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Pseudo skew-symmetric matrices over max-plus algebras

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Abstract. In this paper we consider some properties of symmetric and skew-symmetric matrices over the max-plus algebra. These types of matrices have certain features that are valid in the conventional linear algebra but not in the linear algebra over the max-plus semiring, and vice versa. Taking this into account, we describe a new class of matrices over the max-plus algebra - the class of pseudo skew-symmetric matrices. Pseudo skew-symmetric matrices are square matrices with a zero diagonal whose symmetric elements cannot be negative at the same time. The basic properties of this class of matrices are introduced and proved in this study.

1. Introduction

Matrices play a fundamental role in various mathematical contexts. For the purpose of this paper, we will explore two specific types of matrices named symmetric and skew-symmetric matrices. A square matrix A is symmetric if its transpose A^T equals itself, i.e. $A = A^T$. In simpler terms, the elements along the main diagonal remain unchanged, and the elements above and below the diagonal are mirror images of each other. Symmetric matrices find many applications in optimization, physics, and statistics. A square matrix A is skew-symmetric (or antisymmetric) if its transpose is equal to its negative, i.e. $A^T = -A$. In skew-symmetric matrices, the diagonal elements are always zero. Although less common in real-world applications, skew-symmetric matrices appear in mechanics, electromagnetism, and various fields of mathematics. Understanding these properties of matrices improves our ability to handle complex mathematical challenges. In the present study, we deal with matrices with values in the max-plus algebra, from a similar point of view as in conventional linear algebra. We need to mention that, in this paper, a notion of max-algebra is equal to a notion of max-plus algebra. Vectors and matrices over the max-algebra can be defined naturally, as ordered n-tuples and $m \times n$-tuples of elements from $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, respectively. Other names used in the past for the max-algebra are „path algebra" [12] and „schedule algebra" [21].

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The study of the max-algebra begins in the 1960's with the papers of Cuninghame-Green [15], followed by Carre [12], Gondran and Minoux [23], Vorobyov [28], etc. Basic questions, such as solving systems of equations, the eigenvalue/eigenvector problem, and linear independence in the max-plus sense, are studied in these papers. Strong development of the max-algebra starts from 1985 onwards. For more information, we recommend readers an excellent monograph [6] (notations in this paper are aligned with this book), as well as [3], [4], [7], [16].

Many features from the conventional linear algebra can be transferred to max-algebra by replacing „+" with „$\oplus$" and „ $\cdot$" with „$\otimes$", like eigenvalues and eigenvectors (Chapter 4. in [6]), Cramer's rule and Cayley-Hamilton theorem [27], numerical range of matrices [30], etc. The max-algebra is widely used in many fields such as mathematical physics, optimization, telecommunication networks, control theory, discrete event processes, algebraic geometry, machine scheduling, parallel processing systems, manufacturing systems, and traffic theory (see e.g. [1], [4], [19], [25]). Many problems that are non-linear in conventional algebra can be solved in the max-algebra using a linear approach. Also, many statements and procedures used in conventional linear algebra have corresponding counterparts in the max-algebra.

The paper is organized as follows. After this introductory section, in Section 2 we introduce the most important concepts concerning max-plus semirings (algebras) and complete max-plus semirings, as well as matrices and vectors over them. We mentioned some new tools for dealing with matrices over the max-plus algebra which can not be used in conventional linear algebra. Then, in Section 3, we introduce the concepts of a max-plus symmetric and skew-symmetric matrices and present some characterisations of those matrices. Further, we take into the consideration the conjugate matrix of a given matrix A whose elements are real numbers, and describe the properties of the matrices which are obtained as the max-algebraic sum/max-algebraic product of A and its transpose matrix or its conjugate matrix. In Section 4, pseudo skew-symmetric matrices are defined and named max-algebraic pseudo skew-symmetric matrices or shortly *MAPSS* matrices. Their properties and characteristics of the matrices obtained by applying the max-algebraic operations on *MAPSS* matrices are also presented in this section. Beside, we define and describe a class of weakly *MAPSS* matrices. In the last section, Section 5, we study *MAPSS* and weakly *MAPSS* matrices with infinite entries and introduce their features.

The motivation for defining a new class of matrices - *MAPSS* matrices - has its roots in the paper of Cuninghame-Green and Butkovic, which describes the solving of two-sided max-linear systems of the form $A \otimes x = B \otimes y$, where A, B are matrices and x, y are vectors of compatible sizes. Namely, in [13], the authors have presented an iterative method for solving the two-sided systems of the mentioned form - the so-called Alternating method. If A and B are integer matrices, and one of them, for example A, is finite, while the other matrix does not contain a row nor column whose elements are all equal to $-\infty$, then the number of iterations at the Alternating method is not greater than $(n-1)\,(1 + x(0)^* \otimes A^* \otimes A \otimes x(0))$, where $x(0)$ is random integer vector and n is the column-dimension of A. Thus, $\tilde{A} = A^* \otimes A$ appears in this expression, so we started with the description of the properties of this matrix in order to consider ways to reduce the number of iterations at the Alternating method even more. As a result of the new properties and conclusions we reached, we defined a new class of matrices over max-algebra to which, as we shall see, the matrix $\tilde{A}$ itself belongs.

Also, considering such a class of matrices in conventional linear algebra would not make much sense, because all properties derive from the specific way in which operations in the max-algebra are defined. That is the reason why we studied the mentioned class of matrices over this algebraic structure.

2. Preliminaries

An algebra $\mathbb{S} = (\mathbb{S}, \oplus, \otimes, 0, 1)$ with two binary operations $\oplus$ and $\otimes$ on $\mathbb{S}$, and two constants $0, 1 \in \mathbb{S}$ is a *semiring* if the following conditions hold:

(1) $(\mathbb{S}, \oplus, 0)$ is a commutative semigroup with neutral 0,

(2) $(\mathbb{S}, \otimes, 1)$ is a semigroup with identity 1,

(3) (*distributivity laws*) are satisfied: $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, for all $a, b, c \in \mathbb{S}$,

(4) $0 \otimes a = a \otimes 0 = 0$, for every $a \in \mathbb{S}$.

0 is called the *zero* and 1 the *identity* of the semiring $\mathbb{S}$. The operation $\oplus$ is called *addition*, and the operation $\otimes$ is called *multiplication*. If the multiplication is also commutative, then $\mathbb{S}$ is called a *commutative semiring*. An algebraic structure can be identified with its carrier set, and that is why we denote the carrier set of a semiring $\mathbb{S}$ by the same symbol $\mathbb{S}$.

Typical examples of semirings are fields of real and rational numbers, the ring of integers and the semiring of natural numbers with zero included (with respect to the usual addition and multiplication operations). We refer the reader to [22] for more information on semiring theory.

An element $a \in \mathbb{S}$ is an *idempotent* if $a \oplus a = a$, or equivalently, if $1 \oplus 1 = 1$. In the literature, semirings whose all elements are idempotents are known as *additively idempotent semirings*, *idempotent semirings* or just *dioids*.

Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{S}$ be a semiring. For any $m, n \in \mathbb{N}$, an $m \times n$*-matrix* with entries in $\mathbb{S}$ (or over $\mathbb{S}$) is defined as any mapping $A : \{1 \dots m\} \times \{1 \dots n\} \to \mathbb{S}$, and for arbitrary $(i, j) \in \{1 \dots m\} \times \{1 \dots n\}$, the value a_{ij} is said to be the (i, j)*-element* or (i, j)*-entry* of the matrix A. The set of all $m \times n$-matrices with entries in $\mathbb{S}$ is denoted by $\mathbb{S}^{m\times n}$. Following that, a *vector* of length m over $\mathbb{S}$, for a given $m \in \mathbb{N}$, is a mapping $v : \{1 \dots m\} \to \mathbb{S}$. The notation $\mathbb{S}^m$ is used for the set of all vectors of a length m over $\mathbb{S}$. Thus, we can say that $1 \times n$-matrix is a *row vector* (of length n), and an $m \times 1$-matrix is a *column vector* (of length m). An *identity matrix* of order n is an $n \times n$-matrix I over $\mathbb{S}$ whose (j, k)-element is equal to 1, if $j = k$, and 0, if $j \neq k$, for all $j, k \in \{1 \dots n\}$. The *transpose* of a matrix $A \in \mathbb{S}^{m\times n}$, is a matrix $A^T \in \mathbb{S}^{n\times m}$ whose (i, j)-element is equal to the (j, i)-element of the matrix A, for all $i, j \in \{1, \dots, n\}$.

For $m, n \in \mathbb{N}$, the *matrix addition* is a binary operation on $\mathbb{S}^{m\times n}$ defined for $A, B \in \mathbb{S}^{m\times n}$ such that $C = A \oplus B$ and its entries are

$$c_{ij} = a_{ij} \oplus b_{ij},$$

for all $i \in \{1 \dots m\}$ and $j \in \{1 \dots n\}$. It is an associative and commutative operation on $\mathbb{S}^{m\times n}$, and it can be easily shown that $(\mathbb{S}^{m\times n}, \oplus, O_{m\times n})$ form a commutative semigroup with identity $O_{m\times n}$.

For arbitrary $m, n, p \in \mathbb{N}$, the *matrix product* or *matrix multiplication* is defined between matrices from $\mathbb{S}^{m\times n}$ and $\mathbb{S}^{n\times p}$ as follows: for matrices $A \in \mathbb{S}^{m\times n}$ and $B \in \mathbb{S}^{n\times p}$, their product is a matrix $C = A \otimes B \in \mathbb{S}^{m\times p}$ with entries given by

$$C_{ik} = \sum_{j=1}^{n}{}^{\oplus} a_{ij} \otimes b_{jk}, \tag{1}$$

for all $(i, k) \in \{1 \dots m\} \times \{1 \dots p\}$. Whenever the matrix product is defined it is associative, i.e.,

$$(A \otimes B) \otimes C = A \otimes (B \otimes C),$$

for all $A \in \mathbb{S}^{m\times n}$, $B \in \mathbb{S}^{n\times p}$ and $C \in \mathbb{S}^{p\times k}$.

In this paper we deal with a particular type of semiring (algebra) $\overline{\mathbb{R}} = (\overline{\mathbb{R}}, \oplus, \otimes, -\infty, 0)$, with the carrier set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, and binary operations $\oplus$ and $\otimes$ defined as follows: for arbitrary $a, b \in \overline{\mathbb{R}}$ we put $a \oplus b = \max(a, b)$, and

$$a \otimes b = \begin{cases} a + b & \text{if } a, b \in \mathbb{R}, \\ -\infty & \text{if } a = -\infty \text{ or } b = -\infty. \end{cases} \tag{2}$$

The operation $\oplus = \max$ refers to the usual ordering $\leqslant$ of real numbers extended to $\overline{\mathbb{R}}$ so that $-\infty$ is the least element. This semiring is known as the *max-plus algebra*. The zero of the semiring $\overline{\mathbb{R}}$ is $-\infty$ and the identity is 0. The max-plus algebra $\overline{\mathbb{R}}$ is a commutative dioid which does not have the greatest element.

The operation $\oplus$ is called the max-algebraic addition, and $\otimes$ is the max-algebraic multiplication. For the sake of simplicity, we will call them addition and multiplication, respectively.

Dually, the semiring $\underline{\mathbb{R}} = (\underline{\mathbb{R}}, \oplus', \otimes', +\infty, 0)$ with the carrier set $\underline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, and two binary operations $\oplus'$ and $\otimes'$ defined as follows:

$$a \oplus' b = \min(a, b), \qquad a \otimes' b = \begin{cases} a + b & \text{if } a, b \in \mathbb{R}, \\ +\infty & \text{if } a = +\infty \text{ or } b = +\infty. \end{cases}$$

will be called the *min-plus semiring*. For $\underline{\mathbb{R}}$ we also use the name *min-plus algebra*, or simply *min-algebra*.

When dealing with vectors and matrices, max-algebraic operations are defined as follows. Let A, B and C be matrices of compatible sizes with entries from $\overline{\mathbb{R}}$. Then

$$C = A \oplus B \quad \text{if} \quad c_{ij} = a_{ij} \oplus b_{ij} = \max\left\{a_{ij}, b_{ij}\right\}, \quad \text{for all } i, j, \tag{3}$$

$$C = A \otimes B \quad \text{if} \quad c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k \left\{a_{ik} + b_{kj}\right\}, \quad \text{for all } i, j, \tag{4}$$

$$\theta \otimes A = A \otimes \theta = \left(\theta \otimes a_{ij}\right), \quad \text{for all } \theta \in \overline{\mathbb{R}}. \tag{5}$$

Throughout this paper, max-algebraic sum and max-algebraic product will be denoted by $\sum^{\oplus}$ and $\prod^{\otimes}$, respectively. In addition, N will denote the set with elements $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$.

The identity max-algebraic matrix, denoted by I, is a square matrix of an arbitrary size whose diagonal elements are equal to zero (which is neutral element for $\otimes$) and the off-diagonal elements are all equal to $-\infty$ (neutral for $\oplus$). A diagonal matrix is a square matrix whose off-diagonal elements are equal to $-\infty$ and the diagonal elements are real numbers.

If A is a square matrix, then the expression $A \otimes A \otimes \ldots \otimes A$, in which A appears n times, will be denoted by A^n and called the n-th power of A. The (i, j)-entry of A^n will be denoted by $a_{ij}^{(n)}$.

When it comes to the priority of max-algebraic operations, it is the same as the priority of the corresponding operations in the conventional linear algebra. Thus, the max-algebraic power has the highest priority, while the max-algebraic multiplication has higher priority than the max-algebraic addition.

The max-algebraic operation $\oplus$ is not invertible, but it is idempotent, therefore some new tools have to be provided in the max-algebra, which do not exist in the conventional linear algebra, and which allow solving the main problems. The introduction of new tools is necessary because in the max-algebra there are no inverse matrices, except for the class of generalized permutation matrices [17]. That is why many procedures from the conventional linear algebra cannot be used in the max-algebra. Some of those new tools are maximum cycle mean, transitive closures, conjugation and dual operators, and max-algebraic permanent (see, for example, Sect. 1.6 in [6]).

In the max-algebra, matrices can be associated with weighted digraphs. Using them, here we can define the following matrices: irreducible/reducible matrices, definite and strongly definite, increasing, diagonally dominant, normal and strictly normal matrices. Also, in the max-algebra two matrices can be equivalent, similar and directly similar. Beside, we can talk about diagonal similarity scaling [8].

3. Symmetric and skew-symmetric matrices

In the conventional linear algebra, a matrix A is symmetric if $A = A^T$. A matrix A is skew-symmetric if $A = -A^T$. Symmetric and skew-symmetric matrices are square matrices. A skew-symmetric matrix has zeros on the main diagonal. Also, the following is valid in the conventional linear algebra (clearly, here „+" and „ ·" are the addition and multiplication of matrices in the linear-algebraic sense): if A is a square matrix, then $A + A^T$, $A \cdot A^T$ and $A \cdot (-A^T)$ are symmetric, while $A + (-A^T)$ is a skew-symmetric matrix. We consider whether these properties are also valid in the max-algebra.

Before that, we will state some features of transpose of the matrix and symmetric matrix that are valid in max-algebra. When dealing with the transpose of a matrix, it is not difficult to show that, as in the conventional linear algebra, the following properties hold:

$$\begin{aligned} (A \oplus B)^T &= A^T \oplus B^T, && (6)\\ (\theta \otimes A)^T &= \theta \otimes A^T, && (7)\\ (A \otimes B)^T &= B^T \otimes A^T, && (8) \end{aligned}$$

for all $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $\theta \in \mathbb{R}$.

The following theorem describes the properties of symmetric matrices in the max-algebra. These properties have their analogues in the conventional linear algebra.

Theorem 3.1. *Let $A, B \in \overline{\mathbb{R}}^{n\times n}$ be symmetric matrices and $\theta \in \mathbb{R}$. Then:*

1. *$\theta \otimes A$ is a symmetric matrix;*
2. *$A \oplus B$ is a symmetric matrix;*
3. *If $A \otimes B = B \otimes A$ holds, then $A \otimes B$ is a symmetric matrix;*
4. *If $s \in \mathbb{N}$, then A^s is a symmetric matrix.*

Proof. 1. This follows directly from the definitions of symmetric matrices and the max-algebraic scalar multiplication (Eq. (5)).

2. If A and B are symmetric matrices, we have that

$$a_{ij} = a_{ji} \qquad \text{and} \qquad b_{ij} = b_{ji}, \qquad \text{for all } i, j \in N.$$

Let $C = A \oplus B$. Then, the following holds:

$$c_{ij} = \max\left\{a_{ij}, b_{ij}\right\} = \max\left\{a_{ji}, b_{ji}\right\} = c_{ji}.$$

Therefore, the matrix C is symmetric.

3. Let us use the notation $C = A \otimes B$ and $D = B \otimes A$. Then, we calculate the elements of these matrices as

$$c_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes b_{kj} \qquad \text{and} \qquad d_{ij} = \sum_{k\in N}^{\oplus} b_{ik} \otimes a_{kj}.$$

Since $A \otimes B = B \otimes A$, the equality of the previous expressions holds, i.e.:

$$\sum_{k\in N}^{\oplus} a_{ik} \otimes b_{kj} = \sum_{k\in N}^{\oplus} b_{ik} \otimes a_{kj}.$$

From there, as well as from the commutativity of the operation $\otimes$ in $\overline{\mathbb{R}}$, and the symmetry of the matrices A and B, we get

$$c_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes b_{kj} = \sum_{k\in N}^{\oplus} b_{ik} \otimes a_{kj} = \sum_{k\in N}^{\oplus} a_{kj} \otimes b_{ik} = \sum_{k\in N}^{\oplus} a_{jk} \otimes b_{ki} = c_{ji}.$$

So, $C = A \otimes B$ is a symmetric matrix.

4. Let A be a symmetric matrix, i.e. $a_{ij} = a_{ji}$, for all $i, j \in N$. By the principle of mathematical induction we will prove that $a_{ij}^{(s)} = a_{ji}^{(s)}$.

For $n = 1$, the statement of the theorem is valid, because, according to the condition of the theorem, A is symmetric. Suppose that the statement holds for some $n = p$, $p \in \mathbb{N}$:

$$a_{ij}^{(p)} = a_{ji}^{(p)}. \tag{9}$$

Let us prove then that the statement is also valid for $n = p + 1$. As $A^{p+1} = A^p \otimes A$, as (9) holds, and as $\otimes$ is commutative in $\overline{\mathbb{R}}$, we have

$$a_{ij}^{(p+1)} = \sum_{k\in N}^{\oplus} a_{ik}^{(p)} \otimes a_{kj} = \sum_{k\in N}^{\oplus} a_{ki}^{(p)} \otimes a_{jk} = \sum_{k\in N}^{\oplus} a_{jk} \otimes a_{ki}^{(p)} = a_{ji}^{(p+1)}.$$

We conclude that A^s is a symmetric matrix.

This completes the proof. □

Remark 3.2. If A is symmetric, then according to statement 4. of Theorem 3.1, the matrix A^2 is a symmetric matrix. In addition, it is easy to see that for the diagonal elements of this matrix holds:

$$a_{ii}^{(2)} = 2\max_{k\in N}\{a_{ki}\} = 2\max_{k\in N}\{a_{ik}\}.$$

However, this does not have to be true for all powers of a matrix, as the next example shows.

Example 3.3. Let the symmetric matrix A be given by:

$$A = \begin{pmatrix} 7 & 1 & 3 \\ 1 & 6 & -2 \\ 3 & -2 & 5 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 14 & 8 & 10 \\ 8 & 12 & 4 \\ 10 & 4 & 10 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 21 & 15 & 17 \\ 15 & 18 & 11 \\ 17 & 11 & 15 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 28 & 22 & 24 \\ 22 & 24 & 18 \\ 24 & 18 & 20 \end{pmatrix}, \quad A^5 = \begin{pmatrix} 35 & 29 & 31 \\ 29 & 30 & 25 \\ 31 & 25 & 27 \end{pmatrix}, \quad \ldots$$

Note that A^2, A^3 and A^4 are symmetric matrices, and the same property have matrices A^5, A^6, etc. Also, note that for the diagonal elements of A^2, according to Remark 3.2, holds

$$14 = 2\max\{7,1,3\}; \quad 12 = 2\max\{1,6,-2\}; \quad 10 = 2\max\{3,-2,5\}.$$

Moreover, in this case the same property holds for the diagonal elements of matrices A^3 and A^4 (where 3 and 4, respectively, multiply the maximum):

$$21 = 3\max\{7,1,3\}; \quad 18 = 3\max\{1,6,-2\}; \quad 15 = 3\max\{3,-2,5\}.$$

$$28 = 4\max\{7,1,3\}; \quad 24 = 4\max\{1,6,-2\}; \quad 20 = 4\max\{3,-2,5\}.$$

However, when it comes to the matrix A^5, this property will not be valid (because $a_{33}^{(5)} = 27 \neq 25 = 5\max\{3,-2,5\}$), so this feature does not apply in the general case. ■

In the rest of the paper, for the matrix $A \in \mathbb{R}^{m\times n}$, the matrix $-A^T$ will be denoted by A^*. Hence, for any entry a_{ij}^* of $\in A^*$ it holds $a_{ij}^* = -a_{ji}$). This matrix is usually called the **conjugate matrix** of A and it will be very important in our further considerations (the study of conjugate matrices dates back to the earliest papers in max-algebra and plays a crucial role in solving max-linear systems, see [18]). Note that when dealing with the matrix A^*, we assume that the entries of the matrix A are real numbers. Later, it will be discussed the case when infinite entries appear in the matrix A.

It is easy to see that $(A^*)^* = A$ and $(\theta \otimes A)^* = -\theta \otimes A^*$, for every $\theta \in \mathbb{R}$.

In the next three theorems, we consider the properties of the following matrices: $A \oplus A^T$, $A \oplus A^*$ and $A \otimes A^T$. The matrix $A \otimes A^*$ has some special properties and will be discussed in the next section.

Theorem 3.4. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then $A' = A \oplus A^T$ is a symmetric matrix for whose diagonal elements hold*

$$a'_{ii} = a_{ii}, \quad \textit{for all } i \in N. \tag{10}$$

Proof. For the elements of the matrices A and A^T holds $a^T_{ij} = a_{ji}$, for all $i, j \in N$, so we have

$$a'_{ij} = \max\left\{a_{ij}, a^T_{ij}\right\} = \max\left\{a^T_{ji}, a_{ji}\right\} = a'_{ji}, \quad \text{for all } i, j \in N.$$

Hence, A' is a symmetric matrix.

For the diagonal elements, since $a^T_{ii} = a_{ii}$, it follows

$$a'_{ii} = \max\left\{a_{ii}, a^T_{ii}\right\} = \max\left\{a_{ii}, a_{ii}\right\} = a_{ii}, \quad \text{for all } i \in N,$$

and the proof is complete. □

Theorem 3.5. *Let $A \in \overline{\mathbb{R}}^{n\times m}$. Then $A^\lozenge = A \otimes A^T$ is a symmetric matrix and for its diagonal elements holds*

$$a^\lozenge_{ii} = 2\max_{k\in N}\left\{a_{ik}\right\} = 2\max_{k\in N}\left\{a^T_{ki}\right\}. \tag{11}$$

Proof. If $A \in \overline{\mathbb{R}}^{n\times m}$, then $A^T \in \overline{\mathbb{R}}^{m\times n}$, so $A^\lozenge \in \overline{\mathbb{R}}^{n\times n}$.

We calculate the element $a^\lozenge_{ij}$ as

$$a^\lozenge_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^T_{kj},$$

for all $i, j \in N$. Taking into account the connection between the elements of A and A^T, as well as the commutativity of the operation $\otimes$ for real numbers, we have

$$a^\lozenge_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^T_{kj} = \sum_{k\in N}^{\oplus} a^T_{ki} \otimes a_{jk} = \sum_{k\in N}^{\oplus} a_{jk} \otimes a^T_{ki} = a^\lozenge_{ji},$$

for all $i, j \in N$, so $A^\lozenge$ is symmetric.

For the diagonal elements of the matrix $A^\lozenge$, for every $i \in N$, is valid

$$a^\lozenge_{ii} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^T_{ki} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a_{ik} = \max_{k\in N}\left\{a_{ik} + a_{ik}\right\} = 2\max_{k\in N}\left\{a_{ik}\right\}.$$

The second equality in (11) is obvious, considering $a_{ik} = a^T_{ki}$. □

Theorem 3.6. *Let $A \in \mathbb{R}^{n\times n}$ and $A^\bullet = A \oplus A^*$. Then*

1. *$A^\bullet$ is a square matrix, but neither symmetric nor skew-symmetric;*
2. *$A^\bullet$ cannot have two symmetric negative elements;*
3. *If $a_{ij} \neq a_{ji}$, for all $i, j \in N$, then $A^\bullet$ is a matrix that does not have two equal symmetric elements;*
4. *For the diagonal elements of $A^\bullet$ hold $a^\bullet_{ii} = |a_{ii}|$, for all $i \in N$;*
5. *If A is symmetric, then $a^\bullet_{ij} = \left|a_{ij}\right|$, for all $i, j \in N$.*

Proof. 1. Clearly, $A^\bullet$ is a square matrix. It can be easily checked that the matrix we get when A is max-algebraically multiplied by its negative transpose matrix is neither symmetric nor skew-symmetric.

2. Suppose that the element $a^\bullet_{ij}$ is negative. Then, we have

$$a^\bullet_{ij} = \max\left\{a_{ij}, a^*_{ij}\right\} = \max\left\{a_{ij}, -a_{ji}\right\} < 0.$$

So, it must be $a_{ij} < 0$ and $-a_{ji} < 0$, i.e. $a^*_{ji} > 0$ and $a_{ji} > 0$. Now, we have

$$a^\bullet_{ji} = \max\left\{a_{ji}, a^*_{ji}\right\} > 0.$$

With this we proved that the element, symmetric to the negative element in the matrix $A^\bullet$, must be positive.

3. Element $a^\bullet_{ij}$ is calculated as

$$a^\bullet_{ij} = \max\left\{a_{ij}, a^*_{ij}\right\},$$

for all $i, j \in N$. Without loss of generality, we can suppose that $\max\left\{a_{ij}, a^*_{ij}\right\} = a_{ij}$, i.e. that $a^\bullet_{ij} = a_{ij}$. Then,

$$a_{ij} > a^*_{ij} \quad \Rightarrow \quad -a_{ij} < -a^*_{ij} \quad \Rightarrow \quad a^*_{ji} < a_{ji}.$$

Therefore, we have that

$$a^\bullet_{ji} = \max\left\{a_{ji}, a^*_{ji}\right\} = a_{ji}.$$

Since $a_{ij} \neq a_{ji}$, it follows that $a^\bullet_{ij} \neq a^\bullet_{ji}$.

4. For the diagonal elements of the matrix $A^\bullet$ holds

$$a^\bullet_{ii} = \max\left\{a_{ii}, a^*_{ii}\right\} = \max\left\{a_{ii}, -a_{ii}\right\} = |a_{ii}|, \qquad \text{for all } i \in N.$$

5. If A is symmetric, then we have

$$a^\bullet_{ij} = \max\left\{a_{ij}, a^*_{ij}\right\} = \max\left\{a_{ij}, -a_{ji}\right\} = \max\left\{a_{ij}, -a_{ij}\right\} = \left|a_{ij}\right|,$$

for all $i, j \in N$.

With this, the proof is completed. □

Based on the previous theorems, we conclude that the matrices $A' = A \oplus A^T$ and $A^\Diamond = A \otimes A^T$ are symmetric, like corresponding matrices in conventional linear algebra. The matrix $A^\bullet = A \oplus A^*$ is neither symmetric nor skew-symmetric, in contrast to conventional linear algebra, where the corresponding matrix is skew-symmetric.

Example 3.7. Let matrix $A \in \mathbb{R}^{4\times 4}$ be given:

$$A = \begin{pmatrix} 3.4 & 1 & 2.6 & -3 \\ -1.1 & 6.9 & -2 & 0.8 \\ 10.2 & 4 & 9 & 7.6 \\ 12.3 & 6.7 & -5.2 & -8 \end{pmatrix}.$$

Then, we have

$$A' = \begin{pmatrix} 3.4 & 1 & 10.2 & 12.3 \\ 1 & 6.9 & 4 & 6.7 \\ 10.2 & 4 & 9 & 7.6 \\ 12.3 & 6.7 & 7.6 & -8 \end{pmatrix}, \quad A^\Diamond = \begin{pmatrix} 6.8 & 7.9 & 13.6 & 15.7 \\ 7.9 & 13.8 & 10.9 & 13.6 \\ 13.6 & 10.9 & 20.4 & 22.5 \\ 15.7 & 13.6 & 22.5 & 24.6 \end{pmatrix}, \quad A^\bullet = \begin{pmatrix} 3.4 & 1.1 & 2.6 & -3 \\ -1 & 6.9 & -2 & 0.8 \\ 10.2 & 4 & 9 & 7.6 \\ 12.3 & 6.7 & -5.2 & 8 \end{pmatrix}.$$

We can see that A' and $A^\Diamond$ are symmetric matrices, while $A^\bullet$ is neither symmetric nor skew-symmetric. Also, for diagonal elements of those matrices holds (for example, for $i = 4$):

$$a'_{44} = -8 = a_{44}; \qquad a^\Diamond_{44} = 24.6 = 2\max\left\{12.3, 6.7, -5.2, -8\right\}; \qquad a^\bullet_{44} = 8 = |a_{44}|.$$

which is consistent with the proved properties about diagonal elements from previous theorems. ■

Note that the properties described by the theorems in this section hold dually in min-algebraic theory.

4. Pseudo skew-symmetric matrices

In this section, we introduce and describe a new class of matrices in the max-algebra. The matrix $A \otimes A^*$ also belongs to that class (recall that the matrix A^* denotes $-A^T$), so we describe this matrix first. Here we assume that the entries of a matrix A are real numbers. The case when infinite entries appear in a matrix A will be discussed in Section 5.

Lemma 4.1. *Let $A \in \mathbb{R}^{n\times m}$. Then the matrix $\tilde{A} = A \otimes A^*$ is a square matrix with a zero diagonal.*

Proof. If $A \in \mathbb{R}^{n\times m}$, then the matrix A^* belongs to $\mathbb{R}^{m\times n}$, so $\tilde{A}$ belongs to $\mathbb{R}^{n\times n}$, i.e. it is a square matrix.

For the elements of matrices A and A^* it holds $a^*_{ij} = -a_{ji}$, for all $i, j \in N$. Therefore, for the diagonal elements of the matrix $\tilde{A}$, we have

$$\tilde{a}_{ii} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^*_{ki} = \sum_{k\in N}^{\oplus} a_{ik} \otimes (-a_{ik}) = \max_{k\in N} \{a_{ik} + (-a_{ik})\} = 0, \quad \text{for all } i \in N,$$

which proves the statement. □

In the conventional linear algebra $A \cdot A^*$ is a symmetric matrix. This property is not valid in the max-algebra for $A \otimes A^*$, which can be easily examined.

However, the following holds.

Theorem 4.2. *For a matrix $A \in \mathbb{R}^{n\times m}$, symmetric entries of the matrix $\tilde{A}$ cannot be negative at the same time.*

Proof. Let $\tilde{a}_{ij}$, $i, j \in N$ be a negative entry of the matrix $\tilde{A}$. We calculate this entry as

$$\tilde{a}_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^*_{kj} = \max_{k\in N} \left\{a_{ik} + a^*_{kj}\right\}, \quad i, j \in N$$

Suppose that the greatest summand of this max-algebraic sum is achieved for some $p \in N$:

$$\tilde{a}_{ij} = a_{ip} + a^*_{pj} < 0.$$

Then

$$-(a_{ip} + a^*_{pj}) > 0 \qquad \Leftrightarrow \qquad a^*_{pi} + a_{jp} > 0 \qquad \Leftrightarrow \qquad a_{jp} + a^*_{pi} > 0.$$

The last expression is one of the summands in the max-algebraic sum when calculating the entry $\tilde{a}_{ji}$:

$$\tilde{a}_{ji} = \sum_{k\in N}^{\oplus} a_{jk} \otimes a^*_{ki} = \max \left\{a_{j1} + a^*_{1i}, \ldots, a_{jp} + a^*_{pi}, \ldots, a_{jn} + a^*_{ni}\right\}.$$

Therefore, at least one of the summands in the above sum is positive, so the max-algebraic sum is also positive. This means that $\tilde{a}_{ji} > 0$, which proves that two symmetric entries of $\tilde{A}$ cannot be negative at the same time. □

Based on the proof of this theorem, we can draw the following conclusions:

Corollary 4.3. *If the entry $\tilde{a}_{ij}$ of the matrix $\tilde{A}$ is negative, then the corresponding symmetric entry $\tilde{a}_{ji}$ must be strictly positive, i.e., the case that $\tilde{a}_{ji}$ equals zero is excluded.*

Corollary 4.4. *In the matrix $\tilde{A}$, the off-diagonal entries cannot all be negative.*

Remark 4.5. Note that in the matrix $\tilde{A}$ two symmetric entries can be positive at the same time, i.e., Theorem 4.2 does not assert that symmetric entries in $\tilde{A}$ must necessarily be of different signs.

Example 4.6. Let a matrix $A \in \mathbb{R}^{3\times 3}$ be given by:

$$A = \begin{pmatrix} 15 & 4 & 8 \\ 6 & -9 & -1 \\ 2 & 2 & -10 \end{pmatrix}.$$

Then

$$\tilde{A} = A \otimes A^* = \begin{pmatrix} 0 & \boxed{13} & 18 \\ \boxed{-9} & 0 & 9 \\ -2 & 11 & 0 \end{pmatrix}.$$

We can see that $\tilde{A}$ is a square matrix with zero diagonal.

The entry $\tilde{a}_{21}$ is calculated as:

$$\begin{aligned} \tilde{a}_{21} &= \max\left\{a_{21} + a^*_{11}, a_{22} + a^*_{21}, a_{23} + a^*_{31}\right\} = \max\{6 + (-15), -9 + (-4), -1 + (-8)\} \\ &= \max\{-9, -13, -9\} = -9. \end{aligned}$$

The maximum is reached in $a_{21} + a^*_{11}$ and $a_{23} + a^*_{31}$, and they are negative. Therefore, it will be $-(a_{21} + a^*_{11}) > 0$ and $-(a_{23} + a^*_{31}) > 0$, so $a_{11} + a^*_{12} > 0$ and $a_{13} + a^*_{32} > 0$.

These two expressions appear when calculating the entry $\tilde{a}_{12}$:

$$\tilde{a}_{12} = \max\left\{a_{11} + a^*_{12}, a_{12} + a^*_{22}, a_{13} + a^*_{32}\right\} = \max\{15 + (-6), 4 + 9, 8 + 1\} = \max\{9, 13, 9\} = 13.$$

So, the entry symmetric to the negative entry in the matrix $\tilde{A}$ is positive.

Let us also note that, for example, $\tilde{a}_{32} = 11$ and $\tilde{a}_{23} = 9$. Therefore, symmetric entries of the matrix $\tilde{A}$ can be positive at the same time. ■

Due to certain similarities with the skew-symmetric matrices, we will call the above considered matrices **max-algebraic pseudo skew-symmetric** matrices.

Definition 4.7. A max-algebraic pseudo skew-symmetric matrix, abbreviated as *MAPSS* matrices, is a square matrix with zero diagonal that does not have two symmetric negative entries.

Clearly, $\tilde{A} = A \otimes A^*$ belongs to the class of *MAPSS* matrices, but does not mean that every *MAPSS* matrix is obtained as the max-algebraic product of some matrix and its conjugate matrix.

Also, every square matrix with zero diagonal, whose off-diagonal elements are all non-negative, is a *MAPSS*. It is not difficult to conclude that every skew-symmetric matrix is a *MAPSS* matrix, too. This is not true for symmetric matrices.

Let us also note that the matrix of size $n \times n$ whose all entries are equal to zero, also represents a *MAPSS* matrix, considering that it is a square matrix with a zero diagonal that does not have symmetric negative entries.

In the rest of this section, we consider *MAPSS* matrices with real entries.

It is important to note the following: based on Corollary 4.3, for the matrix $\tilde{A} = A \otimes A^*$ the following holds: if $\tilde{a}_{ij} < 0$, then $\tilde{a}_{ji} > 0$. However, for an arbitrary *MAPSS* matrix M holds: if $m_{ij} < 0$, then $m_{ji} \geqslant 0$, i.e., the possibility of m_{ji} being zero is not excluded.

The next lemma is about the number of negative elements in a *MAPSS* matrix.

Lemma 4.8. *The number of negative entries in a MAPSS matrix $A \in \mathbb{R}^{n\times n}$ cannot be greater than $(n^2 - n)/2$.*

Proof. The total number of entries in the matrix $A \in \mathbb{R}^{n\times n}$ is n^2. Since the diagonal entries are equal to zero and there are n of them, the number of off-diagonal entries is $n^2 - n$. Taking into account that in the *MAPSS* matrices two symmetric entries cannot be negative at the same time, we conclude that at most half of this value can be negative. □

We will now consider whether the properties, which are valid in the conventional linear algebra for symmetric/skew-symmetric matrices, are valid in the max-algebra for *MAPSS* matrices. In the conventional linear algebra, for symmetric/skew-symmetric matrices it holds: the transpose and negative transpose of a symmetric (skew-symmetric) matrix is symmetric (skew-symmetric); if we multiply a symmetric (skew-symmetric) matrix by a scalar, the result will be a symmetric (skew-symmetric) matrix; the sum of two symmetric (skew-symmetric) matrices is a symmetric (skew-symmetric) matrix; the product of two symmetric (skew-symmetric) matrices is not a symmetric (skew-symmetric) matrix.

For *MAPSS* matrices, these properties are described by following statements.

Theorem 4.9. *Let A be a MAPSS matrix. Then A^T is a MAPSS matrix, while A^* is not.*

Proof. If A is a *MAPSS* matrix, it is clear that transpose of this matrix will also be a square matrix with zero diagonal. Moreover, if there are no symmetric elements in A which are negative at the same time, this property will also be valid in A^T, so this matrix is a *MAPSS* matrix.

The matrix A^* is not *MAPSS*. It is a square matrix with zero diagonal, but since A can have two symmetric positive entries, it follows that A^* can have two symmetric negative entries. □

Theorem 4.10. *The result of the max-algebraic multiplication of a MAPSS matrix by a scalar is not MAPSS in a non-trivial case.*

Proof. Let A be a *MAPSS* matrix and θ an arbitrary scalar. The trivial case is when $\theta = 0$. Then, we have $\theta \otimes A = A$, and it is a *MAPSS* matrix.

If $\theta < 0$, the condition that the matrix $\theta \otimes A$ does not have two symmetric negative entries may be disrupted, while for $\theta > 0$, this condition will be valid. But this matrix in both cases will have all entries equal to θ on the main diagonal, so we conclude that $\theta \otimes A$ is not a *MAPSS* matrix. □

Theorem 4.11. *The max-algebraic sum of two MAPSS matrices is also MAPSS.*

Proof. Let $A, B \in \mathbb{R}^{n\times n}$ be *MAPSS* matrices and let $C = A \oplus B$. Clearly, C is a square matrix with zero diagonal. Suppose that C has a negative entry:

$$c_{ij} = \max\left\{a_{ij}, b_{ij}\right\} < 0,$$

for some $i, j \in N$. This means that is $a_{ij} < 0$ and $b_{ij} < 0$. Given that A and B are *MAPSS* matrices, the corresponding symmetric entries will be non-negative: $a_{ji} \geqslant 0$ and $b_{ji} \geqslant 0$. Now, we have:

$$c_{ji} = \max\left\{a_{ji}, b_{ji}\right\} \geqslant 0.$$

Therefore, the matrix C cannot have two symmetric negative entries, so it is a *MAPSS* matrix. □

Theorem 4.12. *The max-algebraic product of two MAPSS matrices is a square matrix that does not contain two symmetric negative entries, but does not have a zero diagonal.*

Proof. Let $A, B \in \mathbb{R}^{n\times n}$ be *MAPSS* matrices and let $D = A \otimes B$. It is clear that D is also from $\mathbb{R}^{n\times n}$. Suppose that D has a negative entry:

$$d_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes b_{kj} = \max\left\{a_{i1} + b_{1j}, \dots, a_{ii} + b_{ij} \dots, a_{in} + b_{nj}\right\} < 0,$$

for some $i, j \in N$. This means that all summands in this sum are negative. Given that A is a zero-diagonal matrix, we have that $a_{ii} = 0$, so it must be $a_{ii} + b_{ij} < 0$, i.e. $b_{ij} < 0$. Since B is a *MAPSS* matrix, we have that $b_{ji} \geqslant 0$. We calculate the entry d_{ji} as

$$d_{ji} = \sum_{k\in N}^{\oplus} a_{jk} \otimes b_{ki} = \max\left\{a_{j1} + b_{1i}, \dots, a_{jj} + b_{ji} \dots, a_{jn} + b_{ni}\right\}.$$

Given that $a_{jj} = 0$ and $b_{ji} \geqslant 0$, we have that at least one summand of the above max-algebraic sum is non-negative. Therefore, the entire sum will be non-negative, i.e., $d_{ji} \geqslant 0$. With this, we have proven that the matrix D cannot have two symmetric negative entries. However, this matrix is not *MAPSS* because the diagonal entries are not equal to zero in the general case, which is not difficult to verify. □

Corollary 4.13. *If A is a MAPSS matrix, then A^p, $p \geqslant 2$, is not a MAPSS matrix.*

Recall that we previously described properties of matrices $A' = A \oplus A^T$, $A^\Diamond = A \otimes A^T$, $A^\bullet = A \oplus A^*$ and $\tilde{A} = A \otimes A^*$ in the max-algebra, where A was an arbitrary or square matrix. We will now describe these matrices under the condition that A is a *MAPSS* matrix.

Theorem 4.14. *Let $A \in \mathbb{R}^{n\times n}$ be a MAPSS matrix. Then the following holds:*

1. $A' = A \oplus A^T$ *is a MAPSS matrix whose entries are all non-negative;*
2. $A^\Diamond = A \otimes A^T$ *is not MAPSS, but its entries are all non-negative;*
3. $A^\bullet = A \oplus A^*$ *is a MAPSS matrix;*
4. $\tilde{A} = A \otimes A^*$ *is a MAPSS matrix.*

Proof. It is clear that the listed matrices are square matrices. It remains to check whether they are with zero diagonal and whether they do not have two symmetric negative entries.

1. From Theorem 4.9, stands that if A is a *MAPSS* matrix, then A^T is a *MAPSS*, too. Also, by Theorem 4.11, the max-algebraic sum of two *MAPSS* matrices is a *MAPSS* matrix, whence $A' = A \oplus A^T$ is a *MAPSS* matrix.

The entry a'_{ij} is calculated by

$$a'_{ij} = \max\left\{a_{ij}, a^T_{ij}\right\} = \max\left\{a_{ij}, a_{ji}\right\}, \qquad \text{for all } i, j \in N.$$

Given that A is a *MAPSS* matrix, we have that a_{ij} and a_{ji} cannot be negative at the same time, i.e., one of them is non-negative, so the maximum is non-negative, too. Therefore, we conclude that all entries in A' are non-negative.

2. It is easy to see that the matrix $A^\Diamond = A \otimes A^T$ is not *MAPSS*, because it does not have a zero diagonal.

Let us suppose that the entry $a^\Diamond_{ij}$ is negative, i.e.,

$$a^\Diamond_{ij} = \sum_{k\in N}^{\oplus} a_{ik} \otimes a^T_{kj} < 0,$$

for some $i, j \in N$. This means that $a_{ik} \otimes a^T_{kj} < 0$, for all $k \in N$.

For $k = i$:

$$a_{ii} + a^T_{ij} < 0.$$

Given A is a *MAPSS* matrix, it has a zero diagonal, so $a_{ii} = 0$. From here, $a^T_{ij} < 0$, i.e., $a_{ji} < 0$.

For $k = j$, we have

$$a_{ij} + a^T_{jj} < 0.$$

Since $a^T_{jj} = a_{jj} = 0$, it follows that $a_{ij} < 0$.

We got that $a_{ij} < 0$ and $a_{ji} < 0$, so since A cannot have two symmetric negative entries, we reject the assumption that $A^\Diamond$ has a negative entry.

3. Since A is a *MAPSS* matrix and A^* has a zero diagonal, the matrix $A^\bullet = A \oplus A^*$ will have zero diagonal, too. Suppose that $a^\bullet_{ij} < 0$, i.e.,

$$a^\bullet_{ij} = \max\left\{a_{ij}, a^*_{ij}\right\} < 0,$$

for some $i, j \in N$. Then, we have that both a_{ij} and a^*_{ij} are negative. Since A is a *MAPSS* matrix and $a_{ij} < 0$, it follows that $a_{ji} \geqslant 0$. Therefore:

$$a^\bullet_{ji} = \max\left\{a_{ji}, a^*_{ji}\right\} \geqslant 0.$$

So, $A^\bullet$ cannot have two symmetric negative entries.

Hence, $A^\bullet$ is a *MAPSS* matrix.

4. This statement holds for an arbitrary matrix A (from Lemma 4.1 and Theorem 4.2), so will also be valid in the case when A is a *MAPSS* matrix.

Thus, the proof is completed. □

Note that, according to Theorem 4.11, we have that the max-algebraic sum of two *MAPSS* matrices is a *MAPSS* matrix, but from the proof of property 3. of the previous theorem, we can conclude more than that: the max-algebraic sum of a *MAPSS* matrix and an arbitrary square matrix with zero diagonal is a *MAPSS* matrix. It is not difficult to conclude that this fact will hold in the general case:

Corollary 4.15. *If A is a MAPSS matrix and $A_1, A_2, \ldots$ are square matrices with zero diagonal (of course, all matrices must be of the same size), then*

$$A \oplus A_1 \oplus A_2 \oplus \ldots$$

is a MAPSS matrix.

Lemma 4.16. *If A is symmetric and MAPSS, then $A^\bullet = A$.*

Proof. If A is symmetric and *MAPSS*, all off-diagonal elements must be non-negative, i.e., $a_{ij} > 0$, for all $i, j \in N$, $i \neq j$. Then, $a^*_{ij} < 0$, for all $i, j \in N$, $i \neq j$. Therefore, we have:

$$a^\bullet_{ij} = \max\left\{a_{ij}, a^*_{ij}\right\} = a_{ij},$$

so the statement is proved. □

4.1. Weakly MAPSS matrices

By the properties proved in Theorem 3.6, $A^\bullet = A \oplus A^*$ (where A is an arbitrary matrix) is a square matrix that cannot have two symmetric negative entries. However, it is not a *MAPSS* matrix because it does not have a zero diagonal, which is not difficult to prove. Nevertheless, based on the theorems of this section, we can draw conclusions about the class of matrices to which $A^\bullet$ belongs, i.e., about the class of square matrices that do not have two symmetric negative entries (diagonal entries can be arbitrary). Let us call this class the class of **weakly** *MAPSS* **matrices**.

Based on the proven claims about *MAPSS* matrices, we now consider whether these claims also hold for weakly *MAPSS* matrices. As we will see, some properties remain unchanged, but others will not hold due to the absence of the zero-diagonal condition.

Clearly, every *MAPSS* matrix is a weakly *MAPSS* matrix.

When it comes to the number of negative entries, there is a difference compared to *MAPSS* matrices.

Lemma 4.17. *The number of negative entries in a weakly MAPSS matrix $A \in \mathbb{R}^{n\times n}$ cannot be greater than $(n^2+n)/2$.*

Proof. In Lemma 4.8, it was shown that the number of negative entries in a *MAPSS* matrix cannot be greater than $(n^2 - n)/2$ (all these elements are off-diagonal). In weakly *MAPSS* matrices, the diagonal entries can be arbitrary (i.e. they can all be negative), so we add n diagonal entries to this number:

$$(n^2 - n)/2 + n = (n^2 + n)/2.$$

This completes the proof. □

Theorem 4.18. *Let A be a weakly MAPSS matrix. Then A^T is weakly MAPSS, while A^* is not.*

Proof. Follows directly from the proof of Theorem 4.9. □

Theorem 4.19. *The result of applying the max-algebraic multiplication of a weakly MAPSS matrix by a scalar is weakly MAPSS if that scalar is non-negative.*

Proof. Based on the proof of the Theorem 4.10, it is simply concluded that if the scalar is non-negative, then the max-algebraic multiplication of a weakly *MAPSS* matrix by a scalar will be a weakly *MAPSS* matrix. If the scalar is negative, the condition that there are no two symmetric negative entries may (but not necessarily) be disrupted, which depends on the value of the scalar and the values of the elements in the matrix. □

Theorem 4.20. *The max-algebraic sum of two weakly MAPSS matrices is also weakly MAPSS.*

Proof. It follows immediately from the Theorem 4.11 and the fact that every *MAPSS* matrix is a weakly *MAPSS*. □

Theorem 4.21. *The max-algebraic product of two weakly MAPSS matrices is not necessarily a weakly MAPSS matrix.*

Proof. This can easily be shown by an counterexample (see Example 4.24 below). □

Theorem 4.22. *The max-algebraic product of two MAPSS matrices is a weakly MAPSS matrix.*

Proof. This claim is proved by Theorem 4.12. □

Theorem 4.23. *If A is weakly MAPSS, then A', $A^\bullet$ and $\tilde{A}$ are weakly MAPSS as well, while the matrix $A^\Diamond$ is not.*

Proof. Based on the proof of Theorem 4.14, it is easy to conclude that the statement holds for matrices A', $A^\bullet$ and $\tilde{A}$. For the matrix $A^\Diamond = A \otimes A^T$, the condition that two symmetric negative entries need to be absent does not have to be satisfied, which is not difficult to show, and hence this matrix is not weakly *MAPSS*. □

Lemma 4.16 is not valid for weakly *MAPSS* matrices, i.e., in the general case $A^\bullet = A \oplus A^* \neq A$, because the diagonal entries in $A^\bullet$ need not be the same as in A.

Example 4.24. Let a weakly *MAPSS* matrix A be given by

$$A = \begin{pmatrix} -11 & 2 & -4 \\ -6 & -18 & -3 \\ 3 & 6 & 9 \end{pmatrix}.$$

Then:

$$A' = \begin{pmatrix} -11 & 2 & 3 \\ 2 & -18 & 6 \\ 3 & 6 & 9 \end{pmatrix}, \quad A^\Diamond = \begin{pmatrix} 4 & -7 & 8 \\ -7 & -6 & 6 \\ 8 & 6 & 18 \end{pmatrix}, \quad A^\bullet = \begin{pmatrix} 11 & 6 & -3 \\ -2 & 18 & -3 \\ 4 & 6 & 9 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 20 & -4 \\ 5 & 0 & -9 \\ 14 & 24 & 0 \end{pmatrix}.$$

As described above, the matrices A', $A^\bullet$ and $\tilde{A}$ are weakly *MAPSS*, while $A^\Diamond$ is not (note that this matrix is actually the max-algebraic product of two weakly *MAPSS* matrices A and A^T, which is consistent with the above statement that the max-algebraic product of two weakly *MAPSS* need not be weakly *MAPSS*). ■

Lemma 4.25. *If one summand in the max-algebraic sum of matrices is weakly MAPSS, then this max-algebraic sum itself is weakly MAPSS.*

Proof. Let S be a matrix which is the max-algebraic sum of a weakly *MAPSS* matrix A and an arbitrary number of other (not necessarily weakly *MAPSS*) matrices $B, C, \ldots$ (all matrices are of the size $n \times n$):

$$S = A \oplus B \oplus C \oplus \ldots$$

Let s_{ij} be a negative entry, i.e.,

$$s_{ij} = \max \left\{ a_{ij}, b_{ij}, c_{ij}, \ldots \right\} < 0,$$

for some $i, j \in N$. It means that all the entries in this maximum must be negative:

$$a_{ij} < 0, \qquad b_{ij} < 0, \qquad c_{ij} < 0, \qquad \ldots$$

Since A is weakly *MAPSS* and $a_{ij} < 0$, it must be $a_{ji} \geqslant 0$. Now, for s_{ji} we have:

$$s_{ji} = \max \left\{ a_{ji}, b_{ji}, c_{ji}, \ldots \right\}.$$

So, here at least one value is non-negative, from where we conclude that s_{ji} is non-negative, too. Therefore, S is a weakly *MAPSS* matrix. □

Corollary 4.26. *If A is a weakly MAPSS matrix, then*

$$\Gamma(A) = A \oplus A^2 \oplus A^3 \oplus \ldots$$

and

$$\Delta(A) = I \oplus A \oplus A^2 \oplus A^3 \oplus \ldots$$

are weakly MAPSS matrices as well.

These matrices are known in max-algebra as *weak* and *strong transitive closure* (or also metric matrix [17] and Kleene star [3]), respectively. Note that the only matrix at $\Delta(A)$ with entries equal to $-\infty$ is the matrix I, however this does not affect the validity of the statement because $-\infty$ is a neutral element for $\oplus$.

4.2. Dual class in min-algebra

Finally, let us recall that for a pair of operations $(\oplus, \otimes)$ we can define a dual pair of operations $(\oplus', \otimes')$. Therefore, just as in the max-algebra we defined a class of square matrices with zero diagonal that do not have two symmetric negative entries, in the min-algebra we can define a class of square matrices with zero diagonal that do not have two symmetric positive entries, i.e., a class of min-algebraic pseudo skew-symmetric matrices (we can use the same notation, *MAPSS*, for the class of min-algebraic pseudo skew-symmetric matrices, if it does not cause confusion). Everything previously said for *MAPSS* matrices in the max-algebra will also be valid for the corresponding dual class in the min-algebra (when we replace „maximum” with „minimum”, „positive” with „negative” etc.).

Example 4.27. For a matrix

$$A = \begin{pmatrix} 1/2 & 3 & 6/5 \\ -2 & 3/2 & -8 \\ 1 & -6 & 11/9 \end{pmatrix}, \qquad \text{we have} \qquad A^* = \begin{pmatrix} -1/2 & 2 & -1 \\ -3 & -3/2 & 6 \\ -6/5 & 8 & -11/9 \end{pmatrix}.$$

Then, with respect to the min-algebraic multiplication, we get

$$\tilde{A} = A \otimes' A^* = \begin{pmatrix} 0 & 3/2 & -1/2 \\ -46/5 & 0 & -83/9 \\ -9 & -15/2 & 0 \end{pmatrix}.$$

As we can see, it is a zero-diagonal square matrix that does not have two symmetric positive entries. All properties described above are valid, in dual form, for this *MAPSS* matrix. ■

At the end of this section, let us note the following: bearing in mind that $(A^T)^T = A$ and $(A^*)^* = A$, it is easy to conclude that all described features are also valid for matrices $A^T \oplus A$, $A^T \otimes A$, $A^* \oplus A$ and $A^* \otimes A$.

Thus, $A^* \otimes A$ is a *MAPSS* matrix as well.

5. *MAPSS* and weakly *MAPSS* matrices with infinite entries

Here we aim our attention to *MAPSS* and weakly *MAPSS* matrices whose entries can be $-\infty$ and $+\infty$. The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ will be extended by the element $+\infty$ and we will write $\overline{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \cup \{+\infty\} = \mathbb{R} \cup \{-\infty, +\infty\}$.

The oredering on $\overline{\mathbb{R}}$ is extended to an ordering on $\overline{\overline{\mathbb{R}}}$, denoted by the same symbol $\leqslant$, so that $+\infty$ is the greatest element, and this naturally extends the max-algebraic addition operation $\oplus$ so that

$$a \oplus +\infty = +\infty \oplus a = +\infty, \quad \text{for all } a \in \overline{\mathbb{R}}.$$

Moreover, the max-algebraic multiplication operation $\otimes$ is extended to $\overline{\overline{\mathbb{R}}}$ by

$$a \otimes +\infty = +\infty \otimes a = +\infty \otimes +\infty = +\infty, \quad \text{for all } a \in \mathbb{R}, \qquad -\infty \otimes +\infty = +\infty \otimes -\infty = -\infty.$$

Besides, $(-\infty)^* = +\infty$ and $(+\infty)^* = -\infty$. Let us also note that at *MAPSS* matrices with infinite entries, apart from the well-known condition $a_{ij} < 0 \Rightarrow a_{ji} \geqslant 0$, also applies: $a_{ij} = -\infty \Rightarrow a_{ji} \geqslant 0$, i.e., we consider $-\infty$ as a negative element.

In the sequel, a vector whose all entries take value $-\infty$ or $+\infty$ will be called an ε-vector. In particular, if an ε-vector is a row (resp. column) vector of some matrix, it will be called an ε-row (resp. ε-column). The ε-vector whose all entries are $-\infty$ will be denoted by ε^-, and the ε-vector whose all entries are $+\infty$ will be denoted by ε^+.

Theorem 5.1. *If $A \in \overline{\overline{\mathbb{R}}}^{n\times m}$, then $\tilde{A} = A \otimes A^*$ is a weakly MAPSS matrix.*

Proof. Let $A \in \overline{\overline{\mathbb{R}}}^{n\times m}$, then $A^* \in \overline{\overline{\mathbb{R}}}^{m\times n}$ and $\tilde{A} \in \overline{\overline{\mathbb{R}}}^{n\times n}$.

We need to prove that the matrix $\tilde{A}$ does not have two symmetric negative entries. Suppose that $\tilde{a}_{ij}$ is a negative entry, for some $i, j \in N$. We distinguish two cases:

1. $-\infty < \tilde{a}_{ij} < 0$. In this case $\tilde{a}_{ij}$ is a negative real number, so according to the proof of Theorem 4.2, $\tilde{a}_{ji}$ must be positive, i.e., $\tilde{a}_{ji} > 0$. This property is preserved even when $-\infty$ appears when calculating $\tilde{a}_{ij}$, because then $+\infty$ will appear when calculating $\tilde{a}_{ji}$, and therefore $\tilde{a}_{ji} = +\infty$. These are again elements with opposite signs.
2. $\tilde{a}_{ij} = -\infty$. Here we have

$$\tilde{a}_{ij} = \max_{k\in N} \left\{a_{ik} + a^*_{kj}\right\} = -\infty,$$

whence we get that $a_{ik} + a^*_{kj} = -\infty$, for all $k \in N$. That means: $a_{ik} = -\infty$ or $a^*_{kj} = -\infty$ (or both), for all $k \in N$, i.e.:

$$\left(a^*_{ki} = +\infty \qquad \vee \qquad a_{jk} = +\infty\right), \qquad \text{for all } k \in N. \tag{12}$$

Now, when calculating $\tilde{a}_{ji}$, we have

$$\tilde{a}_{ji} = \max_{k\in N} \left\{a_{jk} + a^*_{ki}\right\},$$

so, according to (12), all summands in the above maximum are equal to $+\infty$, whence we conclude that $\tilde{a}_{ji} = +\infty$.

This completes the proof. □

Remark 5.2. Based on the proof of the previous theorem, we can conclude that if $\tilde{a}_{ij} = -\infty$ (or $\tilde{a}_{ij} \neq -\infty$, but $-\infty$ appears when calculating $\tilde{a}_{ij}$), then it follows that $\tilde{a}_{ji} = +\infty$. The converse is not valid, i.e., if $\tilde{a}_{ij} = +\infty$, then the corresponding symmetric entry need not be $-\infty$.

Thus, regardless of the existence of infinite entries in a matrix A, the matrix $\tilde{A}$ will be weakly *MAPSS*. In the following theorem, we consider whether the existence of infinite entries can disrupt the condition of existence of zero diagonal in $\tilde{A}$, i.e., under what conditions $\tilde{A}$ is a *MAPSS* matrix.

Theorem 5.3. *The matrix $\tilde{A}$ is MAPSS if and only if A does not have an ε-row.*

Proof. For matrix $A \in \overline{\overline{\mathbb{R}}}^{n\times m}$, based on Theorem 5.1, $\tilde{A} \in \overline{\overline{\mathbb{R}}}^{n\times n}$ is weakly *MAPSS* in the general case. The zero diagonal condition remains to be considered.

($\Rightarrow$) : If A does not have an ε-row, it means that in every row we have at least one entry that is different from $\pm\infty$. Let in i-th row ($i \in N$) there is an entry $a_{ip} \neq \pm\infty$, for some $p \in N$. Recall that $-a_{ip} = a^*_{pi}$. From here, we conclude that $a^*_{pi} \neq \mp\infty$. It is clear then that a_{ip} and a^*_{pi} are opposite real numbers, i.e., $a_{ip} + a^*_{pi} = 0$.

The diagonal entry of the matrix $\tilde{A}$ in the i-th row is calculated by

$$\tilde{a}_{ii} = \max_{k\in N}\left\{a_{ik} + a^*_{ki}\right\} = \max\left\{a_{i1} + a^*_{1i}, \ldots, a_{pi} + a^*_{pi}, \ldots, a_{in} + a^*_{ni}\right\}.$$

In this maximum, we have at least one value that is equal to 0 ($a_{ip} + a^*_{pi} = 0$), while all others are either zeros (if $a_{ik} \neq \pm\infty$, for $k \in N \setminus \{p\}$) or $-\infty$ (if $a_{ik} = \pm\infty$, for $k \in N \setminus \{p\}$). So, the maximal value is zero, and therefore $a_{ii} = 0$. This proves that $\tilde{A}$ has zero diagonal.

($\Leftarrow$) : If $\tilde{A}$ is *MAPSS*, then it has zero diagonal, i.e.,

$$\tilde{a}_{ii} = \max_{k\in N}\left\{a_{ik} + a^*_{ki}\right\} = 0, \qquad \text{for all } i \in N,$$

what means that, for every $i \in N$, there exist some $p \in N$ so that $a_{ip} + a^*_{pi} = 0$, i.e., a_{ip} and a^*_{pi} are opposite real numbers (the case when these entries are equal to $\pm\infty$ is impossible, because then their sum cannot be zero). Therefore, in the i-th row of A, the entry a_{ip} is different from $\pm\infty$, so A does not have an ε-row. □

Corollary 5.4. *The matrix $\tilde{A}$ can be MAPSS if A contains an ε-column.*

Example 5.5. For a given matrix

$$A = \begin{bmatrix} -\infty & -\infty & 3 \\ -5 & -3 & +\infty \\ 2 & 1 & -8 \end{bmatrix}, \qquad \text{we have} \qquad \tilde{A} = \begin{bmatrix} 0 & -\infty & 11 \\ +\infty & 0 & +\infty \\ +\infty & 7 & 0 \end{bmatrix}.$$

The matrix A does not have an ε-row and we can see that $\tilde{A}$ is a *MAPSS* matrix: it has zero diagonal and does not have two symmetric negative entries. Moreover, by Theorem 5.1 and Remark 5.2, the entry symmetric to $\tilde{a}_{12} = -\infty$ is $\tilde{a}_{21} = +\infty$, while the entry symmetric to $\tilde{a}_{23} = +\infty$ need not to be $-\infty$ ($\tilde{a}_{32} = 7$). ■

In the next lemma, ε_1 and ε_2 will denote two different vectors, both with entries $\pm\infty$, but on the different positions, for example $\varepsilon_1 = (-\infty, +\infty, +\infty, -\infty) \neq (+\infty, -\infty, +\infty, -\infty) = \varepsilon_2$.

Lemma 5.6. 1. *If the i-th row of a matrix A is ε^-, then the i-th row of $\tilde{A}$ is ε^-, too.*

2. *If the i-th row of a matrix A is ε^+, then the i-th column of $\tilde{A}$ is ε^-.*
3. *If the i-th row of a matrix A is ε_1, then the i-th row of $\tilde{A}$ is ε_2, where $\varepsilon_1 \neq \varepsilon_2$ and the i-th entry $\varepsilon_2^{(i)}$ of ε_2 is equal to $-\infty$.*

Proof. 1. In the calculation of entries of the i-th row of the matrix $\tilde{A}$ participate the entries of the i-th row of the matrix A, which are all equal to $-\infty$. Thus, all the summands in the corresponding max-algebraic sums are of the form $-\infty + \lambda$, $\lambda \in \overline{\overline{\mathbb{R}}}$, which is equal to $-\infty$ in any case. Then, it is clear that max-algebraic sums themselves (i.e., the elements of the i-th row in $\tilde{A}$) will be $-\infty$ as well.

2. In this case the matrix A^* will have ε^- as a column, so by analogical reasoning like in 1. we get to the proof.

3. Similarly to 1., in the calculation of entries of the i-th row of the matrix $\tilde{A}$, summands in the appropriate max-algebraic sums have form $-\infty + \lambda$ or $+\infty + \lambda$, where $\lambda \in \overline{\overline{\mathbb{R}}}$, which is equal to $-\infty$ or $+\infty$ in any case. Thus, the entries of the i-th row of $\tilde{A}$ are equal to $-\infty$ or $+\infty$, and these entries form the vector ε_2.

For the diagonal entry of $\tilde{A}$ in the i-th row (i.e., the i-th entry of ε_2), stands that the corresponding max-algebraic sum contains only summands of the form $(-\infty) + (+\infty)$ or $(+\infty) + (-\infty)$, which is $-\infty$. Thus, $\varepsilon_2^{(i)} = -\infty$.

Therefore, the theorem is proved. □

Corollary 5.7. *If a matrix A has an ε-row, then $\tilde{A}$ has at least one diagonal entry which is equal to $-\infty$.*

Example 5.8. For a given matrix

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ -\infty & +\infty & -\infty & -\infty \\ 4 & -8 & 9 & 1 \\ -11 & +\infty & -6 & 8 \end{bmatrix}, \qquad \text{we get} \qquad \tilde{A} = \begin{bmatrix} 0 & +\infty & 13 & 14 \\ +\infty & -\infty & +\infty & -\infty \\ 11 & +\infty & 0 & 15 \\ 2 & +\infty & 10 & 0 \end{bmatrix}.$$

In the second row, the matrix A contains the vector $\varepsilon_1 = (-\infty, +\infty, -\infty, -\infty)$, while the matrix $\tilde{A}$ in the same row contains the vector $\varepsilon_2 = (+\infty, -\infty, +\infty, -\infty)$. It is evident that $\varepsilon_1 \neq \varepsilon_2$.

Also, according to Corollary 5.7, matrix $\tilde{A}$ has diagonal element equal to $-\infty$ ($a_{22} = -\infty = \varepsilon_2^{(2)}$). ■

Theorem 5.9. *If one entry in the i-th row of a matrix A is $-\infty$ ($+\infty$), and A has no other entries equal to $\pm\infty$, then all entries in the i-th column (row) of the matrix $\tilde{A}$ are equal to $+\infty$, except the diagonal element, which is equal to zero.*

Proof. Since $A \in \overline{\overline{\mathbb{R}}}^{n\times m}$ does not have an ε-row, then according to Theorem 5.3, $\tilde{A} \in \overline{\overline{\mathbb{R}}}^{n\times n}$ is a *MAPSS* matrix, so it has a zero diagonal.

In the matrix A, let a_{ip} be the entry of the i-th row ($i \in N$) which is equal to $-\infty$, for some $p \in N$. Then, $a^*_{pi} = +\infty$. The entry a^*_{pi} appears in the calculation of entries of the i-th column of the matrix $\tilde{A}$, so the corresponding max-algebraic sums (for the non-diagonal elements) will contain summands of the form $\lambda + (+\infty)$, $\lambda \in \mathbb{R}$, which gives $+\infty$ in any case. Note that here $\lambda \in \mathbb{R}$, because by the condition of the theorem, the only entry of A equal to $-\infty$ is a_{ip}. The case $a_{ip} + a^*_{pi} = -\infty + (+\infty)$ appears in the calculation of the diagonal entry, so here λ is certainly a real number. Thus, all max-algebraic sums contain a summand $+\infty$, and therefore all the non-diagonal entries of the i-th column in $\tilde{A}$ are equal to $+\infty$.

If the entry of the i-th row of the matrix A is equal to $+\infty$, it will participate in the calculation of the entries of the i-th row of $\tilde{A}$, so by analogical reasoning as mentioned we get to the proof in this case. □

One can easily come to a conclusion that the existence of infinite entries in *MAPSS* (weakly *MAPSS*) matrix A will not affect the number of negative entries in that matrix and will not affect the properties of matrices A^T, A^* and $\theta \otimes A$ (where θ is scalar). So, Lemma 4.8 (Lemma 4.17), Theorem 4.9 (Theorem 4.18) and Theorem 4.10 (Theorem 4.19) are valid for *MAPSS* (weakly *MAPSS*) matrices with infinite entries, too.

It remains to consider what happens with the max-algebraic sum and product of two *MAPSS* matrices with infinite entries.

Theorem 5.10. *The max-algebraic sum of two MAPSS matrices with infinite entries is also a MAPSS matrix.*

Proof. Let $C = A \oplus B$, where $A, B \in \overline{\overline{\mathbb{R}}}^{n\times n}$ are *MAPSS* matrices with infinite entries. It is clear that C will also be a square matrix (of size $n \times n$) with zero diagonal.

If c_{ij} is a negative real number, then, according to the proof of Theorem 4.11, the symmetric entry need to be non-negative (the appearance of $-\infty$ in the calculation of c_{ij} will not affect the correctness of the statement, which can be easily concluded). It remains to consider the case when c_{ij} is equal to $-\infty$:

$$c_{ij} = \max\left\{a_{ij}, b_{ij}\right\} = -\infty,$$

for some $i, j \in N$. It follows that $a_{ij} = -\infty$ and $b_{ij} = -\infty$. Given A and B are *MAPSS* matrices, it must be $a_{ji} \geqslant 0$ and $b_{ji} \geqslant 0$. Now, we have

$$c_{ji} = \max\left\{a_{ji}, b_{ji}\right\} \geqslant 0.$$

Hence, the matrix C cannot have two symmetric negative entries, so it is a *MAPSS* matrix. □

The next statement follows immediately from previous theorem.

Theorem 5.11. *The max-algebraic sum of two weakly MAPSS matrices with infinite entries is also a weakly MAPSS matrix.*

Theorem 5.12. *The max-algebraic product of two MAPSS matrices with infinite entries is a weakly MAPSS matrix.*

Proof. Let A and B be *MAPSS* matrices with infinite entries and let denote $D = A \otimes B = \in \overline{\overline{\mathbb{R}}}^{n\times n}$. Clearly, D is a square matrix and it is not difficult to conclude that D does not have zero diagonal.

The case when d_{ij} is negative real number is considered in Theorem 4.12, and the statement holds in this case (if $-\infty$ appear when calculating d_{ij}, then $+\infty$ will appear when calculating d_{ji}. Therefore, we have that $-\infty < d_{ij} < 0$ and $d_{ji} = +\infty$, so the statement is valid).

Let

$$d_{ij} = \max_{k\in N}\left\{a_{ik} + b_{kj}\right\} = -\infty,$$

for some $i, j \in N$. So, $a_{ik} + b_{kj} = -\infty$, for all $k \in N$. For $k = i$ we have $a_{ii} + b_{ij} = -\infty$. Since A is *MAPSS*, $a_{ii} = 0$ and hence $b_{ij} = -\infty$. Now, since B is *MAPSS* and $b_{ij} = -\infty$, it must be $b_{ji} \geqslant 0$. When calculating d_{ji}, we get

$$d_{ji} = \max_{k\in N}\left\{a_{jk} + b_{ki}\right\} = \max\left\{a_{j1} + b_{1i}, \dots, a_{jj} + b_{ji}, \dots, a_{jn} + b_{ni}\right\}.$$

Since $a_{jj} = 0$ and $b_{ji} \geqslant 0$, it follows $a_{jj} + b_{ji} \geqslant 0$. Thus, at least one value in the upper maximum is non-negative, and so d_{ji} is non-negative, too. With this we proved that D cannot have two symmetric negative entries, so it is a weakly *MAPSS* matrix. □

When it comes to weakly *MAPSS* matrices with infinite entries, like for weakly *MAPSS* matrices with real entries, their max-algebraic product is not weakly *MAPSS*, which is easily checked.

Also, the diagonal and identity matrices are neither *MAPSS* nor weakly *MAPSS*, since they contain symmetric entries equal to $-\infty$.

6. Conclusion

In this paper, we defined a new class of matrices over the max-algebra. Many of the described properties can be easily derived from the very definition of these matrices and they are the basis for further considerations. Of course, many new questions emerge that can be considered further. First of all, further examination of the properties of *MAPSS* matrices and their comparison with corresponding results from conventional linear algebra. Also, examining the properties that the digraphs of these matrices will have, and based on those properties making conclusions about some basic concepts related to these matrices, such as the maximum cycle mean, critical cycles or max-algebraic permanent. The next question that can be examined is: for a given matrix $\tilde{A}$, whether and how one can determine a matrix A such that $A \otimes A^*$ is exactly equal to $\tilde{A}$. This can easily lead us to the question of solving max-linear systems of equations in which *MAPSS* matrices appear: how the properties of *MAPSS* matrices will affect solving these systems, etc.

In addition, in recent times, papers have been published that, within the context of max-algebra, consider matrix powers [9], weakly and strongly stable (robust) matrices ([9], [10], [11]), max-numerical range of matrices ([31], [30], [33]), matrix roots [26], complementary basic matrices [14], totally positive matrices [20], etc. The position of *MAPSS* matrices in relation to these topics can also be examined.

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