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A generalization of Szász-type operators including stirling polynomial

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Abstract. A discrete positive linear operators based on Stirling polynomials is studied here. As per our knowledge, the approximations of these operators have not been studied earlier. We proved a convergence theorem for these operators and give the quantitative estimation of the approximation process by using a classical approach and the second modulus of continuity. The paper also contains the graphical representation based on Mathematica which verify the approximation properties of these operators. As per the calculations in the paper, we have observed that the composition of these operators with some integral operators provides us with a summation-integral type operator, which can be a new source of study for further research.

1. Introduction

In mathematics, Stirling numbers arise in a variety of analytic and combinatorial problems. There are two different sets of numbers with this name: Stirling numbers of the first kind and Stirling numbers of the second kind. Also, Lah numbers are sometimes known as Stirling numbers of the third kind. All three types of Stirling numbers describe coefficients in connection with three different sequences of polynomials that frequently arise in combinatorics.

Stirling polynomials are born from investigations into Stirling numbers. They are a family of polynomials which represents important sequences of numbers appearing in combinatorics and analysis, which are closely related to the Stirling numbers, the Bernoulli numbers, and the generalized Bernoulli polynomials. Stirling numbers, in a notation proposed by Jovan Karamata are defined for integers $n, k \geq 0$ and divided in two kinds. Stirling numbers of the first kind are denoted by $s(n,k)=\left[{n \atop k}\right]$, and verbalized by "$n$ cycle k". They count the number of partitions of $[n] = \{1, 2, 3, ..n\}$ into k nonempty cycles. Stirling numbers of the second kind are denoted by $S(n,k)$ and verbalized by "n subset k". Classically, $\left\{{n \atop k}\right\}$ is defined as the number of partitions of $[n] = \{1, 2, 3, ..., k\}$ into k nonempty subsets. The Stirling numbers of the first and second

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kinds $s(n,k)$ and $S(n,k)$ are important in combinatorial analysis, theory of special functions, and number theory.

The generating functions have a lot of bridges between mathematics and other applied sciences such as combinatorics, computer aided geometric design and Machine learning. The generating functions are designed to help us effectively convert problems involving sequences into problems involving functions.

One of famous examples of positive linear operators is Szász operators [23] and it is defined as follows

$$\mathcal{S}_n(\gamma, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} \gamma\left(\frac{k}{n}\right), \; n \in \mathbb{N}, \; x \geq 0, \tag{1}$$

where $\gamma \in C([0,\infty))$ and whenever the above series is convergent. Many researchers studied different types of modifications of Szász operators and certain linear positive operators as in ([6], [13], [14], [15], [18],[1],[16],[21],[19],[5],[17]).

Jakimovski and Leviatan [12] proposed a generalization of Szász operators with Appell polynomials. Let $\mathcal{A}(z) = \sum_{k=0}^{\infty} p_k z^k$, $(p_0 \neq 0)$ be an analytic function in the disc $|z| \leq \rho$ $(\rho > 1)$ and assume that $\mathcal{A}(1) \neq 0$. The Appell polynomial $\mathcal{H}_k(x)$ have the generating functions of the following form

$$\mathcal{A}(u)e^{ux} = \sum_{k=0}^{\infty} \mathcal{H}_k(x) u^k. \tag{2}$$

with the condition $\mathcal{H}_k(x) \geq 0$ for $x \in [0,\infty)$, Jakimovski and Leviatan introduced the linear positive operators $\mathcal{D}_n(\gamma, x)$ via

$$\mathcal{D}_n(\gamma; x) = \frac{e^{-nx}}{\mathcal{A}(1)} \sum_{k=0}^{\infty} \mathcal{H}_k(nx) \gamma\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}. \tag{3}$$

Remark 1.1. *We can see that for $\mathcal{A}(z) = z$ and with the help of definition of generating function defined in equation (2), we easily find $\mathcal{H}_k(x) = \dfrac{x^k}{k!}$ and by using equation (3), we obtained original form of Szász operators as mentioned in (1).*

Then, Ismail [10] established another generalization of the Szász operators (1) and also Jakimovski and Leviatan operators (3) with the help of Sheffer polynomials. Let $\mathcal{B}(z) = \sum_{k=0}^{\infty} \rho_k z^k (\rho_0 \neq 0)$ and $\mathcal{E}(z) = \sum_{k=0}^{\infty} \phi_k z^k (\phi_1 \neq 0)$ be analytic functions in the disc $|z| \leq \rho$ $(\rho \geq 1)$ where, $\rho_k, \phi_k \in \mathbb{R}$. The Sheffer polynomial $\mathcal{S}_k(x)$ have generating function of the type

$$\mathcal{B}(t)e^{x\mathcal{E}(t)} = \sum_{k=0}^{\infty} \mathcal{S}_k(x) t^k, \; |t| \leq \rho. \tag{4}$$

Under the following constraints:

$$\text{For } x \in [0,\infty), \; \mathcal{S}_k(x) \geq 0,$$
$$\mathcal{B}(1) \neq 0, \; \mathcal{E}'(1) = 1.$$

Ismail proposed the approximation properties of linear positive operators given by

$$\mathcal{I}_n(\gamma; x) = \frac{e^{-nx\mathcal{E}(1)}}{\mathcal{B}(1)} \sum_{k=0}^{\infty} \mathcal{S}_k(nx) \gamma\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}. \tag{5}$$

Remark 1.2. *For $\mathcal{E}(t) = t$, one can observe that the generating functions (4) reduce to (2) and from this fact, the operators (7) return to the operators (3).*

Remark 1.3. *By taking, $\mathcal{E}(t) = t$ and $\mathcal{B}(t) = 1$, It is easy to obtained Szász operators from the operators (7).*

In this paper, we aim to construct linear positive operators by using Stirling polynomials. The Stirling polynomials $S_k(x)$ are a family of polynomials that generalize the Bernoulli numbers B_k and the Stirling numbers of the second kind $S(n, k)$. The Stirling polynomials $S_k(x)$ for non-negative integers k are defined by the generating function

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!}, \tag{6}$$

with the condition $S_k(x) \geq 0$, for $x \in [0, \infty)$, we propose a new form of linear positive operators as follows:

$$\mathcal{L}_n(\gamma; x) = \left(1 - \frac{1}{e}\right)^{nx+1} \sum_{k=0}^{\infty} \frac{S_k(nx)}{k!}\gamma\left(\frac{k}{n}\right), \tag{7}$$

where, $x \in [0, \infty)$ and $n \in \mathbb{N}$.

2. Approximation Properties of $\mathcal{L}_n$ Operators

In this section, we investigate the convergence properties of $\mathcal{L}_n(\gamma, x)$ to show the uniform convergence of $\mathcal{L}_n(\gamma, x)$, we derive the operator's value at test functions in the following lemma.

Lemma 2.1. *For all $x \in [0, \infty)$, we have*

1. $\mathcal{L}_n(1; x) = 1$,
2. $\mathcal{L}_n(t; x) = \dfrac{nx+1}{n} - \dfrac{1}{ne}\left(\dfrac{e}{e-1}\right)^{nx+1}$,
3. $\mathcal{L}_n(t^2; x) = \dfrac{nx+1}{n^2}\left(1-\dfrac{1}{e}\right)^{nx+1}\left(nx\left(\dfrac{e}{e-1}\right)^{nx-1}\left(\dfrac{1}{1-e^{-1}} - \dfrac{e^{-1}}{(1-e^{-1})^2}\right)^2\right)$
$+ \dfrac{nx+1}{n^2}\left(1-\dfrac{1}{e}\right)^{nx+1}\left(\dfrac{1}{1-e^{-1}}\right)^{nx}\left(\dfrac{-e^{-1}}{(1-e^{-1})^2} + \dfrac{2e^{-2}}{(1-e^{-1})^3} + \dfrac{1-2e^{-1}}{(1-e^{-1})^2}\right)$

Proof. From generating functions of the Stirling polynomials mentioned in (6), we obtain

$$\sum_{k=0}^{\infty} \frac{kS_k(nx)}{k!} = (nx+1)\left(\frac{e-1}{e}\right)^{-(nx+1)} - \frac{nx+1}{e}\left(\frac{e-1}{e}\right)^{-(nx+2)}, \tag{8}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^2 S_k(nx)}{k!} &= nx(nx+1)\left(\frac{1}{1-e^{-1}}\right)^{nx-1}\left(\frac{1-2e^{-1}}{(1-e^{-1})^2}\right)^2 \\ &\quad + (nx+1)\left(\frac{1}{1-e^{-1}}\right)^{nx}\left[\frac{-e^{-1}}{(1-e^{-1})^2} + \frac{2e^{-2}}{(1-e^{-1})^3} + \frac{1-2e^{-1}}{(1-e^{-1})^2}\right].\end{aligned} \tag{9}$$

By using above two equalities (8) and (9), we can derive the proof. □

Now using above values of the function in Lemma 2.1 and Korovkin's theorem in [3],[2], we establish the uniform convergence of $\mathcal{L}_n$. We define a new collection of functions Δ as follows:

$$\Delta = \left\{\gamma : [0, \infty) \to \mathbb{R}; \lim_{x\to\infty} \frac{\gamma(x)}{1+x^2} \text{ is convergent}\right\}$$

Theorem 2.1. *Let $\gamma \in C([0,\infty)) \cap \Delta$ and $\mathcal{L}_n(\gamma, x)$ be as defined in (7), then*

$$\lim_{n\to\infty} \mathcal{L}_n(\gamma, x) = \gamma(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. As per the Lemma 2.1, one has

$$\lim_{n\to\infty} \mathcal{L}_n(t^i, x) = x^i, \ \ for\, i = 0, 1, 2.$$

The above-mentioned convergences are satisfied uniformly in each compact subset of $[0,\infty)$. Applying the Korovkin-type property (vi) of Theorem 4.1.4 from [3], we can prove the stated result. □

In order to estimate the rate of convergence, we give some definitions and lemmas.

Definition 1. *Let $\gamma \in \bar{C}([0,\infty))$ and $\delta > 0$. The modulus of continuity $w(\gamma, \delta)$ [4] of the function γ is defined by*

$$w(\gamma, \delta) = \sup\{|\gamma(x) - \gamma(y)| : |x - y| \le \delta, \ x, y \in [0,\infty)\},$$

where $\bar{C}([0,\infty))$ is the collection of all uniformly continuous function on $[0,\infty)$.

Definition 2. *[4] The second modulus of continuity of the function $\gamma \in C([a,b])$ is defined by*

$$w_2(\gamma, \delta) = \sup_{0<t\le\delta} \|\gamma(\cdot + 2t) - 2\gamma(\cdot + t) + \gamma(\cdot)\|,$$

where $\|\gamma\| = max\{|\gamma(x)| : x \in [a,b]\}$.

Lemma 2.2. *(Gavrea and Rasşa [7]). Let $\gamma \in C^2([0,a])$ and $(\delta_n)_{n\ge 0}$ be sequence of linear positive operators with the property that $\delta_n(1,x) = 1$. Then,*

$$|\delta_n(\gamma, x) - \gamma(x)| \le \|\gamma^{'}\| \sqrt{\delta_n((s-x)^2, x)} + \frac{1}{2}\|\gamma''\|\delta_n((s-x)^2, x).$$

Lemma 2.3. *(Zhuk [22]). Let $\gamma \in C([a,b])$ and $\gamma \in C((0, \frac{b-a}{2}))$. Let γ_p be second-order Steklov function attached to the function γ. Then the following inequalities are satisfied.*

1. $\|\gamma_p - \gamma\| \le \dfrac{3}{4} w_2(\gamma, p)$.
2. $\|\gamma_p''\| \le \dfrac{3}{2p^2} w_2(\gamma, p)$.

Lemma 2.4. *For $x \in [0,\infty)$, one has*

$$\begin{aligned}
\mathcal{L}_n((t-x)^2; x) = {} & \frac{nx+1}{n^2}\left(1 - \frac{1}{e}\right)^{nx+1}\left(nx\left(\frac{e}{e-1}\right)^{nx-1}\left(\frac{1}{1-e^{-1}} - \frac{e^{-1}}{(1-e^{-1})^2}\right)^2\right) \\
& + \frac{nx+1}{n^2}\left(1 - \frac{1}{e}\right)^{nx+1}\left(\frac{1}{1-e^{-1}}\right)^{nx}\left(\frac{-e^{-1}}{(1-e^{-1})^2} + \frac{2e^{-2}}{(1-e^{-1})^3} + \frac{1-2e^{-1}}{(1-e^{-1})^2}\right) \\
& - \left(\frac{nx+1}{n} - \frac{1}{ne}\left(\frac{e}{e-1}\right)^{nx+1}\right)2x + x^2.
\end{aligned}$$

Proof. Using the property of linearity for the operators $\mathcal{L}_n$, one can write

$$\mathcal{L}_n((t-x)^2; x) = \mathcal{L}_n(t^2; x) - 2x\mathcal{L}_n(t; x) + x^2\mathcal{L}_n(1; x).$$

With the help of Lemma 2.1, one can prove the equality stated in the lemma 2.4. □

For class of positive linear operators, we will use the modulus of continuity and second order modulus of continuity to established a result on error estimation which has connection with the rate of convergence. Now, we will calculate the rate of convergence theorem in the following theorems.

Theorem 2.2. *Let $\gamma \in \bar{C}([0,\infty)) \cap \Delta$. $\mathcal{L}_n$ operators satisfies the following inequality.*

$$|\mathcal{L}_n(\gamma, x) - \gamma(x)| \leq 2w(\gamma, \sqrt{\lambda_n(x)}) \tag{10}$$

where

$$\begin{aligned}\lambda := \lambda_n(x) &= \mathcal{L}_n((t-x)^2, x) \\ &= \frac{nx+1}{n^2}\left(1-\frac{1}{e}\right)^{nx+1}\left(nx\left(\frac{e}{e-1}\right)^{nx-1}\left(\frac{1}{1-e^{-1}} - \frac{e^{-1}}{(1-e^{-1})^2}\right)^2\right) \\ &+ \frac{nx+1}{n^2}\left(1-\frac{1}{e}\right)^{nx+1}\left(\frac{1}{1-e^{-1}}\right)^{nx}\left(\frac{-e^{-1}}{(1-e^{-1})^2} + \frac{2e^{-2}}{(1-e^{-1})^3} + \frac{1-2e^{-1}}{(1-e^{-1})^2}\right) \\ &- \left(\frac{nx+1}{n} - \frac{1}{ne}\left(\frac{e}{e-1}\right)^{nx+1}\right)2x + x^2.\end{aligned} \tag{11}$$

Proof. By Lemma 2.1 and with the help of property of modulus of continuity, we deduce

$$\begin{aligned}|\mathcal{L}_n(\gamma, x) - \gamma(x)| &= \left|\left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\left(\gamma\left(\frac{k}{n}\right) - \gamma(x)\right)\right| \\ &\leq \left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\left|\gamma\left(\frac{k}{n}\right) - \gamma(x)\right| \\ &\leq \left\{1 + \left(1-\frac{1}{e}\right)^{nx+1}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\left|\frac{k}{n} - x\right|\right\} w(\gamma, \delta).\end{aligned}$$

with the help of Cauchy-Schwarz inequality and then by applying Lemma 2.4, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\left|\frac{k}{n} - x\right| &\leq \left\{\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\right\}^{\frac{1}{2}}\left\{\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\left|\frac{k}{n} - x\right|^2\right\}^{\frac{1}{2}} \\ &= \left(1-\frac{1}{e}\right)^{-(nx+1)}\left(\mathcal{L}_n((t-x)^2, x)\right)^{\frac{1}{2}}\end{aligned}$$

Considering the last inequality, we obtain

$$|\mathcal{L}_n(\gamma, x) - \gamma(x)| \leq \left\{1 + \frac{1}{\delta}\sqrt{\lambda_n(x)}\right\} w(\gamma, \delta),$$

where $\lambda_n(x)$ is given by (11). In above inequality, by choosing $\delta = \sqrt{\lambda_n(x)}$, we get the required result. □

Theorem 2.3. *For $\gamma \in C([0,a])$, the following estimate*

$$|\mathcal{L}_n(\gamma, x) - \gamma(x)| \leq \frac{2}{a}\|\gamma\|h^2 + \frac{3}{4}(a + 2 + h^2)w_2(\gamma, h)$$

holds, where

$$h = h_n(x) = (\mathcal{L}_n((s-x)^2, x))^{\frac{1}{4}}.$$

Proof. Let γ_h be the second-order Steklov function attached to the funciton γ. With considering inequality $\mathcal{L}_n(1, x) = 1$, we have

$$
\begin{aligned}
|\mathcal{L}_n(\gamma, x) - \gamma(x)| &\leq |\mathcal{L}_n(\gamma - \gamma_h, x)| + |\mathcal{L}_n(\gamma_h, x) - \gamma_h(x)| + |\gamma_h(x) - \gamma(x)| \\
&\leq 2\|\gamma_h - \gamma\| + |\mathcal{L}_n(\gamma, x) - \gamma_h(x)|.
\end{aligned}
$$

We use the fact that $\gamma_h \in C^2([0, a])$, it follows from Lemma 2.2, one can observe

$$
|\mathcal{L}_n(\gamma, x) - \gamma(x)| \leq \|\gamma^{'}\| \sqrt{\mathcal{L}_n((t-x)^2, x)} + \frac{1}{2}\|\gamma^{''}\| \mathcal{L}_n((t-x)^2, x).
$$

If one combines Landau inequality with Lemma 2.3, we can write

$$
\begin{aligned}
\|\gamma_h^{'}\| &\leq \frac{2}{a}\|\gamma_h\| + \frac{a}{2}\|\gamma_h^{'}\| \\
&\leq \frac{2}{a}\|\gamma\| + \frac{3a}{4}\frac{1}{h^2} w_2(\gamma, h).
\end{aligned}
$$

From the last inequality and Lemma 2.3and because by taking $h = (\mathcal{L}_n((s-x)^2, x))^{\frac{1}{4}}$

$$
|\mathcal{L}_n(\gamma_h, x) - \gamma_h(x)| \leq \frac{2}{a}\|\gamma\| h^2 + \frac{3a}{4} w_2(\gamma, h) + \frac{3}{4} h^2 w_2(\gamma, h).
$$

and from Lemma 2.3, gives the proof of theorem. □

3. Representation of $\mathcal{L}_n$ operators with connection of Stirling numbers of first and second kinds

The Stirling numbers of the first and second kinds $s(n, k)$ and $S(n, k)$ are important in combinatorial analysis, theory of special functions, and number theory. They can be generated by the rising factorial

$$
(x)_n = \prod_{k=0}^{n-1}(x+k) = \sum_{k=0}^{n} s(n,k) x^k
$$

and the exponential function

$$
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}.
$$

The Stirling polynomials $S_k(x)$ are a family of polynomials that generalize the Bernoulli numbers B_k and the Stirling numbers of the second kind $S(n, k)$. The Stirling polynomials $S_k(x)$ for nonnegative integers k are defined by the generating function

$$
\left(\frac{t}{1 - e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!}
$$

For value of $t = 1$, we have established the new positive linear operators based on equation(7) and studied its approximation properties and the explicit expressions

$$
\begin{aligned}
S_k(x) &= (-1)^k \sum_{j=0}^{k} (-1)^j S(k+j, j) \frac{\binom{x+j}{j}\binom{x+k+1}{k-j}}{\binom{k+j}{j}} \\
&= \sum_{j=0}^{k} (-1)^j s(k+j+1, j+1) \frac{\binom{x-k}{j}\binom{x-k-j-1}{k-j}}{\binom{k+j}{j}},
\end{aligned}
$$

which come from Lagrange's interpolation formula, are known. For more information on $S_k(x)$, see the papers [24],[20] and the closely related references therein. The authors Feng Qi. et al. in [20] established the result on the explicit form of Stirling polynomials in terms of Stirling number of first and second kinds $s(n,k)$ and $S(n,k)$

$$S_k(x) = (-1)^k k! \sum_{m=0}^{k} \left(\sum_{l=m}^{k} \frac{s(l+1,m+1)}{(k+l)!} \sum_{i=0}^{l} (-1)^i \binom{k+l}{l-i} S(k+i,i) \right) x^m, \text{ for } k \geq 0. \tag{12}$$

With the help of Stirling numbers of first and second kind and the operators defined in (12), we can get the different representation of the established operators as mentoined below.

$$\begin{aligned} &\mathcal{L}_n(f,x) \\ &= \left(1-\frac{1}{e}\right)^{nx+1} \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k k! \sum_{m=0}^{k} \left(\sum_{l=m}^{k} \frac{s(l+1,m+1)}{(k+l)!} \sum_{i=0}^{l} (-1)^l (-1)^i \binom{k+l}{l-i} S(k+i,i) \right) x^m f\left(\frac{k}{n}\right). \end{aligned} \tag{13}$$

4. Modified form preserving e^{Ax}

In this section, we proposed the modified form of the operators defined in (7) to preserve the exponential functions with the condition $S_k(x) \geq 0$ is defined by

$$\bar{\mathcal{L}}_n(f,x) = \left(1-\frac{1}{e}\right)^{n\alpha_n(x)+1} \sum_{k=0}^{\infty} \frac{S_k(n\alpha_n(x))}{k!} f\left(\frac{k}{n}\right). \tag{14}$$

Suppose $\bar{T}_n$ preserve e^{Ax}, then

$$e^{Ax} = \left(\frac{e^{\frac{2A}{n}+1} - e^{\frac{2A}{n}}}{e^{\frac{A}{n}+1} - e} \right)^{n\alpha_n x+1}$$

implying

$$\alpha_n(x) = \frac{1}{n} \left(\frac{Ax}{log\left(\frac{e^{\frac{2A}{n}+1} - e^{\frac{2A}{n}}}{e^{\frac{A}{n}+1} - e} \right)} - 1 \right)$$

Remark 4.1. *By simple computation after replacing x by $\alpha_n(x)$ in Lemma 2.4 and taking the limits for $A > 0$, we have the following outcomes*

$$\lim_{n\to\infty} n\bar{\mathcal{L}}_n((t-\alpha_n),x) = 1 - \frac{1}{e},$$
$$\lim_{n\to\infty} n\bar{\mathcal{L}}_n((t-\alpha_n)^2,x) = 0.$$

5. Graphical Representation

This section deals with graphical representation on some exponential functions and the linear functions of the cases treated in Theorem2.1. Here we have used the term by term convergence of the operators to the functions as mentioned in the table. The table contains the term of operators, its color and the function.

Example 5.1. *As per the condition mentioned in theorem 2.1, we take $\gamma(x) = e^{-2x}$, for $x \in [0,1]$ and $n = 5, 10$, the approximation to the function f by the operators $\mathcal{L}_n$ is illustrated in Fig (1) with Table (1) and Fig (2) with Table (2). From the graph, we note that the term by term approximation of function γ is achieved by the proposed operators $\mathcal{L}_n$.*

Term	Color	Function
Original Function	Red	$\gamma(x) = e^{-2x}$
1^{st}*term*	Blue	$\gamma(x) = \left(1 - \frac{1}{e}\right)^{5x+1} S_0(5x) f\left(\frac{0}{5}\right)$
2^{nd}*term*	Black	$1^{st}term + \left(1 - \frac{1}{e}\right)^{5x+1} S_1(5x) f\left(\frac{1}{5}\right)$
3^{rd}*term*	Purple	$2^{nd}term + \frac{1}{2!}\left(1 - \frac{1}{e}\right)^{5x+1} S_2(5x) f\left(\frac{2}{5}\right)$
4^{th}*term*	Green	$3^{rd}term + \frac{1}{3!}\left(1 - \frac{1}{e}\right)^{5x+1} S_3(5x) f\left(\frac{3}{5}\right)$

Table 1

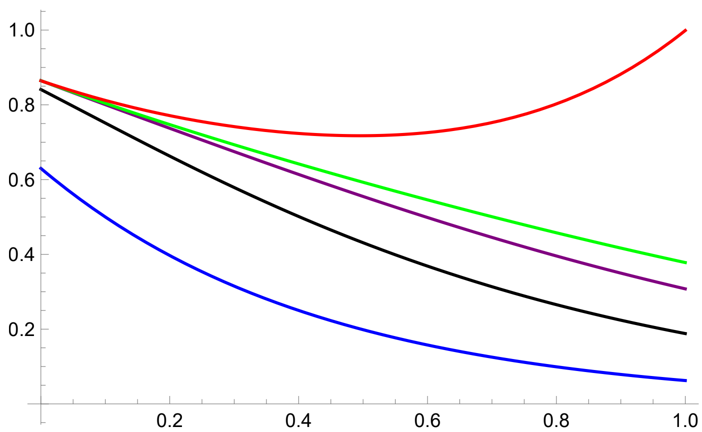

Figure 1: $\gamma(x) = e^{-2x}, n = 5$

Term	Color	Function
Original Function	Red	$\gamma(x) = e^{-2x}$
1^{st}*term*	Blue	$\gamma(x) = \left(1 - \frac{1}{e}\right)^{10x+1} S_0(10x) f\left(\frac{0}{10}\right)$
2^{nd}*term*	Black	$1^{st}term + \left(1 - \frac{1}{e}\right)^{10x+1} S_1(10x) f\left(\frac{1}{10}\right)$
3^{rd}*term*	Purple	$2^{nd}term + \frac{1}{2!}\left(1 - \frac{1}{e}\right)^{10x+1} S_2(10x) f\left(\frac{2}{10}\right)$
4^{th}*term*	Green	$3^{rd}term + \frac{1}{3!}\left(1 - \frac{1}{e}\right)^{10x+1} S_3(10x) f\left(\frac{3}{10}\right)$

Table 2

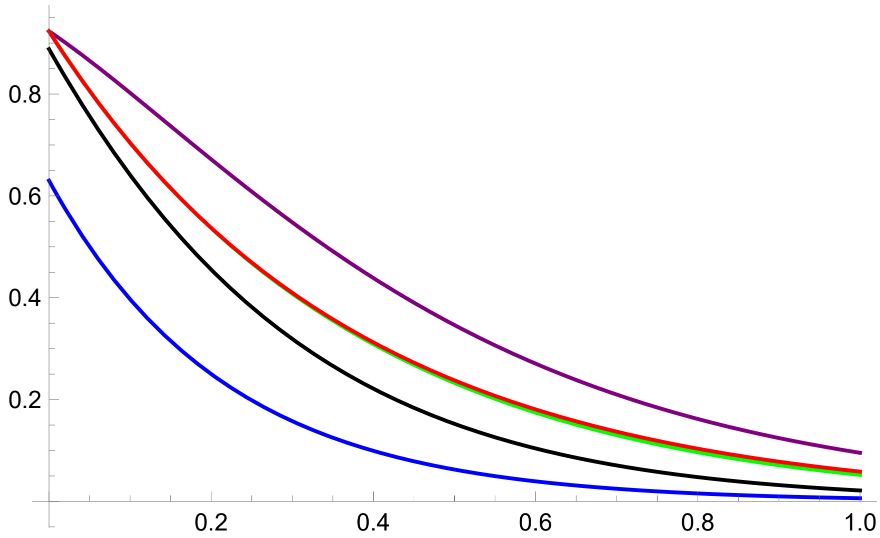

Figure 2: $\gamma(x) = e^{-2x}, n = 10$

Example 5.2. *We take $\gamma(x) = x$ with the condition that f satisfy the result given in Theorem (2.1) for $x \in [0, 0.3]$ and $n = 5, 10$, the approximation of function γ by the established operators $\mathcal{L}_n$ is illustrated in the Fig (3) with Table (3) and Fig(4) with Table (4). From the graphical presentation, it is clear that the approximation of function f is successfully achieved though the operators $\mathcal{L}_n$.*

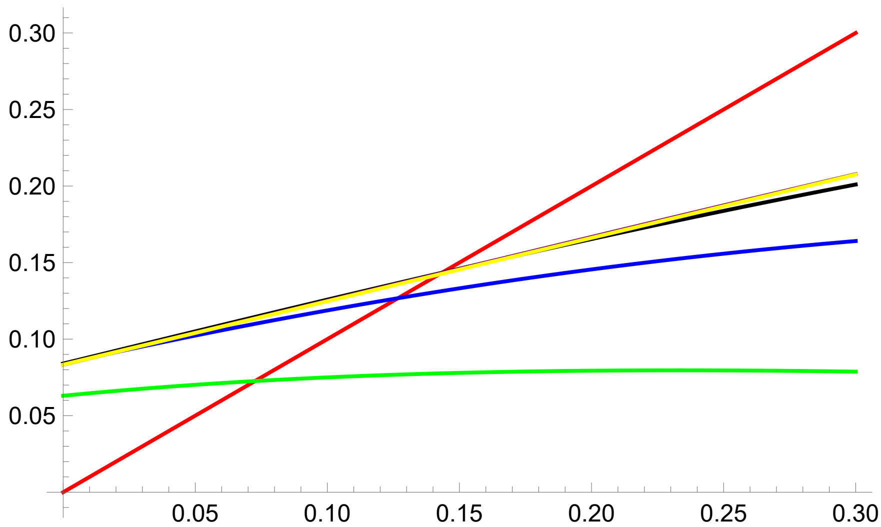

Figure 3: $\gamma(x) = x, n = 5$

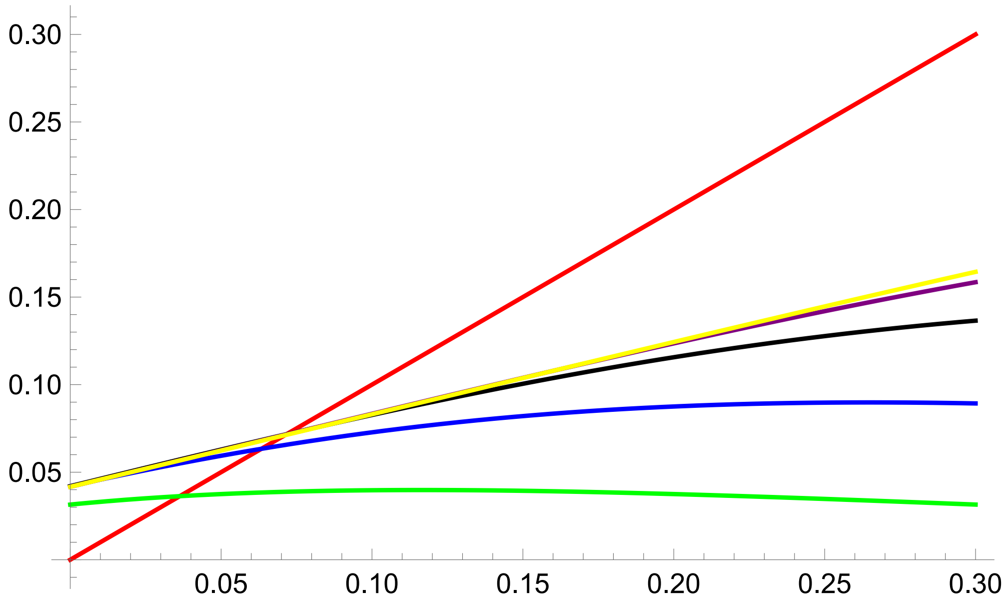

Figure 4: $\gamma(x) = x, n = 10$

Term	Color	Function
Original Function	Red	$\gamma(x) = x$
$1^{st}term$	Blue	$\gamma(x) = \left(1-\frac{1}{e}\right)^{5+1} S_0(5x) f\left(\frac{0}{5}\right)$
$2^{nd}term$	Black	$1^{st}term+\left(1-\frac{1}{e}\right)^{5x+1} S_1(5x) f\left(\frac{1}{5}\right)$
$3^{rd}term$	Purple	$2^{nd}term+\frac{1}{2!}\left(1-\frac{1}{e}\right)^{5x+1} S_2(5x) f\left(\frac{2}{5}\right)$
$4^{th}term$	Green	$3^{rd}term+\frac{1}{3!}\left(1-\frac{1}{e}\right)^{5x+1} S_3(5x) f\left(\frac{3}{5}\right)$

Table 3

Term	Color	Function
Original Function	Red	$\gamma(x) = x$
$1^{st}term$	Blue	$\gamma(x) = \left(1-\frac{1}{e}\right)^{10x+1} S_0(10x) f\left(\frac{0}{10}\right)$
$2^{nd}term$	Black	$1^{st}term+\left(1-\frac{1}{e}\right)^{10x+1} S_1(10x) f\left(\frac{1}{10}\right)$
$3^{rd}term$	Purple	$2^{nd}term+\frac{1}{2!}\left(1-\frac{1}{e}\right)^{10x+1} S_2(10x) f\left(\frac{2}{10}\right)$
$4^{th}term$	Green	$3^{rd}term+\frac{1}{3!}\left(1-\frac{1}{e}\right)^{10x+1} S_3(10x) f\left(\frac{3}{10}\right)$

Table 4

6. Further new Summation-Integral type operators

In this section, we introduce new operators linked with the Stirling polynomials. We take Post-Widder operators [11], defined by

$$(P_n\gamma)(x) = \frac{n^n}{x^n}\frac{1}{\Gamma(n)}\int_0^\infty e^{\frac{-nt}{x}}t^{n-1}\gamma(t)dt.$$

The composition of $\mathcal{L}_n$ with the Post-Widder operators P_n, we get a new approximation operator $\mathcal{K}_n$ as follows.

$$(\mathcal{K}_n f)(x) = (P_n \circ \mathcal{L}_n)(x)$$

Definition 3. *The new operators can be represented as follows*

$$(\mathcal{K}_n f)(x) = \frac{n^n}{x^n}\frac{1}{\Gamma(n)}\left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\gamma\left(\frac{k}{n}\right)\int_0^\infty e^{-\frac{nt}{x}}t^{n-1}dt.$$

The semi-exponetial Post-Widder operators [9] is defined by

$$(\mathcal{P}_n^\beta\gamma)(x) = \frac{n^n}{x^n e^{\beta x}}\sum_{v=0}^{\infty}\frac{(n\beta)^v}{v!\Gamma(n+v)}\int_0^\infty e^{-\dfrac{nt}{x}}t^{n+v-1}\gamma(t)dt.$$

The composition of $\mathcal{P}_n^\beta\gamma$ and $\mathcal{L}_n$ gives us another new operators $\mathcal{K}_n^\beta$, as follows

$$(\mathcal{K}_n^\beta\gamma)(x) = (\mathcal{P}_n^\beta \circ \mathcal{L})(x).$$

Definition 4. *The new positive linear operators can be defined as follows*

$$(\mathcal{K}_n^\beta\gamma)(x) = \frac{n^n}{x^n e^{\beta x}}\left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\gamma\left(\frac{k}{n}\right)\sum_{v=0}^{\infty}\frac{(n\beta)^v}{v!\Gamma(n+v)}\int_0^\infty e^{-\frac{nt}{x}}t^{n+v-1}dt.$$

The integral operators due to Rathore (See [8] and references therein) is defined by

$$(\mathcal{W}_n\gamma)(x) = \frac{n^{nx}}{\Gamma(nx)}\int_0^\infty e^{-nt}t^{nx-1}\gamma(t)dt.$$

We take a composition of $\mathcal{L}_n$ with the Rathore operators $\mathcal{W}_n$, to obtain another new approximation operators $\prime_n$ as follows

$$(O_n\gamma)(x) = (\mathcal{W}_n \circ \mathcal{L}_n\gamma)(x).$$

Definition 5. *The new variant of positive linear operators can be defineds as follows*

$$(O_n f)(x) = \frac{n^{nx}}{\Gamma(nx)}\left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\gamma\left(\frac{k}{n}\right)\int_0^\infty e^{-nt}t^{nx-1}dt.$$

The semi-exponential Ismail-May operators related to x^3 is defined by

$$(Q_n\gamma)(x) = \sqrt{\frac{n}{2\pi}}e^{\frac{n}{x}}\int_0^\infty t^{-\frac{3}{2}}e^{-\left(\frac{n}{2t}+\frac{nt}{2x^2}\right)}\gamma(t)dt.$$

We take composition of $\mathcal{L}_n$ with the operators Q_n, to obtain new operators $\mathcal{M}_n$ as follows

$$(\mathcal{M}_n\gamma)(x) = (Q_n \circ \mathcal{L}_n\gamma)(x).$$

Definition 6. *The new operators can be represented as follows*

$$(\mathcal{M}_n\gamma)(x) = \sqrt{\frac{n}{2\pi}}e^{\frac{n}{x}}\left(1-\frac{1}{e}\right)^{nx+1}\sum_{k=0}^{\infty}\frac{S_k(nx)}{k!}\gamma\left(\frac{k}{n}\right)\int_0^\infty t^{-\frac{3}{2}}e^{-\left(\frac{n}{2t}+\frac{nt}{2x^2}\right)}dt.$$

7. Conclusion

The paper contains a study of discrete positive linear operator based on Stirling polynomials. Using a classical method and the second modulus of continuity, we provided the quantitative estimation of the approximation process and demonstrated a convergence theorem for these operators. In Section 3, we also gives the representation of newly defined operators based on the connection with the Stirling numbers of first kind and second kind. Additionally, the study includes a graphical representation based on Mathematica that confirms these operators' approximation properties. According to the calculations in the paper, we have shown that combining these operators with a few integral operators yields an operator of the summation-integral type, which may serve as a novel area of investigation for future research. One potential area for further exploration is the direct convergence of the operators, which, due to the complexity of the form of operators was not fully addressed in the graphical part. Instead we employed a term-by-term convergence approach, which provided valuable insights into the convergence properties.

8. Upcoming discoveries

In this article, we have proposed Szász type summation-integral operators $\mathcal{K}_n$(Defintion 3), $\mathcal{K}_n^{\beta}$(Definition 4), $\mathcal{O}_n$(Definition 5) and $\mathcal{M}_n$(Definition 6). More approximation properties and different modification of these operators can be studied which leads to better approximation by means of rate of convergence.

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- Competing interests
  There are no competing interests that authors need to declare.

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