpha} f(a^t b^{1-t}) + \frac{m(2^\alpha - 1)}{2^\alpha} f((a^{1-t} b^t)^{1/m}) - \frac{1}{4}F(a^t b^{1-t} - a^{1-t} b^t)$$

for $t \in [0, 1]$. Integrating on both sides gives

$$f(\sqrt{ab}) = \int_0^1 f(\sqrt{ab})\,\mathrm{d}t \le \int_0^1 \left[\frac{1}{2^\alpha} f(a^t b^{1-t}) + \frac{m(2^\alpha - 1)}{2^\alpha} f((a^{1-t} b^t)^{1/m}) - \frac{1}{4}F(a^t b^{1-t} - a^{1-t} b^t)\right]\mathrm{d}t. \quad (3.2)$$

Making the integral transformation $u = 1 - t$ for $t \in [0, 1]$, we obtain

$$\int_0^1 f((a^{1-t}b^t)^{1/m})\,\mathrm{d}t = \int_0^1 f((a^t b^{1-t})^{1/m})\,\mathrm{d}t.$$

Letting $x = a^t b^{1-t}$ for $0 \le t \le 1$ yields

$$\begin{aligned}
&\int_0^1 \Big[\frac{1}{2^\alpha} f(a^t b^{1-t}) + \frac{m(2^\alpha - 1)}{2^\alpha} f((a^{1-t}b^t)^{1/m}) - \frac{1}{4} F(a^t b^{1-t} - a^{1-t}b^t)\Big]\,\mathrm{d}t \\
&= \frac{1}{2^\alpha}\int_0^1 \frac{f(a^t b^{1-t})a^t b^{1-t}}{a^t b^{1-t}}\,\mathrm{d}t + \frac{m(2^\alpha - 1)}{2^\alpha}\int_0^1 \frac{f((a^t b^{1-t})^{1/m})a^t b^{1-t}}{a^t b^{1-t}}\,\mathrm{d}t - \frac{1}{4}\int_0^1 \frac{F(a^t b^{1-t} - a^{1-t}b^t)a^t b^{1-t}}{a^t b^{1-t}}\,\mathrm{d}t \quad (3.3)\\
&= \frac{1}{2^\alpha(\ln b - \ln a)}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x + \frac{m(2^\alpha - 1)}{2^\alpha(\ln b - \ln a)}\int_a^b \frac{f(x^{1/m})}{x}\,\mathrm{d}x - \frac{1}{4(\ln b - \ln a)}\int_0^1 \frac{F(x - \frac{ab}{x})}{x}\,\mathrm{d}x.
\end{aligned}$$

From the inequalities (3.2) to (3.3), the inequality (3.1) follows immediately.

On the other hand, putting $x = a^t b^{1-t}$ for $0 \le t \le 1$, by the (α, m)-GA-F-convexity of $f(x)$ on $\mathbb{R}_+$, we obtain

$$\begin{aligned}
\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x = \int_0^1 f(a^t b^{1-t})\,\mathrm{d}t &= \int_0^1 f(a^t b^{\frac{1}{m}\cdot m(1-t)})\,\mathrm{d}t \\
&\le \int_0^1 [t^\alpha f(a) + m(1 - t^\alpha) f(b^{1/m}) - t(1-t)F(a-b)]\,\mathrm{d}t \\
&= \frac{f(a) + m\alpha f(b^{1/m})}{\alpha + 1} - \frac{1}{6}F(a - b).
\end{aligned}$$

The proof of Theorem 5 is complete. □

Letting $m = 1$ in Theorem 5, we have

Corollary 1. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $\alpha \in (0, 1]$. Suppose that $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is an α-GA-F-convex function on $[a, b]$ such that $f \in L_1([a, b])$ with $a < b$. Then*

$$f(\sqrt{ab}) + \frac{1}{4(\ln b - \ln a)}\int_0^1 \frac{F(x - \frac{ab}{x})}{x}\,\mathrm{d}x \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \le \frac{f(a) + \alpha f(b)}{\alpha + 1} - \frac{1}{6}F(a - b).$$

Theorem 6. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $(\alpha, m) \in (0, 1]^2$. Suppose that $f : (0, b^*] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable such that $a, b \in (0, b^*]$ with $a < b$ and $\max\{b, b^{1/m}\} \le b^*$. If $|f'|^q$ is (α, m)-GA-F-convex on $(0, \max\{b, b^{1/m}\}]$ for $q \ge 1$ and $f' \in L_1((0, \max\{b, b^{1/m}\}])$, then*

$$\begin{aligned}
\left|\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\{Q^{1-1/q}(a, b)[S(a, b; \alpha)|f'(a)|^q + mT(a, b; \alpha)|f'(b^{1/m})|^q \\
&- R(a, b)F(a - b^{1/m})]^{1/q} + Q^{1-1/q}(b, a)[mT(a, b; \alpha)|f'(a^{1/m})|^q + S(a, b; \alpha)|f'(b)|^q - R(a, b)F(b - a^{1/m})]^{1/q}\}, \quad (3.4)
\end{aligned}$$

where $Q(x, y)$, $S(x, y; \alpha)$, $T(x, y; \alpha)$, $R(x, y)$are defined as in Lemma 3.

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned}
\left|\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\int_0^1 t[a^t b^{1-t}|f'(a^t b^{1-t})| + a^{1-t}b^t|f'(a^{1-t}b^t)|]\,\mathrm{d}t \\
&\le \frac{\ln b - \ln a}{2}\Big(\int_0^1 ta^t b^{1-t}\,\mathrm{d}t\Big)^{1-1/q}\Big[\int_0^1 ta^t b^{1-t}|f'(a^t b^{1-t})|^q\,\mathrm{d}t\Big]^{1/q} \quad (3.5)\\
&+ \frac{\ln b - \ln a}{2}\Big(\int_0^1 ta^{1-t}b^t\,\mathrm{d}t\Big)^{1-1/q}\Big[\int_0^1 ta^{1-t}b^t|f'(a^{1-t}b^t)|^q\,\mathrm{d}t\Big]^{1/q}.
\end{aligned}$$

Using the (α, m)-GA-F-convexity of $|f'|^q$ on $(0, \max\{b, b^{1/m}\}]$ and Lemma 3, we obtain

$$\begin{aligned}\int_0^1 ta^t b^{1-t}|f'(a^t b^{1-t})|^q \,\mathrm{d}t &= \int_0^1 ta^t b^{1-t}|f'(a^t b^{\frac{1}{m}\cdot m(1-t)})|^q \,\mathrm{d}t \\ &\le \int_0^1 ta^t b^{1-t}[t^\alpha|f'(a)|^q + m(1-t^\alpha)|f'(b^{1/m})|^q - t(1-t)F(a - b^{1/m})] \,\mathrm{d}t \\ &= S(a, b; \alpha)|f'(a)|^q + mT(a, b; \alpha)|f'(b^{1/m})|^q - R(a, b)F(a - b^{1/m})\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_0^1 ta^{1-t} b^t|f'(a^{1-t} b^t)|^q \,\mathrm{d}t &\le \int_0^1 ta^{1-t} b^t[m(1-t^\alpha)|f'(a^{1/m})|^q + t^\alpha|f'(b)|^q - t(1-t)F(b - a^{1/m})] \,\mathrm{d}t \\ &= mT(a, b; \alpha)|f'(a^{1/m})|^q + S(a, b; \alpha)|f'(b)|^q - R(a, b)F(b - a^{1/m}).\end{aligned} \tag{3.7}$$

Substituting the inequalities (3.6) and (3.7) into the inequality (3.5) leads to the inequality (3.4). The proof of Theorem 6 is complete. □

In Theorem 6, when $m = 1$, we have

Corollary 2. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $\alpha \in (0, 1]$. Suppose that $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable for $a, b \in \mathbb{R}_+$ with $a < b$. If $|f'|^q$ is $(\alpha, 1)$-GA-F-convex on $[a, b]$ for $q \ge 1$ and $f' \in L_1([a, b])$, then*

$$\begin{aligned}\left|\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\{Q^{1-1/q}(a, b)[S(a, b; \alpha)|f'(a)|^q + T(a, b; \alpha)|f'(b)|^q \\ &- R(a, b; \alpha)F(a - b)]^{1/q} + Q^{1-1/q}(b, a)[T(a, b; \alpha)|f'(a)|^q + S(a, b; \alpha)|f'(b)|^q - R(a, b)F(b - a)]^{1/q}\},\end{aligned}$$

where $Q(x, y)$, $S(x, y; \alpha)$, $T(x, y; \alpha)$, $R(x, y)$ are defined as in Lemma 3.

In Theorem 6, if $\alpha = m = 1$, we obtain

Corollary 3. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function. Suppose that $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable for $a, b \in \mathbb{R}_+$ with $a < b$. If $|f'|^q$ is GA-F-convex on $[a, b]$ for $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\{Q^{1-1/q}(a, b)[N(a, b)|f'(a)|^q + M(a, b)|f'(b)|^q \\ &- R(a, b)F(a - b)]^{1/q} + Q^{1-1/q}(b, a)[M(b, a)|f'(a)|^q + N(b, a)|f'(b)|^q - R(a, b)F(b - a)]^{1/q}\},\end{aligned}$$

where

$$\begin{aligned}N(x, y) &\triangleq \int_0^1 t(1-t)x^t y^{1-t}\,\mathrm{d}t = \frac{x + y - 2L(x, y)}{(\ln y - \ln x)}, \\ M(x, y) &\triangleq \int_0^1 t^2 x^t y^{1-t}\,\mathrm{d}t = \frac{2L(x, y) - 2x - x(\ln y - \ln x)}{(\ln y - \ln x)^2},\end{aligned} \tag{3.8}$$

the logarithmic mean $L(u, v)$ is defined by (1.2)*, and $Q(x, y)$, $R(x, y)$, $K(x, y)$ are defined as in Lemma 3.*

Theorem 7. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $(\alpha, m) \in (0, 1]^2$. Suppose that $f : (0, b^*] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable such that $a, b \in (0, b^*]$ with $a < b$ and $\max\{b, b^{1/m}\} \le b^*$. If $|f'|^q$ is (α, m)-GA-F-convex on $(0, \max\{b, b^{1/m}\}]$ for $q > 1$ and $f \in L_1((0, \max\{b, b^{1/m}\}])$, then*

$$\begin{aligned}\left|\frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}[S(a^q, b^q; \alpha - 1)|f'(a)|^q \\ &+ m[L(a^q, b^q) - S(a^q, b^q; \alpha - 1)]|f'(b^{1/m})|^q - N(a^q, b^q)F(a - b^{1/m})]^{1/q},\end{aligned} \tag{3.9}$$

where $L(u, v)$ is defined by (1.2)*, $S(x, y; \alpha)$ is defined as in Lemma 3, and $N(x, y)$ is defined by* (3.8).

Proof. From Lemma 2 and Hölder's integral inequality, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\right|\le\frac{\ln b-\ln a}{2}\int_0^1|1-2t|a^tb^{1-t}|f'(a^tb^{1-t})|\,\mathrm{d}t$$
$$\le\frac{\ln b-\ln a}{2}\left(\int_0^1|1-2t|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 a^{qt}b^{q(1-t)}|f'(a^tb^{1-t})|^q\,\mathrm{d}t\right]^{1/q},\quad(3.10)$$

where

$$\int_0^1|1-2t|^{q/(q-1)}\,\mathrm{d}t=\frac{q-1}{2q-1}.\quad(3.11)$$

Making use of the (α,m)-GA-F-convexity of $|f'|^q$ on $(0,\max\{b,b^{1/m}\}]$ and Lemma 3, we obtain

$$\int_0^1 a^{qt}b^{q(1-t)}|f'(a^tb^{1-t})|^q\,\mathrm{d}t\le\int_0^1 a^{qt}b^{q(1-t)}[t^\alpha|f'(a)|^q+m(1-t^\alpha)|f'(b^{1/m})|^q-t(1-t)F(a-b^{1/m})]\,\mathrm{d}t$$
$$=S(a^q,b^q;\alpha-1)|f'(a)|^q+m[L(a^q,b^q)-S(a^q,b^q;\alpha-1)]|f'(b^{1/m})|^q-N(a^q,b^q)F(a-b^{1/m}).\quad(3.12)$$

Combining two inequalities (3.11) and (3.12) with (3.10), the inequality (3.9) follows readily. The proof of Theorem 7 is complete. □

Putting $m=1$ in Theorem 7, we derive

Corollary 4. *Let $F:\mathbb{R}\to\mathbb{R}_0$ be a fixed function and $\alpha\in(0,1]$. Suppose that $f:[a,b]\subseteq\mathbb{R}_+\to\mathbb{R}$ is differentiable for $a,b\in\mathbb{R}_+$ with $a<b$. If $|f'|^q$ is $(\alpha,1)$-GA-F-convex on $[a,b]$ for $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\right|\le\frac{\ln b-\ln a}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times[S(a^q,b^q;\alpha-1)|f'(a)|^q+[L(a^q,b^q)-S(a^q,b^q;\alpha-1)]|f'(b)|^q-N(a^q,b^q)F(a-b)]^{1/q},$$

where $L(u,v)$ is defined by (1.2)*, $S(x,y;\alpha)$ is defined as in Lemma 3, and $N(x,y)$ is defined by* (3.8)*.*

Theorem 8. *Let $F:\mathbb{R}\to\mathbb{R}_0$ be a given function and $(\alpha,m)\in(0,1]^2$. Suppose that $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is differentiable and $a,b\in I$ with $a<b$. If $|f'|^q$ is (α,m)-GA-F-convex on I for $q>1$ and $f\in L_1((0,\max\{b,b^{1/m}\}])$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\right|\le\frac{\ln b-\ln a}{2}[L(a^{q/(q-1)},b^{q/(q-1)})-\frac{1}{2}L(a^{q/2(q-1)},b^{q/2(q-1)})]^{1-1/q}$$
$$\times\left[\frac{(2^\alpha\alpha+1)|f'(a)|^q}{2^\alpha(\alpha+1)(\alpha+2)}+\frac{2[2^\alpha(\alpha^2+\alpha+1)-2]|f'(b^{1/m})|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)}-\frac{F(a-b^{1/m})}{16}\right]^{1/q},\quad(3.13)$$

where $L(u,v)$ is defined as in Lemma 3.

Proof. Using Lemma 2 and Hölder's integral inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\right|\le\frac{\ln b-\ln a}{2}\int_0^1|1-2t|a^tb^{1-t}|f'(a^tb^{1-t})|\,\mathrm{d}t$$
$$\le\frac{\ln b-\ln a}{2}\left(\int_0^1|1-2t|(a^tb^{1-t})^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1|1-2t||f'(a^tb^{1-t})|^q\,\mathrm{d}t\right]^{1/q}.\quad(3.14)$$

By the (α,m)-GA-F-convexity of $|f'|^q$ on $(0,\max\{b,b^{1/m}\}]$, we have

$$\int_0^1|1-2t||f'(a^tb^{1-t})|^q\,\mathrm{d}t\le\int_0^1|1-2t|[t^\alpha|f'(a)|^q+m(1-t^\alpha)|f'(b^{1/m})|^q-t(1-t)F(a-b^{1/m})]\,\mathrm{d}t$$
$$=\frac{(2^\alpha\alpha+1)|f'(a)|^q}{2^\alpha(\alpha+1)(\alpha+2)}+\frac{m[2^\alpha(\alpha^2+\alpha+1)-2]|f'(b^{1/m})|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)}-\frac{F(a-b^{1/m})}{16}\quad(3.15)$$

and

$$\int_0^1 |1-2t|(a^t b^{1-t})^{q/(q-1)}\,\mathrm{d}t = L(a^{q/(q-1)}, b^{q/(q-1)}) - \frac{L(a^{q/2(q-1)}, b^{q/2(q-1)})}{2}. \tag{3.16}$$

Substituting (3.15) and (3.16) into the inequality (3.14) arrives at the inequality (3.13). The proof of Theorem 8 is thus complete. □

Taking $m = 1$ in Theorem 8, we obtain

Corollary 5. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $\alpha \in (0, 1]$. Suppose that $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable for $a, b \in \mathbb{R}_+$ with $a < b$. If $|f'|^q$ is $(\alpha, 1)$-GA-F-convex on $[a, b]$ for $q > 1$ and $f' \in L_1([a, b])$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\Big[L(a^{q/(q-1)}, b^{q/(q-1)}) - \frac{1}{2}L(a^{q/2(q-1)}, b^{q/2(q-1)})\Big]^{1-1/q} \\ &\quad\times\left[\frac{(2^\alpha\alpha+1)|f'(a)|^q}{2^\alpha(\alpha+1)(\alpha+2)} + \frac{[2^\alpha(\alpha^2+\alpha+1)-2]|f'(b)|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)} - \frac{F(a-b)}{16}\right]^{1/q},\end{aligned} \tag{3.17}$$

where $L(u, v)$ is defined as in Lemma 3.

Theorem 9. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a given function and $(\alpha, m) \in (0, 1]^2$. Suppose that $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable and $a, b \in I$ with $a < b$. If $|f'|^q$ is (α, m)-GA-F-convex on I for $q > 1$ and $f \in L_1((0, \max\{b, b^{1/m}\}])$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}L^{1-1/q}(a^{q/(q-1)}, b^{q/(q-1)})\bigg\{\bigg[\frac{|f'(a)|^q}{q+\alpha+1} + \frac{m|f'(b^{1/m})|^q}{(q+\alpha+1)(q+1)} \\ &\quad - \frac{F(a-b^{1/m})}{(q+5)(q+1)}\bigg]^{1/q}\bigg[\frac{m|f'(a^{1/m})|^q}{(q+\alpha+1)(q+1)} + \frac{|f'(b)|^q}{q+\alpha+1} - \frac{F(b-a^{1/m})}{(q+5)(q+1)}\bigg]^{1/q}\bigg\},\end{aligned} \tag{3.18}$$

where $L(u, v)$ is defined as in Lemma 3.

Proof. In view of Lemma 1 and Hölder's integral inequality, we acquire

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| &\le \frac{\ln b - \ln a}{2}\int_0^1 t[a^t b^{1-t}|f'(a^t b^{1-t})| + a^{1-t}b^t|f'(a^{1-t}b^t)|]\,\mathrm{d}t \\ &\le \frac{\ln b - \ln a}{2}\bigg[\int_0^1 (a^t b^{1-t})^{q/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_0^1 t^q|f'(a^t b^{1-t})|^q\,\mathrm{d}t\bigg]^{1/q} \\ &\quad + \frac{\ln b - \ln a}{2}\bigg[\int_0^1 (a^{1-t}b^t)^{q/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_0^1 t^q|f'(a^{1-t}b^t)|^q\,\mathrm{d}t\bigg]^{1/q}.\end{aligned} \tag{3.19}$$

Since $|f'|^q$ is (α, m)-GA-F-convex on $(0, \max\{b, b^{1/m}\}]$, we have

$$\begin{aligned}\int_0^1 t^q|f'(a^t b^{1-t})|^q\,\mathrm{d}t &\le \int_0^1 t^q[t^\alpha|f'(a)|^q + m(1-t^\alpha)|f'(b^{1/m})|^q - t(1-t)F(a-b^{1/m})]\,\mathrm{d}t \\ &= \frac{|f'(a)|^q}{q+\alpha+1} + \frac{m|f'(b^{1/m})|^q}{(q+\alpha+1)(q+1)} - \frac{F(a-b^{1/m})}{(q+5)(q+1)}\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\int_0^1 t^q|f'(a^{1-t}b^t)|^q\,\mathrm{d}t &\le \int_0^1 t^q[m(1-t^\alpha)|f'(a^{1/m})|^q + t^\alpha|f'(b)|^q - t(1-t)F(b-a^{1/m})]\,\mathrm{d}t \\ &= \frac{m|f'(a^{1/m})|^q}{(q+\alpha+1)(q+1)} + \frac{|f'(b)|^q}{q+\alpha+1} - \frac{F(b-a^{1/m})}{(q+5)(q+1)}.\end{aligned} \tag{3.21}$$

Straightforward computation gives

$$\int_0^1 (a^t b^{1-t})^{q/(q-1)}\,\mathrm{d}t = \int_0^1 (a^{1-t} b^t)^{q/(q-1)} = L(a^{q/(q-1)}, b^{q/(q-1)}). \tag{3.22}$$

Substituting (3.20), (3.21), and (3.22) into the inequality (3.19) arrives at the inequality (3.18). The proof of Theorem 9 is thus complete. □

Putting $m = 1$ in Theorem 9, we obtain

Corollary 6. *Let $F : \mathbb{R} \to \mathbb{R}_0$ be a fixed function and $\alpha \in (0, 1]$. Suppose that $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable for $a, b \in \mathbb{R}_+$ with $a < b$. If $|f'|^q$ is $(\alpha, 1)$-GA-F-convex on $[a, b]$ for $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| \le \frac{\ln b - \ln a}{2} L^{1-1/q}(a^{q/(q-1)}, b^{q/(q-1)}) \left\{\left[\frac{|f'(a)|^q}{q+\alpha+1} + \frac{|f'(b)|^q}{(q+\alpha+1)(q+1)} - \frac{F(a-b)}{(q+5)(q+1)}\right]^{1/q} \left[\frac{|f'(a)|^q}{(q+\alpha+1)(q+1)} + \frac{|f'(b)|^q}{q+\alpha+1} - \frac{F(b-a)}{(q+5)(q+1)}\right]^{1/q}\right\},$$

where $L(u, v)$ is defined as in Lemma 3.

4. Conclusions

In this paper, we defined the notion of (α, m)-GA-F-convex functions in Definition 5 and, via the integral identity (2.2) and other analytic techniques such as those in Lemma 3, established those integral inequalities (3.4), (3.9), and (3.13) in Theorems 6 to 8 of the Hermite–Hadamard type for geometric-arithmetic-F-convex functions defined by the inequality (2.1) in Definition 5.

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