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A novel perspective on the Bullen and Milne inequalities through the use of multiplicatively absolute value

Bouharket Benaissa[a], Noureddine Azzouz[b], Hüseyin Budak[c,d], Artion Kashuri[e,*]

[a] *Faculty of Material Sciences, University of Tiaret, Algeria*
[b] *Faculty of Sciences , University Center Nour Bachir El Bayadh, Algeria*
[c] *Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India*
[d] *Department of Mathematics, Faculty of Sciences and Art, Kocaeli University, Kocaeli 41001, Türkiye*
[e] *Department of Mathematical Engineering, Polytechnic University of Tirana, 1001 Tirana, Albania*

Abstract. This article formulates a new version of multiplicative Bullen and Milne inequalities derived from constraints on multiplicative Riemann-Liouville fractional integrals. We establish these further inequalities under the multiplicative absolute value and on the assumption that the function is multiplicatively h-convex. Additionally, these results are presented for multiplicatively P-functions and multiplicatively s-convex functions. The final section introduces a novel definition of a multiplicative Lipschitzian function, accompanied by examples, properties and theorems.

1. Introduction

The Hermite-Hadamard inequality is well-known for determining the integral mean of a convex function. Let $\Upsilon : I \to \mathbb{R}$ be a convex function on I and $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, then

$$\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \leq \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \Upsilon(x)\,dx \leq \frac{\Upsilon(\sigma_1)+\Upsilon(\sigma_2)}{2}. \tag{1}$$

In [14], Bullen improved the right side of (1) by the following inequality, which is known as Bullen's inequality:

$$\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \Upsilon(t)\,dt \leq \frac{1}{2}\left[\frac{\Upsilon(\sigma_1)+\Upsilon(\sigma_2)}{2} + \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right] \leq \frac{\Upsilon(\sigma_1)+\Upsilon(\sigma_2)}{2}.$$

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* Corresponding author: Artion Kashuri

Email address: a.kashuri@fimif.edu.al (Artion Kashuri)

ORCID iD: https://orcid.org/0000-0002-1195-6169 (Bouharket Benaissa), https://orcid.org/0000-0003-0658-2438 (Noureddine Azzouz), https://orcid.org/0000-0001-8843-955X (Hüseyin Budak), https://orcid.org/0000-0003-0115-3079 (Artion Kashuri)

In 2016, the authors presented an estimate of Bullen-type inequalities for functions whose first derivatives absolute values are convex [18, Remark 4.2]:

$$\left|\frac{1}{2}\left[\frac{\Upsilon(\sigma_1)+\Upsilon(\sigma_2)}{2}+\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Upsilon(t)\,dt\right|\leq\frac{(\sigma_2-\sigma_1)\ [|\Upsilon'(\sigma_1)|+|\Upsilon'(\sigma_2)|]}{16}.$$

In 2022, Djenaoui and Meftah presented a Milne inequality for convex functions with Riemann integral as follows [15, Corollary 2.4.]:

$$\left|\frac{1}{3}\left[2\Upsilon(\sigma_1)-\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)+2\Upsilon(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Upsilon(t)\,dt\right|\leq\frac{5(\sigma_2-\sigma_1)}{24}\left(|\Upsilon'(\sigma_1)|+|\Upsilon'(\sigma_2)|\right). \tag{2}$$

In 2023, Budak et al. presented new Milne-type inequalities for fractional integrals for convex functions using Riemann-Liouville fractional operators [13, Theorem 2]:

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Upsilon(\sigma_1)-\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)+2\Upsilon(\sigma_2)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)+\mathfrak{I}^{\alpha}_{\sigma_1^+}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right]\right|\\&\quad\leq\frac{(\sigma_2-\sigma_1)}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(|\Upsilon'(\sigma_1)|+|\Upsilon'(\sigma_2)|\right).\end{aligned}$$

The Bullen and Milne inequalities are useful for estimating the average value of a convex function from both sides and ensuring that it may be integrated. Because of their dual quality, the inequalities developed by Bullen and Milne are very helpful in a wide range of situations related to optimization and the calculus of variations. By making use of these inequalities, researchers have the opportunity to gain valuable knowledge about the behavior of convex functions across certain ranges. Readers who are interested in searching for recent research that has been published recently about Bullen and Milne inequality may consult the following sources: [6–11, 17, 20].

The following is a multiplicative Bullen-type inequality for $*$-convex functions that was established in [12, Theorem 5]:

Theorem 1.1. *Let $\Upsilon:[\sigma_1,\sigma_2]\to\mathbb{R}^+$ be a multiplicative differentiable function on $[\sigma_1,\sigma_2]$. If Υ^* is multiplicatively convex function on $[\sigma_1,\sigma_2]$, then*

$$\begin{aligned}&\left|\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{4}}\left\{{*\mathcal{I}}^{\alpha}_{(\sigma_1)^+}f\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot{*\mathcal{I}}^{\alpha}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}\Upsilon(\sigma_2)\right\}^{-2^{\alpha-1}\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}\right|\\&\leq\left[\Upsilon^*(\sigma_1)\cdot\Upsilon^*(\sigma_2)\right]^{\frac{\sigma_2-\sigma_1}{2}\left(\frac{2-\alpha^2-3\alpha}{8(\alpha+1)(\alpha+2)}+\frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)}\left[\Upsilon^*\left(\frac{\sigma_1+\sigma_2}{2}\right)\right]^{\frac{\sigma_2-\sigma_1}{2}\left(\frac{2-\alpha}{4(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)}.\end{aligned}$$

In the context of integral inequality within multiplicative calculus, readers may turn to sources [6, 11, 19, 20] for examples.

A variant of conventional Newtonian calculus called multiplicative or non-Newtonian calculus was introduced by Grosman and Katz in [16]. Instead of adding and subtracting, multiplication and division are the main operations used. In their exhaustive study, Bashirov et al. [3] presented a rigorously organized multiplicative calculus. The operators for multiplying the derivative ($*$derivative) and the integral ($*$integral) were then demonstrated.

Definition 1.2. *Given a function $\Upsilon:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}^+$, the multiplicative derivative (or $*$derivative) of Υ, denoted by Υ^*, is given by:*

$$\Upsilon^*(x)=\lim_{h\to 0}\left(\frac{\Upsilon(x+h)}{\Upsilon(x)}\right)^{\frac{1}{h}}.$$

Remark 1.3. *For a positive differentiable function Υ, a corresponding multiplicative derivative Υ^* exists, and the relationship between Υ^* and Υ' can be expressed using the following formulas:*

$$\begin{aligned} &\Upsilon^*(t) = \exp\{(\ln\circ\Upsilon)'(t)\}, \\ or \quad & \\ &\ln\circ\Upsilon^* = (\ln\circ\Upsilon)'. \end{aligned} \tag{3}$$

We cite in the following theorem some properties of multiplicative derivatives.

Theorem 1.4. *Let Υ and Φ be positive $*$differentiable functions. If c is an arbitrary constant, then functions $c\Upsilon, \Upsilon\Phi, \Upsilon+\Phi, \frac{\Upsilon}{\Phi}, \Upsilon^{\Phi}$ and $\Upsilon\circ\Phi$ are $*$differentiable, and*

D-1 $(c\Upsilon)^*(t) = \Upsilon^*(t)$.

D-2 $(\Upsilon\,\Phi)^*(t) = \Upsilon^*(t)\,\Phi^*(t)$.

D-3 $(\Upsilon+\Phi)^*(t) = \Upsilon^*(t)^{\dfrac{\Upsilon(t)}{\Upsilon(t)+\Phi(t)}} \cdot \Phi^*(t)^{\dfrac{\Phi(t)}{\Upsilon(t)+\Phi(t)}}$.

D-4 $\left(\dfrac{\Upsilon}{\Phi}\right)^*(t) = \dfrac{\Upsilon^*(t)}{\Phi^*(t)}$.

D-5 $(\Upsilon^{\Phi})^*(t) = \Upsilon^*(t)^{\Phi(t)} \cdot \Upsilon(t)^{\Phi'(t)}$.

D-6 $(\Upsilon\circ\Phi)^*(t) = \Upsilon^*(\Phi(t))^{\Phi'(t)}$.

Definition 1.5. *For a function $\Upsilon : I_0 \subseteq \mathbb{R} \to \mathbb{R}^+$, the multiplicative integral of Υ, represented by $\int_{\sigma_1}^{\sigma_2}(\Upsilon(x))^{dx}$, is defined as:*

$$\int_{\sigma_1}^{\sigma_2}(\Upsilon(x))^{dx} = \exp\left\{\int_{\sigma_1}^{\sigma_2}\ln(\Upsilon(x))dx\right\}. \tag{4}$$

Example 1.6. *For $C \in \mathbb{R}$:*

$$\begin{aligned} &\int_{\sigma_1}^{\sigma_2}(C)^{dx} = C^{\sigma_2-\sigma_1}. \\ &\int_{\sigma_1}^{\sigma_2}\left(C^{(x-\sigma_1)^{\alpha-1}}\right)^{dx} = \exp\left\{\ln C\int_{\sigma_1}^{\sigma_2}(x-\sigma_1)^{\alpha-1}\,dx\right\} = C^{\frac{(\sigma_2-\sigma_1)^{\alpha}}{\alpha}}. \end{aligned} \tag{5}$$

The next theorem relates different properties of multiplicative integrals.

Theorem 1.7. *If Υ and Φ are positive and Riemann integrable on the interval $[\sigma_1,\sigma_2] \subset I^\circ$, then Υ and Φ are $*$-integrable on $[\sigma_1,\sigma_2]$, and*

I-1 $\displaystyle\int_{\sigma_1}^{\sigma_2}((\Upsilon(x))^p)^{dx} = \left(\int_{\sigma_1}^{\sigma_2}(\Upsilon(x))^{dx}\right)^p; \quad p \in \mathbb{R}.$

I-2 $\displaystyle\int_{\sigma_1}^{\sigma_2}(\Upsilon(x)\cdot\Phi(x))^{dx} = \int_{\sigma_1}^{\sigma_2}(\Upsilon(x))^{dx}\cdot\int_{\sigma_1}^{\sigma_2}(\Phi(x))^{dx}.$

I-3 $\displaystyle\int_{\sigma_1}^{\sigma_2}\left(\frac{\Upsilon(x)}{\Phi(x)}\right)^{dx} = \frac{\int_{\sigma_1}^{\sigma_2}(\Upsilon(x))^{dx}}{\int_{\sigma_1}^{\sigma_2}(\Phi(x))^{dx}}.$

I-4 $\displaystyle\int_{\sigma_1}^{c} (\Upsilon(x))^{dx} \cdot \int_{c}^{\sigma_2} (\Upsilon(x))^{dx} = \int_{\sigma_1}^{\sigma_2} (\Upsilon(x))^{dx}; \quad \sigma_1 \leq c \leq \sigma_2.$

I-5 $\displaystyle\int_{\sigma_1}^{\sigma_1} (\Upsilon(x))^{dx} = 1$ *and* $\displaystyle\int_{\sigma_1}^{\sigma_2} (\Upsilon(x))^{dx} = \left(\int_{\sigma_2}^{\sigma_1} (\Upsilon(x))^{dx}\right)^{-1}.$

Theorem 1.8. *[3, Theorem 6] (Multiplicative integration by parts): Let Υ be a positive, multiplicatively differentiable function on I° and $g : I^\circ \to \mathbb{R}$ differentiable and $[\sigma_1, \sigma_2] \subset I^\circ$, then the function $(\Upsilon^*)^g$ is integrable, and we have*

$$\int_{\sigma_1}^{\sigma_2} \left((\Upsilon^*(x))^{g(x)}\right)^{dx} = \frac{(\Upsilon(\sigma_2))^{g(\sigma_2)}}{(\Upsilon(\sigma_1))^{g(\sigma_1)}} \cdot \frac{1}{\displaystyle\int_{\sigma_1}^{\sigma_2} \left((\Upsilon(x))^{g'(x)}\right)^{dx}}. \tag{6}$$

The next process involves the definition of multiplicative h-convexity [21, Definition 2.2].

Definition 1.9. *Let $h : J \supset (0, 1) \to \mathbb{R}$ be a non-negative function and $h \neq 0$. We say that the function $\Upsilon : I^\circ \to \mathbb{R}_+^*$ is multiplicatively h-convex (∗h-convex) if for all $x, y \in [\sigma_1, \sigma_2] \subset I^\circ$ and $t \in [0, 1]$, we have*

$$\Upsilon(t\,x + (1-t)\,y) \leq [\Upsilon(x)]^{h(t)}\ [\Upsilon(y)]^{h(1-t)}. \tag{7}$$

If inequality (7) is reversed, then Υ is said to be multiplicatively h-concave (∗h-concave).

Remark 1.10. *Based on the previously mentioned definition, we get the following relations:*

- *If Υ and Φ are two multiplicatively h-convex functions, then the product $\Upsilon \cdot \Phi$ is also a multiplicatively h-convex function.*
- *If Υ is a multiplicatively h-convex function, then $\frac{1}{\Upsilon}$ is a multiplicatively h-concave function.*
- *If Φ is a multiplicatively h-concave function, then $\frac{1}{\Phi}$ is a multiplicatively h-convex function.*
- *If Υ is a multiplicatively h-convex function and Φ is a multiplicatively h-concave function, then the quotient $\frac{\Upsilon}{\Phi}$ is a multiplicatively h-convex function.*
- *If Υ and Φ are two multiplicatively h-convex functions, the quotient $\frac{\Upsilon}{\Phi}$ is not necessarily a multiplicatively h-convex function. For example, let $\Phi = \Upsilon \cdot \psi$, where ψ is a multiplicatively h-convex function. This results gives $\frac{\Upsilon}{\Phi} = \frac{1}{\psi}$, which is a multiplicatively h-concave function.*

We are examining various forms of multiplicative convexity in this context.

1. The multiplicatively ∗s-convex functions are obtained by setting $h(t) = t^s$, $s \in (0, 1]$ in (7) [21, Definition 2.3].

$$\Upsilon(t\,x + (1-t)\,y) \leq [\Upsilon(x)]^{t^s}\ [\Upsilon(y)]^{(1-t)^s}.$$

2. By setting $h(t) = 1$ in (7), we derive multiplicatively ∗P-functions [21, Definition 2.5].

$$\Upsilon(t\,x + (1-t)\,y) \leq \Upsilon(x) \cdot \Upsilon(y).$$

3. By replacing $h(t) = t$ in (7), we derive the concept of multiplicatively convex functions, see [22].

$$\Upsilon(t\,x + (1-t)\,y) \leq [\Upsilon(x)]^{t}\ [\Upsilon(y)]^{1-t}.$$

For additional details, consult [21–23].

Remark 1.11. *Since the function* $\ln(.)$ *is concave, for* $\Upsilon(x), \Upsilon(y) > 0$ *and* $t \in [0,1]$, *we obtain*

$$t \,\ln \Upsilon(x) + (1-t)\,\ln \Upsilon(y) \leq \ln(t\,\Upsilon(x) + (1-t)\,\Upsilon(y)).$$

Therefore

$$[\Upsilon(x)]^t \cdot [\Upsilon(y)]^{1-t} \leq t\,\Upsilon(x) + (1-t)\,\Upsilon(y). \tag{8}$$

This signifies that a multiplicatively convex function has convexity, although the converse is not necessarily valid.

In [4, 5], the concept of a B-function was introduced as follows:

Definition 1.12. *Let* $g : [0,\infty) \to \mathbb{R}$ *be a nonnegative function. The function* g *is called a B-function if*

$$g(t-a) + g(b-t) \leq 2g\left(\frac{a+b}{2}\right), \tag{9}$$

where $a < t < b$ *with* $a, b \in [0,\infty)$.

In particular, taking $a = 0$ and $b = 1$ in (9), we obtain the inequality:

$$g(\alpha) + g(\beta) \leq 2g\left(\frac{1}{2}\right), \tag{10}$$

where $\alpha + \beta = 1$, $\alpha \in [0,1]$. Examples of such a function g satisfying the inequality (10) can be provided by $g_1(t) = 1$, $g_2(t) = t$ and $g_3(t) = t^s$ with $s \in (0,1]$.

Multiplicative Riemann-Liouville fractional integrals of order $\alpha > 0$, defined for an integrable function $\Upsilon : [a,b] \to \mathbb{R}_+^*$, were presented in [1]:

$$_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon(x) = \exp\left\{\mathcal{I}_{\sigma_1^+}^{\alpha} \ln\circ\Upsilon(x)\right\} \quad \text{and} \quad {}_*\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon(x) = \exp\left\{\mathcal{I}_{\sigma_2^-}^{\alpha} \ln\circ\Upsilon(x)\right\}, \tag{11}$$

where $\mathcal{I}_{\sigma_1^+}^{\alpha}$ and $\mathcal{I}_{\sigma_2^-}^{\alpha}$ represent the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, defined regarding of the Euler's Gamma function $\Gamma(\cdot)$ as follows:

$$\mathcal{I}_{\sigma_1^+}^{\alpha}\Upsilon(x) = \frac{1}{\Gamma(\alpha)}\int_{\sigma_1}^{x}(x-t)^{\alpha-1}\Upsilon(t)\,dt \quad \text{and} \quad \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\sigma_2}(t-x)^{\alpha-1}\Upsilon(t)\,dt. \tag{12}$$

Remark 1.13. *Combining* (11) *and* (12), *we have*

$$_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon(x) = \int_{\sigma_1}^{x}\left[(\Upsilon(t))^{\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}}\right]^{dt} \quad , \quad {}_*\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon(x) = \int_{x}^{\sigma_2}\left[(\Upsilon(t))^{\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}}\right]^{dt}. \tag{13}$$

The main objective of this study is to determine which of Bullen's and Milne's inequalities can be established within the context of multiplicative calculus for positive multiplicatively h-convex functions. To accomplish this, we use the multiplicative absolute value computation in conjunction with a recent property. Additionally, a new lemma on monotonic functions in the context of multiplicative calculus and a new version of multiplicative Bullen and Milne inequalities through monotonic functions is established. A new definition of multiplicatively Lipschitzian functions is presented to us, along with some current examples, and a new property is also included. At the end of this paper, we present new results on Bullen and Milne inequalities that include multiplicatively Lipschitzian functions.

2. Preliminaries

Definition 2.1. *[2, p 44] The multiplicative absolute value is represented by $|.|^*$ and is defined as follows:*

$$|x|^* = \begin{cases} x & \text{if } x \geq 1; \\ \frac{1}{x} & \text{if } 0 < x < 1. \end{cases}$$

Property 2.2. *For all $k \geq 0$ and $x > 0$, we have*

$$(|x|^*)^k = \left|x^k\right|^*. \tag{14}$$

Proof.

- The equality (14) becomes obvious when $k = 0$.
- The exponential function with a positive base x yields: for every $k > 0$, we get

  $$\begin{cases} x^k \geq 1 \Longleftrightarrow x \geq 1 \\ x^k < 1 \Longleftrightarrow x < 1. \end{cases}$$

  Hence

  $$\left|x^k\right|^* = \begin{cases} x^k & \text{if } x^k \geq 1 \\ \frac{1}{x^k} & \text{if } 0 < x^k < 1 \end{cases} = \begin{cases} x^k & \text{if } x \geq 1 \\ \frac{1}{x^k} & \text{if } 0 < x < 1 \end{cases} = (|x|^*)^k.$$

□

Remark 2.3. *Based on the preceding Definition 2.1, we can derive the following observations:*

- *The absolute value $|A - B|$ for real numbers A and B is analogous to the multiplicative absolute value $\left|\frac{x}{y}\right|^*$ in the context of positive real numbers x and y.*
- *Using the notation $\left|\frac{x}{y}\right|^*$ indicates that $\frac{x}{y} > 1$ or $\frac{x}{y} < 1$, which corresponds to the conditions $x > y$ or $x < y$, respectively.*
- *For all positive real numbers x and y, we have: $\left|\frac{x}{y}\right|^* = \left|\frac{y}{x}\right|^*$ and $\left|\frac{x}{y}\right|^* \geq 1$.*
- *In multiplicative calculus, the notation $\left|\frac{x}{y}\right|$ does not have significance for positive real numbers x and y.*

The following lemmas are required for the establishment of our principal results.

Lemma 2.4.

- *For the real number A, we have*

  $$\left|\exp\{A\}\right|^* = \exp|A|. \tag{15}$$

- *If φ is a positive, multiplicatively function on I°, the following equality is valid:*

  $$\left|\ln \circ \varphi\right| = \ln \circ \left|\varphi\right|^*. \tag{16}$$

Proof.

- When $A \geq 0$, the equality (15) is obvious.

- Let $A < 0$, we get

$$\left|\exp\{A\}\right|^{*} = \frac{1}{\exp\{A\}} = \exp\{-A\} = \exp|A|.$$

1. If $\varphi \geq 1$, we have

$$\left|\ln\circ\varphi\right| = \ln\circ\varphi = \ln\circ\left|\varphi\right|^{*}.$$

2. If $0 < \varphi < 1$, we get

$$\left|\ln\circ\varphi\right| = -\ln\circ\varphi = \ln\circ\frac{1}{\varphi} = \ln\circ\left|\varphi\right|^{*}.$$

□

Throughout all of the paper, I° denotes to be an open real interval.

Lemma 2.5. *Let Υ be a positive, multiplicatively differentiable function on I° with $[\sigma_1,\sigma_2] \subset I^{\circ}$, $u : [0,1] \to [\sigma_1,\sigma_2]$ be a differentiable function and $g : I^{\circ} \to \mathbb{R}$ be a differentiable function, then $(\Upsilon^{*}\circ u)^{g}$ is integrable function, and we have*

$$\int_0^1 \left((\Upsilon^{*}(u(t)))^{u'(t)\cdot g(t)}\right)^{dt} = \frac{((\Upsilon\circ u)(1))^{g(1)}}{((\Upsilon\circ u)(0))^{g(0)}}\cdot\frac{1}{\int_0^1\left(((\Upsilon\circ u)(x))^{g'(x)}\right)^{dx}}. \tag{17}$$

Proof. Using the property $(\Upsilon\circ u)^{*}(t) = (\Upsilon^{*}(u(t)))^{u'(t)}$, we can obtain the following result:

$$\int_0^1 \left((\Upsilon^{*}(u(t)))^{u'(t)\cdot g(t)}\right)^{dt} = \int_0^1\left([(\Upsilon\circ u)^{*}(t)]^{g(t)}\right)^{dt}.$$

The application of integration by parts to multiplicative integrals results gives

$$\int_0^1 \left((\Upsilon^{*}(u(t)))^{u'(t)\cdot g(t)}\right)^{dt} = \frac{((\Upsilon\circ u)(1))^{g(1)}}{((\Upsilon\circ u)(0))^{g(0)}}\cdot\frac{1}{\int_0^1\left(((\Upsilon\circ u)(x))^{g'(x)}\right)^{dx}}.$$

□

3. Basic identities

The following lemmas are required in order to correctly establish our results.

Lemma 3.1. *Let $\Upsilon : I^{\circ} \to \mathbb{R}^{+}$ be a multiplicative differentiable function on $[\sigma_1,\sigma_2] \subset I^{\circ}$. If Υ^{*} is multiplicatively integrable on $[\sigma_1,\sigma_2]$, then*

$$\begin{aligned}&\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{2}\right)^{\frac{1}{4}}\left\{{}_{\sigma_1^{+}}\mathcal{I}_{*}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_{*}\mathcal{I}_{\sigma_2^{-}}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}\\&=\left(\int_0^1\left[\left(\Upsilon^{*}\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{2t^{\alpha}-1}{4}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{2}}\left(\int_0^1\left[\left(\Upsilon^{*}\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-\frac{2t^{\alpha}-1}{4}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{2}}.\end{aligned}\tag{18}$$

Proof. Employing the equality (17) yields

$$
\begin{aligned}
J_1 &= \left(\int_0^1 \left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1 + \left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{2t^{\alpha}-1}{4}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{2}} = \int_0^1 \left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1 + \left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{\sigma_2-\sigma_1}{2}\cdot\frac{2t^{\alpha}-1}{4}}\right]^{dt} \\
&= \frac{(\Upsilon(\sigma_2))^{\frac{1}{4}}}{\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{4}}} \cdot \frac{1}{\int_0^1 \left[\left(\Upsilon\left(\left(\frac{1-t}{2}\right)\sigma_1 + \left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{1}{2}\alpha t^{\alpha-1}}\right]^{dt}} \\
&= \frac{(\Upsilon(\sigma_2))^{\frac{1}{4}}\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\int_0^1 \frac{1}{2}\alpha t^{\alpha-1}\cdot \ln\circ\Upsilon\left(\left(\frac{1-t}{2}\right)\sigma_1 + \left(\frac{1+t}{2}\right)\sigma_2\right)dt\right\}} \\
&= \frac{\left(\Upsilon(\sigma_2)\,\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\frac{1}{2}\alpha\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\left(\tau-\frac{\sigma_1+\sigma_2}{2}\right)^{\alpha-1}\cdot\ln\circ\Upsilon(\tau)d\tau\right\}} \\
&= \frac{\left(\Upsilon(\sigma_2)\,\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\frac{1}{2}\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\Gamma(\alpha+1)\ \Im^{\alpha}_{\sigma_2^-}(\ln\circ\Upsilon)\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}} \\
&= \left(\Upsilon(\sigma_2)\,\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{\frac{1}{4}}\left({}_*\Im^{\alpha}_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}. \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \left(\int_0^1 \left[\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1 + \left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-\frac{2t^{\alpha}-1}{4}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{2}} = \int_0^1 \left[\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1 + \left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-\frac{\sigma_2-\sigma_1}{2}\cdot\frac{2t^{\alpha}-1}{4}}\right]^{dt} \\
&= \frac{(\Upsilon(\sigma_1))^{\frac{1}{4}}}{\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{4}}} \cdot \frac{1}{\int_0^1 \left[\left(\Upsilon\left(\left(\frac{1+t}{2}\right)\sigma_1 + \left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{\frac{1}{2}\alpha t^{\alpha-1}}\right]^{dt}} \\
&= \frac{\left(\Upsilon(\sigma_1)\,\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\int_0^1 2\,\alpha t^{\alpha-1}\cdot \ln\circ\Upsilon\left(\left(\frac{1+t}{2}\right)\sigma_1 + \left(\frac{1-t}{2}\right)\sigma_2\right)dt\right\}} \\
&= \frac{\left(\Upsilon(\sigma_1)\,\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\frac{1}{2}\alpha\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}\left(\frac{\sigma_1+\sigma_2}{2}-\tau\right)^{\alpha-1}\cdot\ln\circ\Upsilon(\tau)d\tau\right\}} \\
&= \frac{\left(\Upsilon(\sigma_1)\,\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{4}}}{\exp\left\{\frac{1}{2}\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\Gamma(\alpha+1)\ \Im^{\alpha}_{\sigma_1^+}(\ln\circ\Upsilon)\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}} \\
&= \left(\Upsilon(\sigma_1)\,\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{\frac{1}{4}}\left({}_{\sigma_1^+}\Im^{\alpha}_{*}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}. \qquad (20)
\end{aligned}
$$

Multiplying above equalities (19) and (20) gives us the desired result. □

Lemma 3.2. *Let $\Upsilon : I^\circ \to \mathbb{R}^+$ be a multiplicative differentiable function on $[\sigma_1, \sigma_2] \subset I^\circ$. If Υ^* is multiplicatively integrable on $[\sigma_1, \sigma_2]$, then*

$$\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}\left\{{}_{\sigma_1^+}\mathcal{I}^{\alpha}_{*}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_* \mathcal{I}^{\alpha}_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}$$
$$=\left(\int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{t^\alpha+\frac{1}{3}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{4}}\left(\int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-t^\alpha-\frac{1}{3}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{4}}. \tag{21}$$

Proof. Employing the equality (17) yields

$$\begin{aligned}
B_1 &= \left(\int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{t^\alpha+\frac{1}{3}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{4}}\\
&= \int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{\sigma_2-\sigma_1}{2}\cdot\frac{3t^\alpha+1}{6}}\right]^{dt}\\
&= \frac{(\Upsilon(\sigma_2))^{\frac{2}{3}}}{\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{6}}}\cdot\frac{1}{\int_0^1\left[\left(\Upsilon\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)^{\frac{1}{2}\alpha t^{\alpha-1}}\right]^{dt}}\\
&= \frac{(\Upsilon(\sigma_2))^{\frac{2}{3}}\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{6}}}{\exp\left\{\int_0^1\frac{1}{2}\alpha t^{\alpha-1}\cdot\ln\circ\Upsilon\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)dt\right\}}\\
&= \frac{\left((\Upsilon(\sigma_2))^2\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}}{\exp\left\{\frac{1}{2}\alpha\left(\frac{2}{\sigma_2-\sigma_1}\right)^\alpha\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\left(\tau-\frac{\sigma_1+\sigma_2}{2}\right)^{\alpha-1}\cdot\ln\circ\Upsilon(\tau)d\tau\right\}}\\
&= \frac{\left((\Upsilon(\sigma_2))^2\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}}{\exp\left\{\frac{1}{2}\left(\frac{2}{\sigma_2-\sigma_1}\right)^\alpha\Gamma(\alpha+1)\;\mathfrak{I}^\alpha_{\sigma_2^-}(\ln\circ\Upsilon)\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}}\\
&= \left((\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}\left({}_*\mathfrak{I}^\alpha_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}.
\end{aligned} \tag{22}$$

$$\begin{aligned}
B_2 &= \left(\int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-t^\alpha-\frac{1}{3}}\right]^{dt}\right)^{\frac{\sigma_2-\sigma_1}{4}}\\
&= \int_0^1\left[\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-\frac{\sigma_2-\sigma_1}{2}\cdot\frac{3t^\alpha+1}{6}}\right]^{dt}\\
&= \frac{(\Upsilon(\sigma_1))^{\frac{2}{3}}}{\left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{\frac{1}{6}}}\cdot\frac{1}{\int_0^1\left[\left(\Upsilon\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{\frac{1}{2}\alpha t^{\alpha-1}}\right]^{dt}}
\end{aligned}$$

$$= \frac{(\Upsilon(\sigma_1))^{\frac{2}{3}} \left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{6}}}{\exp\left\{\int_0^1 2\,\alpha t^{\alpha-1}\cdot\ \ln\circ\Upsilon\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)dt\right\}}$$

$$= \frac{\left((\Upsilon(\sigma_2))^2 \left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}}{\exp\left\{\frac{1}{2}\,\alpha\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}\left(\frac{\sigma_1+\sigma_2}{2}-\tau\right)^{\alpha-1}\cdot\ln\circ\Upsilon(\tau)d\tau\right\}}$$

$$= \frac{\left((\Upsilon(\sigma_2))^2 \left(\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}}{\exp\left\{\frac{1}{2}\left(\frac{2}{\sigma_2-\sigma_1}\right)^{\alpha}\Gamma(\alpha+1)\ \mathfrak{I}^{\alpha}_{\sigma_1^+}(\ln\circ\Upsilon)\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}}.$$

Hence

$$B_2 = \left((\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\frac{1}{2}}\right)^{\frac{1}{3}}\left({}_{\sigma_1^+}\mathfrak{I}^{\alpha}_{*}\,\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-\dfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}. \tag{23}$$

Multiplying above equalities (22) and (23) gives us the desired result. □

4. Multiplicative Bullen and Milne inequalities

Theorem 4.1. *Let h be a B-function. If Υ is a positive differentiable function on the interval on I° with $[\sigma_1,\sigma_2]\subset I^\circ$ and $|\Upsilon^*|^*$ be $*h$-convex where h satisfying the condition (10), then the next multiplicative Bullen inequality holds:*

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}^{\alpha}_{*}\Upsilon(\frac{\sigma_1+\sigma_2}{2})\cdot_* \mathcal{I}^{\alpha}_{\sigma_2^-}\Upsilon(\frac{\sigma_1+\sigma_2}{2})\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)(\Upsilon(\frac{\sigma_1+\sigma_2}{2}))^2\right)^{\frac{1}{4}}}\right|^* \leq \left[|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)C_\alpha}{4}}, \tag{24}$$

where

$$C_\alpha := \left(\frac{1}{2}\right)^{\alpha}\left(\frac{2\alpha}{\alpha+1}\right)+\left(\frac{1-\alpha}{\alpha+1}\right). \tag{25}$$

Proof. Applying the identity (18) and the equalities (3) gives

$$\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}\left\{{}_{\sigma_1^+}\mathcal{I}^{\alpha}_{*}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}^{\alpha}_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}$$

$$= \int_0^1\left(\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-1}\right]^{\frac{\sigma_2-\sigma_1}{2}\cdot\frac{2t^{\alpha}-1}{4}}\right)^{dt}$$

$$= \exp\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{8}(2t^{\alpha}-1)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right\}.$$

Using the multiplicative absolute value and the equalities (15), and (16) results gives

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}^{\alpha}_{*}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}^{\alpha}_{\sigma_2^-}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^*$$

$$= \exp\left|\int_0^1 \frac{(\sigma_2-\sigma_1)}{8}\left(2t^\alpha-1\right)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right|$$

$$\le \exp\left\{\int_0^1 \left|\frac{(\sigma_2-\sigma_1)}{8}\left(2t^\alpha-1\right)\right|\left[\left|\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right|+\left|\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right|\right]dt\right\}$$

$$= \exp\left\{\int_0^1 \ln\left[\left|\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right|^*\cdot\left|\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right|^*\right]^{\frac{\sigma_2-\sigma_1}{8}|2t^\alpha-1|}dt\right\}.$$

Considering h satisfies the inequality (10) and $|\Upsilon^*|^*$ is $*h$-convex, then we obtain

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^* \le \exp\left\{\int_0^1 \ln\left[|\Upsilon^*(\sigma_1)|^*\cdot|\Upsilon^*(\sigma_2)|^*\right]^{2h\left(\frac{1}{2}\right)\frac{\sigma_2-\sigma_1}{8}|2t^\alpha-1|}dt\right\}$$

$$= \left[|\Upsilon^*(\sigma_1)|^*\cdot|\Upsilon^*(\sigma_2)|^*\right]^{h\left(\frac{1}{2}\right)\frac{\sigma_2-\sigma_1}{4}\int_0^1|2t^\alpha-1|dt}.$$

Since

$$\int_0^1 |1-2t^\alpha|\,dt = \int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(1-2t^\alpha)dt+\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1(2t^\alpha-1)dt = \left(\frac{1}{2}\right)^\alpha\left(\frac{2\alpha}{\alpha+1}\right)\left(\frac{1-\alpha}{\alpha+1}\right),$$

we obtain the desired inequality (24). □

Theorem 4.2. *Let h be a B-function. If Υ is a positive differentiable function on the interval on I° with $[\sigma_1,\sigma_2]\subset I^\circ$ and $|\Upsilon^*|^*$ be $*h$-convex where h satisfying the condition (10), then the next multiplicative Milne inequality holds:*

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \le \left(|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right)^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)(\alpha+4)}{6(\alpha+1)}}. \tag{26}$$

Proof. Applying the identity (21) and the equalities (3) gives

$$\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}$$

$$= \int_0^1\left(\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-1}\right]^{\frac{(\sigma_2-\sigma_1)}{4}\left(t^\alpha+\frac{1}{3}\right)}\right)^{dt}$$

$$= \exp\left\{\int_0^1 \frac{(\sigma_2-\sigma_1)}{4}\left(t^\alpha+\frac{1}{3}\right)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right\}.$$

Using the multiplicative absolute value and the equality (15), and (16) results gives

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^*$$

$$= \exp\left|\int_0^1 \frac{(\sigma_2-\sigma_1)}{4}\left(t^\alpha+\frac{1}{3}\right)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right|$$

$$\leq \exp\left\{\int_0^1 \left|\frac{(\sigma_2-\sigma_1)}{4}\left(t^\alpha+\frac{1}{3}\right)\right|\left[\left|\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right|+\left|\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right|\right]dt\right\}$$
$$=\exp\left\{\int_0^1 \ln\left[\left|\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right|^*\cdot\left|\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right|^*\right]^{\frac{\sigma_2-\sigma_1}{4}\left(t^\alpha+\frac{1}{3}\right)}dt\right\}.$$

Considering h satisfies the inequality (10) and $|\Upsilon^*|^*$ is $*h$-convex, then we obtain

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^*\leq\exp\left\{\int_0^1\ln\left[|\Upsilon^*(\sigma_1)|^*\cdot|\Upsilon^*(\sigma_2)|^*\right]^{2h\left(\frac{1}{2}\right)\frac{\sigma_2-\sigma_1}{4}\left(t^\alpha+\frac{1}{3}\right)}dt\right\}$$
$$=\left[|\Upsilon^*(\sigma_1)|^*\cdot|\Upsilon^*(\sigma_2)|^*\right]^{h\left(\frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)}{2}\int_0^1\left(t^\alpha+\frac{1}{3}\right)dt}.$$

Hence, we get the desired inequality (26). □

By replacing $\alpha = 1$ through the above Theorem 4.1 and Theorem 4.2, we obtain the other results.

Corollary 4.3. *Let h be a B-function. If Υ is a positive multiplicatively h-convex function on $[\sigma_1,\sigma_2]$, then the following multiplicative Bullen and Milne inequalities hold:*

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^*\leq\left[|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{8}}. \tag{27}$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^*\leq\left[|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right]^{\frac{5(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{12}}. \tag{28}$$

4.1. *Bullen and Milne inequalities via multiplicative s-convex functions*

By replacing $h(t) = t^s$, where $s \in (0,1]$, into the inequalities (24), (26), (27) and (28), we obtain the next results.

Corollary 4.4. *Let $s \in (0,1]$ and f be a positive multiplicatively s-convex function on I° with $[\sigma_1,\sigma_2]\subset I^\circ$, then the following multiplicative inequalities are valid:*

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^\alpha\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^*\leq\left[|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right]^{\frac{(\sigma_2-\sigma_1)C_\alpha}{4\cdot 2^s}}. \tag{29}$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^*\leq\left[|\Upsilon^*(\sigma_2)|^*\cdot|\Upsilon^*(\sigma_1)|^*\right]^{\frac{\sigma_2-\sigma_1}{8\cdot 2^s}}. \tag{30}$$

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left(|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right)^{\frac{(\sigma_2-\sigma_1)(\alpha+4)}{6(\alpha+1)\cdot 2^s}}. \tag{31}$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left[|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right]^{\frac{5(\sigma_2-\sigma_1)}{12\cdot 2^s}}. \tag{32}$$

Remark 4.5. *We get new multiplicative Bullen and Milne inequalities for $*$-convex functions when we set $s = 1$ in the inequalities (29), (30), (31) and (32).*

4.2. Bullen and Milne inequalities via multiplicative P-functions

Taking $h(t) = 1$ in (24), (26), (27) and (28) gives the multiplicative Bullen inequalities for $*P$-functions.

Corollary 4.6. *If Υ is a positive multiplicatively P-functions on on I° with $[\sigma_1, \sigma_2] \subset I^\circ$, then the following multiplicative Bullen inequalities hold:*

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^* \leq \left[|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right]^{\frac{(\sigma_2-\sigma_1)C_\alpha}{4}}.$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^* \leq \left[|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right]^{\frac{\sigma_2-\sigma_1}{8}}.$$

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_*\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left(|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right)^{\frac{(\sigma_2-\sigma_1)(\alpha+4)}{6(\alpha+1)}}.$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left[|\Upsilon^*(\sigma_2)|^* \cdot |\Upsilon^*(\sigma_1)|^*\right]^{\frac{5(\sigma_2-\sigma_1)}{12}}.$$

5. Multiplicatively Bullen and Milne inequalities through monotonic functions

Lemma 5.1. *Let Υ be a positive and differentiable function on $I^\circ \subseteq \mathbb{R}$ with $[\sigma_1, \sigma_2] \subset I^\circ$, then we have*

1. *Υ is an increasing function on $[\sigma_1, \sigma_2]$ if and only if $\Upsilon^*(x) > 1$ for every $x \in [\sigma_1, \sigma_2]$.*

2. *Υ is a decreasing function on $[\sigma_1, \sigma_2]$ if and only if $0 < \Upsilon^*(x) < 1$ for every $x \in [\sigma_1, \sigma_2]$.*

Proof. Consider Υ to be a positive and differentiable function defined on $I^\circ \subseteq \mathbb{R}$ such that $[\sigma_1, \sigma_2] \subset I^\circ$. Applying Remark 1.3, for every $x \in [\sigma_1, \sigma_2]$, we obtain the condition $\Upsilon^*(x) > 1$ if and only if $\frac{\Upsilon'(x)}{\Upsilon(x)} > 0$. This implies that $\Upsilon'(x) > 0$, indicating that Υ is an increasing function. The same holds true for the second property. □

From the Remark 1.3 and the Definition 2.1, we deduce

$$|\Upsilon^*|^* = \begin{cases} \Upsilon^* & \text{if } \Upsilon^* \geq 1; \\ \frac{1}{\Upsilon^*} & \text{if } 0 < \Upsilon^* < 1. \end{cases} \tag{33}$$

The combination of the statements from (33) and Lemma 5.1 in Theorem 4.1 and Corollary 4.3 produces the next corollaries for the cases $\Upsilon > 1$ and $\Upsilon < 1$, respectively.

Corollary 5.2. *Let h be a B-function and Υ be a positive and differentiable function on $I^\circ \subseteq \mathbb{R}$ with $[\sigma_1, \sigma_2] \subset I^\circ$. If Υ is increasing function on $[\sigma_1, \sigma_2]$ and Υ^* is multiplicatively h-convex function (*h-convex) on $[\sigma_1, \sigma_2]$, then the following inequalities hold:*

$$\left| \frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_*^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2) \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2 \right)^{\frac{1}{4}}} \right|^* \leq \left[\Upsilon^*(\sigma_2) \cdot \Upsilon^*(\sigma_1) \right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)C_\alpha}{4}}.$$

$$\left| \frac{\left(\int_{\sigma_1}^{\sigma_2} [(\Upsilon(t))]^{dt} \right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2) \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2 \right)^{\frac{1}{4}}} \right|^* \leq \left[\Upsilon^*(\sigma_2) \cdot \Upsilon^*(\sigma_1) \right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{8}}.$$

$$\left| \frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_*^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2 \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1} \right)^{\frac{1}{3}}} \right|^* \leq \left(\Upsilon^*(\sigma_2) \cdot \Upsilon^*(\sigma_1) \right)^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)(\alpha+4)}{6(\alpha+1)}}.$$

$$\left| \frac{\left(\int_{\sigma_1}^{\sigma_2} [(\Upsilon(t))]^{dt} \right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2 \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1} \right)^{\frac{1}{3}}} \right|^* \leq \left[\Upsilon^*(\sigma_2) \cdot \Upsilon^*(\sigma_1) \right]^{\frac{5(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{12}}.$$

Corollary 5.3. *Let h be a B-function and Υ be a positive and differentiable function on $I^\circ \subseteq \mathbb{R}$ with $[\sigma_1, \sigma_2] \subset I^\circ$. If Υ is decreasing function on $[\sigma_1, \sigma_2]$ and Υ^* is multiplicatively h-concave function (*h-concave) on $[\sigma_1, \sigma_2]$, then the following inequalities hold:*

$$\left| \frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_*^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha} \Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2) \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2 \right)^{\frac{1}{4}}} \right|^* \leq \left[\frac{1}{\Upsilon^*(\sigma_2) \cdot \Upsilon^*(\sigma_1)} \right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)C_\alpha}{4}}.$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^* \leq \left[\frac{1}{\Upsilon^*(\sigma_2)\cdot\Upsilon^*(\sigma_1)}\right]^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{8}}.$$

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left(\frac{1}{\Upsilon^*(\sigma_2)\cdot\Upsilon^*(\sigma_1)}\right)^{\frac{(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)(\alpha+4)}{6(\alpha+1)}}.$$

$$\left|\frac{\left(\int_{\sigma_1}^{\sigma_2}[(\Upsilon(t))]^{dt}\right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^* \leq \left[\frac{1}{\Upsilon^*(\sigma_2)\cdot\Upsilon^*(\sigma_1)}\right]^{\frac{5(\sigma_2-\sigma_1)h\left(\frac{1}{2}\right)}{12}}.$$

Remark 5.4.

- *Choosing $h(t)=t^s$ where $s\in(0,1]$ in Corollary 5.2 and Corollary 5.3 yields conclusions related to multiplicative s-convex functions.*
- *Taking $h(t)=1$ in Corollary 5.2 and Corollary 5.3 produces consequences through multiplicative P-functions.*

6. Bullen and Milne inequalities through multiplicatively Lipschitzian functions

Definition 6.1. *A function $\Upsilon : I^\circ \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ defined on the interval $[\sigma_1,\sigma_2]\subset I^\circ$ is called a multiplicatively k-Lipschitzian (∗k-Lipschitzian) on $[\sigma_1,\sigma_2]$ if there exists a positive real constant k such that for all $x,y\in[\sigma_1,\sigma_2]$, we have*

$$\left|\frac{\Upsilon(x)}{\Upsilon(y)}\right|^* \leq \left|\left(\frac{x}{y}\right)^k\right|^*. \tag{34}$$

Example 6.2. *Let $0<p<k$ be two real numbers.*

1. *The function $\Upsilon_1(x)=x^p$ is a multiplicatively k-Lipschitzian on $[\sigma_1,\sigma_2]\subset\mathbb{R}^+$.*
2. *The function $\Upsilon_2(x)=e^x$ is a multiplicatively p-Lipschitzian on $[\sigma_1,\sigma_2]\subset]0,p[$.*
3. *The function $\Upsilon_3(x)=\ln x$ is a multiplicatively p-Lipschitzian on $[\sigma_1,\sigma_2]\subset]e^{\frac{1}{p}},+\infty[$.*

Proof. Since $\phi_1(x)=\frac{1}{x^{k-p}}$ is a non-increasing function on $]0,+\infty[$, then for all $x,y\in[\sigma_1,\sigma_2]\subset\mathbb{R}^+$ with $y<x$, we get

$$\phi_1(x)\leq\phi_1(y)\Longleftrightarrow\frac{x^p}{y^p}\leq\frac{x^k}{y^k}.$$

So,

$$\left|\frac{\Upsilon_1(x)}{\Upsilon_1(y)}\right|^* \leq \left|\left(\frac{x}{y}\right)^k\right|^*.$$

Given that $\phi_2(x) = \frac{e^x}{x^p}$ is a non-increasing function on $]0, p[$, then for all $x, y \in [\sigma_1, \sigma_2] \subset]0, p[$ with $y < x$, we have

$$\frac{e^x}{e^y} \leq \frac{x^p}{y^p}.$$

Therefore

$$\left|\frac{\Upsilon_2(x)}{\Upsilon_2(y)}\right|^* \leq \left|\left(\frac{x}{y}\right)^p\right|^*.$$

Since $\phi_3(x) = \frac{\ln x}{x^p}$ is a non-increasing function on $]e^{\frac{1}{p}}, +\infty[$, then for al $x, y \in [\sigma_1, \sigma_2] \subset]e^{\frac{1}{p}}, +\infty[$ with $y < x$, we obtain

$$\frac{\ln x}{\ln y} \leq \frac{x^p}{y^p}.$$

Hence,

$$\left|\frac{\Upsilon_3(x)}{\Upsilon_3(y)}\right|^* \leq \left|\left(\frac{x}{y}\right)^p\right|^*.$$

□

Property 6.3. *Since* $\ln(\cdot)$ *is a L-Lipschitzian function on the interval* $[\sigma_1, \sigma_2] \subset \mathbb{R}^+$*, the following property can be established:*

If Φ *is a multiplicatively k-Lipschitzian function, then* $\ln \circ \Phi$ *is a Lk-Lipschitzian function.*

Proof. Suppose that $\Phi : I^\circ \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is a multiplicatively k-Lipschitzian function on $[\sigma_1, \sigma_2] \subset I^\circ$, then

$$\left|\frac{\Phi(x)}{\Phi(y)}\right|^* \leq \left|\left(\frac{x}{y}\right)^k\right|^*.$$

Using (15), gives

$$\exp\left|(\ln \circ \Phi)(x) - (\ln \circ \Phi)(y)\right| \leq \exp\left|k(\ln x - \ln y)\right|.$$

So,

$$\left|(\ln \circ \Phi)(x) - (\ln \circ \Phi)(y)\right| \leq kL\left|x - y\right|.$$

□

Theorem 6.4. *Let* Υ *be a positive differentiable function on the interval on* I° *with* $[\sigma_1, \sigma_2] \subset I^\circ$*. If* Υ^* *is a multiplicatively k-Lipschitzian function (*k-Lipschitzian) on* $[\sigma_1, \sigma_2]$*, then*

$$\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right) \cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^* \leq \exp\left\{\frac{Lk\,(\sigma_2 - \sigma_1)^2\, C_\alpha}{8}\right\}, \tag{35}$$

where

$$C_\alpha := \left(\frac{1}{2}\right)^\alpha\left(\frac{2\alpha}{\alpha+1}\right) + \left(\frac{1-\alpha}{\alpha+1}\right).$$

Moreover,

$$\left|\frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_{*}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_{*}\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^{*}\leq\exp\left\{\frac{Lk\,(\sigma_2-\sigma_1)^2\,(\alpha+4)}{12(\alpha+1)}\right\}. \tag{36}$$

Proof. Applying the identity (18) and the equalities (3) gives

$$\begin{aligned}
&\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}\left\{ {}_{\sigma_1^+}\mathcal{I}_{*}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_{*}\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}\\
&=\int_0^1\left(\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-1}\right]^{\frac{\sigma_2-\sigma_1}{2}\cdot\frac{2t^{\alpha}-1}{4}}\right)^{dt}\\
&=\exp\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{8}(2t^{\alpha}-1)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right\}\\
&=\exp\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{8}(2t^{\alpha}-1)\left[\left(\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right.\right.\\
&\qquad\left.\left.-\left(\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\right)\right]dt\right\}.
\end{aligned}$$

Using the multiplicative absolute value and the equality (15) results gives

$$\begin{aligned}
&\left|\frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_{*}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_{*}\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^{*}\\
&=\exp\left|\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{8}(2t^{\alpha}-1)\left[\left(\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-\left(\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right]dt\right\}\right|\\
&\leq\exp\left\{\int_0^1\left|\frac{(\sigma_2-\sigma_1)}{8}(2t^{\alpha}-1)\right|\left[\left|\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right|\right.\right.\\
&\qquad\left.\left.+\left|\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right|\right]dt\right\}.
\end{aligned}$$

Since $\ln\circ\Upsilon^*$ is a Lk-Lipschitzian function, we derive

$$\begin{aligned}
&\left|\frac{\left\{ {}_{\sigma_1^+}\mathcal{I}_{*}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_{*}\mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2)\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2\right)^{\frac{1}{4}}}\right|^{*}\\
&\leq\exp\left\{\frac{kL(\sigma_2-\sigma_1)}{8}\int_0^1|(2t^{\alpha}-1)|\left[\left|\left(\frac{1+t}{2}\right)(\sigma_2-\sigma_1)\right|+\left|\left(\frac{1-t}{2}\right)(\sigma_2-\sigma_1)\right|\right]dt\right\}\\
&=\exp\left\{\frac{kL(\sigma_2-\sigma_1)^2}{8}\int_0^1|(2t^{\alpha}-1)|\,dt\right\}.
\end{aligned}$$

Now, using the identity (21) and the equalities (3) gives

$$\begin{aligned}
&\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}\\
&=\int_0^1\left(\left[\left(\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)\right)\left(\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right)^{-1}\right]^{\frac{\sigma_2-\sigma_1}{4}\left(t^{\alpha}+\frac{1}{3}\right)}\right)^{dt}\\
&=\exp\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{4}\left(t^{\alpha}+\frac{1}{3}\right)\left[\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)\right]dt\right\}\\
&=\exp\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{4}\left(t^{\alpha}+\frac{1}{3}\right)\left[\left(\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right.\right.\\
&\qquad\left.\left.-\left(\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right]dt\right\}.
\end{aligned}$$

Applying the multiplicative absolute value and the equality (15) results gives

$$\begin{aligned}
&\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^*\\
&=\exp\left|\left\{\int_0^1\frac{(\sigma_2-\sigma_1)}{4}\left(t^{\alpha}+\frac{1}{3}\right)\left[\left(\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-\left(\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right)\right]dt\right\}\right|\\
&\le\exp\left\{\int_0^1\left|\frac{(\sigma_2-\sigma_1)}{4}\left(t^{\alpha}+\frac{1}{3}\right)\right|\left[\left|\ln\circ\Upsilon^*\left(\left(\frac{1-t}{2}\right)\sigma_1+\left(\frac{1+t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right|\right.\right.\\
&\qquad\left.\left.+\left|\ln\circ\Upsilon^*\left(\left(\frac{1+t}{2}\right)\sigma_1+\left(\frac{1-t}{2}\right)\sigma_2\right)-\ln\circ\Upsilon^*(\sigma_1)\right|\right]dt\right\}.
\end{aligned}$$

Given $\ln\circ\Upsilon^*$ is a Lk-Lipschitzian function, we derive

$$\begin{aligned}
&\left|\frac{\left\{{}_{\sigma_1^+}\mathcal{I}_*^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\cdot_* \mathcal{I}_{\sigma_2^-}^{\alpha}\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right\}^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2\left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^{-1}\right)^{\frac{1}{3}}}\right|^*\\
&\le\exp\left\{\frac{kL(\sigma_2-\sigma_1)}{4}\int_0^1\left(t^{\alpha}+\frac{1}{3}\right)\left[\left|\left(\frac{1+t}{2}\right)(\sigma_2-\sigma_1)\right|+\left|\left(\frac{1-t}{2}\right)(\sigma_2-\sigma_1)\right|\right]dt\right\}\\
&=\exp\left\{\frac{kL(\sigma_2-\sigma_1)^2}{4}\int_0^1\left(t^{\alpha}+\frac{1}{3}\right)dt\right\}.
\end{aligned}$$

□

By putting $\alpha=1$ in Theorem 6.4, we derive the next result.

Corollary 6.5. *Let Υ be a positive differentiable function on the interval on I° with $[\sigma_1,\sigma_2]\subset I^\circ$. If Υ^* is a multiplicatively k-Lipschitzian function (*k-Lipschitzian) on $[\sigma_1,\sigma_2]$, then the following multiplicative inequalities* *hold:*

$$\left| \frac{\left(\int_{\sigma_1}^{\sigma_2} [(\Upsilon(t))]^{dt} \right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left(\Upsilon(\sigma_1)\Upsilon(\sigma_2) \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2} \right) \right)^2 \right)^{\frac{1}{4}}} \right|^* \leq e^{\frac{k(\sigma_2-\sigma_1)^2}{16}},$$

and

$$\left| \frac{\left(\int_{\sigma_1}^{\sigma_2} [(\Upsilon(t))]^{dt} \right)^{\frac{1}{\sigma_2-\sigma_1}}}{\left((\Upsilon(\sigma_1)\Upsilon(\sigma_2))^2 \left(\Upsilon\left(\frac{\sigma_1+\sigma_2}{2} \right) \right)^{-1} \right)^{\frac{1}{3}}} \right|^* \leq e^{\frac{5k(\sigma_2-\sigma_1)^2}{24}}.$$

7. Conclusion

The present research established the Bullen and Milne inequalities in the context of multiplicative calculus for positive multiplicatively h-convex functions using the multiplicative absolute value. A pair of new lemmas on monotonic functions in the context of multiplicative calculus and a new definition of multiplicatively Lipschitz functions have allowed for the development of new versions of the multiplicative Bullen and Milne inequalities through monotonic functions and multiplicatively Lipschitz functions. These results pave the way for new and fascinating directions in multiplicative calculus and its applications to different types of convexity.

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