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On a fully coupled system of nonlinear multi-term fractional differential equations with integral-multipoint boundary conditions

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Abstract. We introduce and study a new class of fully coupled integral-multipoint boundary value problems for a system of nonlinear multi-term fractional differential equations. The existence criterion for solutions to the given problem is obtained via Leray-Schauder's alternative, while the uniqueness of its solutions is accomplished by utilizing Banach's fixed point theorem. We also discuss the Ulam–Hyers stability for the problem at hand. Examples are included to illustrate the main results. Some new results appearing as special cases of the present ones are also indicated.

1. Introduction

The tools of fractional calculus are found to be quite useful in the mathematical modeling of real-world problems of natural and social sciences, for example, see [17, 29]. Further application details include chaos synchronization [34], neural networks [33], relaxation filtration processes [9], ecology [23], blood flow in small-lumen arterial vessels [7], epithelial cells [12], etc.

Nonlocal nonlinear fractional order boundary value problems have been extensively studied during the last two decades. One can find some interesting and useful results for fractional boundary value problems in a recent monograph [4], while the basic details of fractional calculus can be found in the text [22]. A differential equation containing more than one fractional-order differential operator is known as a multi-term fractional differential equation. Examples include the Bagley-Torvik equation [30], and the generalized Basset equation [25]. For some recent work on multi-term fractional differential equations supplemented with different boundary conditions, we refer the reader to the papers [2, 3, 5, 6, 13, 15, 16, 21, 26].

A great interest has also been observed in studying the stability phenomenon of fractional differential equations; for instance, see [10, 19, 32]. The concept of Ulam–Hyers stability [27, 31] is effectively applied

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in several diverse disciplines, for example, see [20, 24, 28]. The Ulam's stability for a system of coupled fractional differential equations with integral boundary conditions is investigated in [11]. In [35], the author discussed the Ulam–Hyers stability for a nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions. For some recent work on Ulam–Hyers stability for fractional differential equations, for instance, see the articles [1, 8, 14] and the references cited therein.

In this paper, motivated by [5, 26], we formulate and investigate a new integral-multipoint fully coupled boundary value problem for a system of nonlinear multi-term fractional differential equations

$$\begin{cases} \sum_{i=1}^{p} \omega_i {}^cD^{\alpha_i} x(t) = f_1(t, x(t), y(t)), \ \alpha_\zeta \in (p-1, p], \ p \geq 2, \ 0 < \alpha_i \leq 1, \ i \neq \zeta, \ \omega_\zeta \neq 0, \ \omega_i \in \mathbb{R}, \ t \in [0,1], \\ \sum_{j=1}^{q} \rho_j {}^cD^{\beta_j} y(t) = f_2(t, x(t), y(t)), \ \beta_\mu \in (q-1, q], \ q \geq 2, \ 0 < \beta_j \leq 1, \ j \neq \mu, \ \rho_\mu \neq 0, \ \rho_j \in \mathbb{R}, \ t \in [0,1], \end{cases} \tag{1}$$

complemented with nonlocal integral-multipoint boundary conditions

$$\begin{cases} x(0) = 0, \ x^{(\kappa_1)}(0) = 0, \ \kappa_1 = 1, 2, 3, \ldots, p-2, \\ y(0) = 0, \ \ y^{(\kappa_2)}(0) = 0, \ \ \kappa_2 = 1, 2, 3, \ldots, q-2, \\ x(1) = v_1 \int_0^1 y(s)ds - \sum_{l=1}^{m} a_l y(\sigma_l), \quad v_1, \ a_l \ \in \mathbb{R}, \ \sigma_l \in (0,1), \\ y(1) = v_2 \int_0^1 x(s)ds - \sum_{l=1}^{m} b_l x(\sigma_l), \ v_2, \ b_l \ \in \mathbb{R}, \ \sigma_l \in (0,1), \end{cases} \tag{2}$$

where ${}^cD^{\alpha_i}$ and ${}^cD^{\beta_j}$ denote the Caputo fractional derivatives of order α_i and β_j respectively, and $f_1, f_2 \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Here, we emphasize that it is the first article dealing with a coupled system of nonlinear multi-term fractional differential equations, equipped with coupled integral-multipoint boundary conditions.

The rest of the paper is organized as follows. Some basic definitions and a result dealing with the linear version of the problem (1)-(2) are presented in Section 2. The main results are proved in Section 3. Examples illustrating the results obtained in Section 3 are presented in Section 4. We discuss the Ulam–Hyers stability for the problem (1)-(2) in Section 5. The paper concludes with some interesting observations.

2. A preliminary result

We begin this section with some basic concepts of fractional calculus [22].

Definition 2.1. *For $\varphi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$, denoted by $I_{a^+}^{\alpha}\varphi$, is defined as*

$$I_{a^+}^{\alpha}\varphi\,(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\, \varphi\,(s)ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2. *Let $\varphi, \varphi^{(m)} \in L_1[a,b], a, b \in \mathbb{R}$ and $\alpha \in (m-1, m]$, $m \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order α, denoted by $D_{a^+}^{\alpha}\varphi$, is defined as*

$$D_{a^+}^{\alpha}\varphi\,(t) = \frac{d^m}{dt^m} I_{a^+}^{1-\alpha}\varphi\,(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha}\, \varphi\,(s)ds,$$

while the Caputo fractional derivative ${}^cD^{\alpha}_{a^+}\varphi$ of order $\alpha \in (m-1, m]$ is defined by

$$ {}^cD^{\alpha}_{a^+}\varphi(t) = D^{\alpha}_{a^+}\left[\varphi(t) - \varphi(a) - \varphi'(a)\frac{(t-a)}{1!} - \ldots - \varphi^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right]. $$

Remark 2.3. *In the literature, the (left) Caputo fractional derivative for a function $\varphi \in AC^m[a,b]$ of order $\alpha \in (m-1, m]$ is also defined as*

$$ {}^cD^{q}_{a^+}\varphi(t) = \int_a^t \frac{(t-s)^{m-q-1}}{\Gamma(m-q)}\varphi^{(m)}(s)ds. $$

Lemma 2.4. *[22] Let $m-1 < \alpha \le m$, then*

$$ I^{\alpha}_{a^+}{}^cD^{\alpha}_{a^+}\varphi(t) = \varphi(t) - \sum_{k=0}^{m-1}\frac{\varphi^{(k)}(a)}{k!}(t-a)^k, $$

with $\varphi^{(0)}(a) = \varphi(a)$.

Lemma 2.5. *If $0 < \beta \le 1$, $m-1 < \alpha \le m$ and $m \ge 2$, then*

$$ I^{\alpha}_{0^+}{}^cD^{\beta}_{0^+}\varphi(t) = I^{\alpha-\beta}_{0^+}\varphi(t) - \frac{\varphi(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} $$

Proof. *Using semi-group property of Riemann-Liouville integral operators and Lemma 2.4, we get*

$$ I^{\alpha}_{0^+}{}^cD^{\beta}_{0^+}\varphi(t) = I^{\alpha-\beta}_{0^+}(I^{\beta}_{0^+}{}^cD^{\beta}_{0^+}\varphi(t)) = I^{\alpha-\beta}_{0^+}\Big(\varphi(t) - \varphi(0)\Big) = I^{\alpha-\beta}_{0^+}\varphi(t) - \frac{\varphi(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. $$

□

In the current work, we write the Riemann–Liouville fractional integral operator $I^q_{a^+}$ and the Caputo fractional derivative operator ${}^cD^q_{a^+}$ with $a = 0$ as I^q and ${}^cD^q$, respectively.
In the following lemma, we solve the linear version of the system (1) complemented with the boundary data (2).

Lemma 2.6. *Let $\vartheta_1, \vartheta_2 \in C([0,1], \mathbb{R})$, and $\Delta = AB + 1 \neq 0$, then the unique solution of the linear system*

$$ \begin{cases} \sum_{i=1}^{p}\omega_i{}^cD^{\alpha_i}x(t) = \vartheta_1(t),\ \alpha_\zeta \in (p-1,p],\ p \ge 2,\ 0 < \alpha_i \le 1,\ i \neq \zeta,\ \omega_\zeta \neq 0,\ \omega_i \in \mathbb{R},\ t \in [0,1], \\ \sum_{j=1}^{q}\rho_j{}^cD^{\beta_j}y(t) = \vartheta_2(t),\ \beta_\mu \in (q-1,q],\ q \ge 2,\ 0 < \beta_j \le 1,\ j \neq \mu,\ \rho_\mu \neq 0,\ \rho_j \in \mathbb{R},\ t \in [0,1], \end{cases} \tag{3} $$

subject to the boundary conditions (2) is given by

$$ \begin{aligned} x(t) = \frac{t^{p-1}}{\Delta}\Bigg[&\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}\vartheta_2(u)duds \\ &- \sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}y(u)duds - \sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}\vartheta_2(s)ds \\ &+ \sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}y(s)ds - \frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}\vartheta_1(s)ds \end{aligned} $$

$$+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds+A\bigg(\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}\vartheta_1(u)duds$$
$$-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}x(u)duds-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}\vartheta_1(s)ds$$
$$+\sum_{l=1}^{m}b_l\sum_{i=1,i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}\vartheta_2(s)ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}y(s)ds\bigg)\bigg]$$
$$+\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}\vartheta_1(s)ds-\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds \tag{4}$$

and

$$y(t)=\frac{t^{q-1}}{\Delta}\bigg[-B\bigg(\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}\vartheta_2(u)duds$$
$$-\sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}y(u)duds-\sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}\vartheta_2(s)ds$$
$$+\sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}y(s)ds-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}\vartheta_1(s)ds$$
$$+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds\bigg)+\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}\vartheta_1(u)duds$$
$$-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}x(u)duds-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}\vartheta_1(s)ds$$
$$+\sum_{l=1}^{m}b_l\sum_{i=1,i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}\vartheta_2(s)ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}y(s)ds\bigg]$$
$$+\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^t(t-s)^{\beta_\mu-1}\vartheta_2(s)ds-\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^t(t-s)^{\beta_\mu-\beta_j-1}y(s)ds, \tag{5}$$

where

$$A=\frac{v_1}{q}-\sum_{l=1}^{m}a_l\sigma_l^{q-1},\ \ B=\frac{-v_2}{p}+\sum_{l=1}^{m}b_l\sigma_l^{p-1}. \tag{6}$$

Proof. Operating the integral operators I^{α_ζ} and I^{β_μ} respectively on the first and second equations in (3) and using Lemma 2.5, we get

$$\begin{cases} x(t) = \Big(1 + \sum_{i=1, i\neq\zeta}^{p} \frac{\omega_i t^{\alpha_\zeta - \alpha_i}}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i + 1)}\Big) c_1 + c_2 t + \cdots + c_p t^{p-1} + \frac{1}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^t (t-s)^{\alpha_\zeta - 1} \vartheta_1(s) ds \\ - \sum_{i=1,\ i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i)} \int_0^t (t-s)^{\alpha_\zeta - \alpha_i - 1} x(s) ds, \\ y(t) = \Big(1 + \sum_{j=1, j\neq\mu}^{q} \frac{\rho_j t^{\beta_\mu - \beta_j}}{\rho_\mu \Gamma(\beta_\zeta - \beta_j + 1)}\Big) e_1 + e_2 t + \cdots + e_q t^{q-1} + \frac{1}{\rho_\mu \Gamma(\beta_\mu)} \int_0^t (t-s)^{\beta_\mu - 1} \vartheta_2(s) ds \\ - \sum_{j=1,\ j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu \Gamma(\beta_\mu - \beta_j)} \int_0^t (t-s)^{\beta_\mu - \beta_j - 1} y(s) ds, \end{cases} \tag{7}$$

where c_i and $e_j \in \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, are unknown arbitrary constants.
Combining (7) with the conditions $x(0) = 0$, $y(0) = 0$, $x^{(\kappa_1)}(0) = 0$, $\kappa_1 = 1, 2, 3, \ldots, p-2$, and $y^{(\kappa_2)}(0) = 0$, $\kappa_2 = 1, 2, 3, \ldots, q-2$, we find that $c_i = 0$ for $i = 1, 2, \ldots, p-1$ and $e_j = 0$ for $j = 1, 2, \ldots, q-1$.
Thus, (7) takes the form:

$$\begin{cases} x(t) = c_p t^{p-1} + \frac{1}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^t (t-s)^{\alpha_\zeta - 1} \vartheta_1(s) ds - \sum_{i=1,\ i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i)} \int_0^t (t-s)^{\alpha_\zeta - \alpha_i - 1} x(s) ds, \\ y(t) = e_q t^{q-1} + \frac{1}{\rho_\mu \Gamma(\beta_\mu)} \int_0^t (t-s)^{\beta_\mu - 1} \vartheta_2(s) ds - \sum_{j=1,\ j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu \Gamma(\beta_\mu - \beta_j)} \int_0^t (t-s)^{\beta_\mu - \beta_j - 1} y(s) ds. \end{cases} \tag{8}$$

Now, using (8) in the last two conditions of (2), we find a system of equations in terms of the unknown constants c_p and e_q as

$$\begin{cases} c_p - A e_q = J_1, \\ B c_p + e_q = J_2, \end{cases} \tag{9}$$

where A and B are given in (6) and

$$\begin{aligned} J_1 &= \frac{v_1}{\rho_\mu \Gamma(\beta_\mu)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - 1} \vartheta_2(u) du ds \\ &- \sum_{j=1,\ j\neq\mu}^{q} \frac{v_1 \rho_j}{\rho_\mu \Gamma(\beta_\mu - \beta_j)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - \beta_j - 1} y(u) du ds - \sum_{l=1}^{m} \frac{a_l}{\rho_\mu \Gamma(\beta_\mu)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - 1} \vartheta_2(s) ds \\ &+ \sum_{l=1}^{m} a_l \sum_{j=1, j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu \Gamma(\beta_\mu - \beta_j)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - \beta_j - 1} y(s) ds - \frac{1}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^1 (1-s)^{\alpha_\zeta - 1} \vartheta_1(s) ds \\ &+ \sum_{i=1,\ i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i)} \int_0^1 (1-s)^{\alpha_\zeta - \alpha_i - 1} x(s) ds, \\ J_2 &= \frac{v_2}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta - 1} \vartheta_1(u) du ds \\ &- \sum_{i=1,\ i\neq\zeta}^{p} \frac{v_2 \omega_i}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta - \alpha_i - 1} x(u) du ds - \sum_{l=1}^{m} \frac{b_l}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^{\sigma_l} (\sigma_l - s)^{\alpha_\zeta - 1} \vartheta_1(s) ds \\ &+ \sum_{l=1}^{m} b_l \sum_{i=1, i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta \Gamma(\alpha_\zeta - \alpha_i)} \int_0^{\sigma_l} (\sigma_l - s)^{\alpha_\zeta - \alpha_i - 1} x(s) ds - \frac{1}{\rho_\mu \Gamma(\beta_\mu)} \int_0^1 (1-s)^{\beta_\mu - 1} \vartheta_2(s) ds \end{aligned}$$

$$+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}y(s)ds. \tag{10}$$

Solving the system (9) for c_p and e_q, we obtain

$$c_p=\frac{1}{\Delta}\Big(J_1+AJ_2\Big),\quad e_q=\frac{1}{\Delta}\Big(-BJ_1+J_2\Big). \tag{11}$$

Inserting the above values of c_p and e_q in (8) and using (6), we get the solution (4) and (5). The converse of the lemma follows by direct computation. □

3. Main Results

By Lemma 2.6, the problem (1)-(2) can be transformed into a fixed point problem as $(x,y)=\mathcal{F}(x,y)$, where $\mathcal{F}:\Omega\times\Omega\to\Omega\times\Omega$ is an operator defined by

$$\mathcal{F}(x,y)(t)=\begin{pmatrix}\mathcal{F}_1(x,y)(t)\\ \mathcal{F}_2(x,y)(t)\end{pmatrix}, \tag{12}$$

with

$$\begin{aligned}
\mathcal{F}_1(x,y)(t) \;=\; & \frac{t^{p-1}}{\Delta}\Bigg[\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}f_2(u,x(u),y(u))duds\\
&-\sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}y(u)duds\\
&-\sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}f_2(s,x(s),y(s))ds\\
&+\sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}y(s)ds\\
&-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds\\
&+A\Bigg(\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}f_1(u,x(u),y(u))duds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}x(u)duds\\
&-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&+\sum_{l=1}^{m}b_l\sum_{i=1,i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds\\
&-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}f_2(s,x(s),y(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}y(s)ds\Big)\Big]\\
&+\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds, \qquad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_2(x,y)(t) \;=\;& \frac{t^{q-1}}{\Delta}\Big[-B\Big(\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}f_2(u,x(u),y(u))duds\\
&-\sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}y(u)duds\\
&-\sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}f_2(s,x(s),y(s))ds\\
&+\sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}y(s)ds\\
&-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds\Big)\\
&+\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}f_1(u,x(u),y(u))duds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}x(u)duds\\
&-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s)))ds\\
&+\sum_{l=1}^{m}b_l\sum_{i=1,\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}x(s)ds\\
&-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}f_2(s,x(s),y(s))ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}y(s)ds\Big]\\
&+\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^t(t-s)^{\beta_\mu-1}f_2(s,x(s),y(s))ds\\
&-\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^t(t-s)^{\beta_\mu-\beta_j-1}y(s)ds. \qquad (14)
\end{aligned}
$$

Here, $\Omega\times\Omega$ is the Banach space equipped with the norm $\|(x,y)\| = \|x\| + \|y\|, (x,y) \in \Omega\times\Omega$, where Ω is the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the supremum norm

$\|x\| = \sup_{t\in[0,1]} |x(t)|$. Observe that the fixed points of the operator $\mathcal{F}$ are solution to the problem (1)-(2).

In the sequel, we need the assumptions:

(H_1) There exist real constants $\widehat{m}_p, \widehat{n}_p \geq 0,\ p = 1,2$, and $\widehat{m}_0, \widehat{n}_0 > 0$ such that

$$|f_1(t,x,y)| \leq \widehat{m}_0 + \widehat{m}_1|x| + \widehat{m}_2|y|, \quad |f_2(t,x,y)| \leq \widehat{n}_0 + \widehat{n}_1|x| + \widehat{n}_2|y|, \forall x,y \in \mathbb{R};$$

(H_2) For all $t \in [0,1], x_p, y_p \in \mathbb{R},\ p = 1,2$, there exist positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$|f_1(t,x_2,y_2) - f_1(t,x_1,y_1| \leq \mathcal{L}_1(|x_2 - x_1| + |y_2 - y_1|),$$
$$|f_2(t,x_2,y_2) - f_2(t,x_1,y_1| \leq \mathcal{L}_2(|x_2 - x_1| + |y_2 - y_1|).$$

In the forthcoming analysis, we use the notation:

$$\begin{aligned}
Q_1 &= \frac{1}{|\Delta|}\left[\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)} + \frac{|A||v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta+2)} + \sum_{l=1}^{m}\frac{|A||b_l|\sigma_l^{\alpha_\zeta}}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\right] + \frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)},\\
Q_2 &= \frac{1}{|\Delta|}\left[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu+2)} + \sum_{l=1}^{m}\frac{|a_l|\sigma_l^{\beta_\mu}}{|\rho_\mu|\Gamma(\beta_\mu+1)} + \frac{|A|}{|\rho_\mu|\Gamma(\beta_\mu+1)}\right],\\
Q_3 &= \frac{1}{|\Delta|}\sum_{i=1,i\neq\zeta}^{p}\left[\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)} + \frac{|A||v_2||\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+2)}\right]\\
&\quad + \frac{|A|}{|\Delta|}\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|\sigma_l^{\alpha_\zeta-\alpha_i}}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)} + \sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)},\\
Q_4 &= \frac{1}{|\Delta|}\left[\sum_{j=1,j\neq\mu}^{q}\frac{|v_1||\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+2)} + \sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|\sigma_l^{\beta_\mu-\beta_j}}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\right.\\
&\quad \left.+ \sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j||A|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\right],\\
\widehat{Q}_1 &= \frac{1}{|\Delta|}\left[\frac{|B|}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)} + \frac{|v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta+2)} + \sum_{l=1}^{m}\frac{|b_l|\sigma_l^{\alpha_\zeta}}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\right],\\
\widehat{Q}_2 &= \frac{1}{|\Delta|}\left[\frac{|B||v_1|}{|\rho_\mu|\Gamma(\beta_\mu+2)} + \sum_{l=1}^{m}\frac{|a_l||B|\sigma_l^{\beta_\mu}}{|\rho_\mu|\Gamma(\beta_\mu+1)} + \frac{1}{|\rho_\mu|\Gamma(\beta_\mu+1)}\right] + \frac{1}{|\rho_\mu|\Gamma(\beta_\mu+1)},\\
\widehat{Q}_3 &= \frac{1}{|\Delta|}\sum_{i=1,i\neq\zeta}^{p}\left[\frac{|B||\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)} + \frac{|v_2||\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+2)}\right]\\
&\quad + \frac{1}{|\Delta|}\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|\sigma_l^{\alpha_\zeta-\alpha_i}}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)},\\
\widehat{Q}_4 &= \frac{1}{|\Delta|}\left[\sum_{j=1,j\neq\mu}^{q}\frac{|B||v_1||\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+2)} + \sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|B||\rho_j|\sigma_l^{\beta_\mu-\beta_j}}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\right.\\
&\quad \left.+ \sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\right] + \sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}.
\end{aligned} \tag{15}$$

Now, we proceed to present our main results. In our first result, we establish the existence of solutions for the problem (1)-(2) with the aid of the Leray-Schauder alternative [18].

Theorem 3.1. *If $f_1, f_2 : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition (H_1), then the problem (1)-(2) has at least one solution on $[0,1]$, provided that $0 < \max\{\mathcal{Z}_1, \mathcal{Z}_2\} < 1$, where*

$$\begin{aligned} \mathcal{Z}_1 &= \widehat{m}_1(Q_1 + \widehat{Q}_1) + \widehat{n}_1(Q_2 + \widehat{Q}_2) + (Q_3 + \widehat{Q}_3), \\ \mathcal{Z}_2 &= \widehat{m}_2(Q_1 + \widehat{Q}_1) + \widehat{n}_2(Q_2 + \widehat{Q}_2) + (Q_4 + \widehat{Q}_4), \end{aligned} \tag{16}$$

and Q_p, $\widehat{Q}_p$, $p = 1, 2, 3, 4$, are given in (15).

Proof. Let us first establish that the operator $\mathcal{F} : \Omega\times\Omega \to \Omega\times\Omega$ defined in (12) is completely continuous. Notice that continuity of the operator $\mathcal{F}$ follows from that of the functions f_1 and f_2. Let $\Lambda_{\tilde{\rho}} = \{(x, y) \in \Omega\times\Omega : \|(x, y)\| \le \tilde{\rho}\}$ with

$$\tilde{\rho} \ge \frac{\widehat{m}_0(Q_1 + \widehat{Q}_1) + \widehat{n}_0(Q_2 + \widehat{Q}_2)}{1 - \max\{\mathcal{Z}_1, \widehat{\mathcal{Z}}_2\}}, \tag{17}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are given in (16). For any $x, y \in \Lambda_{\tilde{\rho}}$, let

$$\begin{aligned} |f_1(t, x, y)| &\le \widehat{m}_0 + \widehat{m}_1\|x\| + \widehat{m}_2\|y\| = \mathcal{K}_1, \\ |f_2(t, x, y)| &\le \widehat{n}_0 + \widehat{n}_1\|x\| + \widehat{n}_2\|y\| = \mathcal{K}_2. \end{aligned} \tag{18}$$

Then, for any $(x, y) \in \Lambda_{\tilde{\rho}}$, we obtain

$$\begin{aligned} \|\mathcal{F}_1(x, y)\| \le \sup_{t\in[0,1]} \Bigg\{ \frac{t^{p-1}}{|\Delta|} \Bigg[& \frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu)} \int_0^1 \int_0^s (s-u)^{\beta_\mu-1} |f_2(u, x(u), y(u))| du ds \\ & + \sum_{j=1,\ j\neq\mu}^{q} \frac{|v_1\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)} \int_0^1 \int_0^s (s-u)^{\beta_\mu-\beta_j-1} |y(u)| du ds \\ & + \sum_{l=1}^{m} \frac{|a_l|}{|\rho_\mu|\Gamma(\beta_\mu)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu-1} |f_2(s, x(s), y(s))| ds \\ & + \sum_{l=1}^{m} |a_l| \sum_{j=1, j\neq\mu}^{q} \frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu-\beta_j-1} |y(s)| ds \\ & + \frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)} \int_0^1 (1-s)^{\alpha_\zeta-1} |f_1(s, x(s), y(s))| ds \\ & + \sum_{i=1,\ i\neq\zeta}^{p} \frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)} \int_0^1 (1-s)^{\alpha_\zeta-\alpha_i-1} |x(s)| ds \\ & + |A| \Bigg(\frac{|v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta-1} |f_1(u, x(u), y(u))| du ds \\ & + \sum_{i=1,\ i\neq\zeta}^{p} \frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta-\alpha_i-1} |x(u)| du ds \\ & + \sum_{l=1}^{m} \frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)} \int_0^{\sigma_l} (\sigma_l - s)^{\alpha_\zeta-1} |f_1(s, x(s), y(s))| ds \\ & + \sum_{l=1}^{m} |b_l| \sum_{i=1, i\neq\zeta}^{p} \frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)} \int_0^{\sigma_l} (\sigma_l - s)^{\alpha_\zeta-\alpha_i-1} |x(s)| ds \\ & + \frac{1}{|\rho_\mu|\Gamma(\beta_\mu)} \int_0^1 (1-s)^{\beta_\mu-1} |f_2(s, x(s), y(s))| ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}|y(s)|ds\Bigg)\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}|f_1(s,x(s),y(s))|ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\Bigg\}\\
&\le\mathcal{K}_1\Bigg(\frac{1}{|\Delta|}\Bigg[\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}+\frac{|A||v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta+2)}+\sum_{l=1}^{m}\frac{|A||b_l|\sigma_l^{\alpha_\zeta}}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Bigg)+\frac{\mathcal{K}_2}{|\Delta|}\Bigg[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu+2)}+\sum_{l=1}^{m}\frac{|a_l|\sigma_l^{\beta_\mu}}{|\rho_\mu|\Gamma(\beta_\mu+1)}+\frac{|A|}{|\rho_\mu|\Gamma(\beta_\mu+1)}\Bigg]\\
&+\Bigg(\frac{1}{|\Delta|}\sum_{i=1,i\neq\zeta}^{p}\Bigg[\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}+\frac{|A||v_2||\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+2)}\Bigg]\\
&+\frac{|A|}{|\Delta|}\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|\sigma_l^{\alpha_\zeta-\alpha_i}}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}+\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}\Bigg)\|x\|\\
&+\frac{1}{|\Delta|}\Bigg[\sum_{j=1,j\neq\mu}^{q}\frac{|v_1||\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+2)}+\sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|\sigma_l^{\beta_\mu-\beta_j}}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\\
&+\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j||A|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\Bigg]\|y\|\\
&\le\mathcal{K}_1Q_1+\mathcal{K}_2Q_2+Q_3\|x\|+Q_4\|y\|,
\end{aligned}
$$

where $Q_p,\ p=1,2,3,4$, are given in (15). Inserting the values of $\mathcal{K}_1$ and $\mathcal{K}_2$ from (18) into the above inequality, we get

$$
\|\mathcal{F}_1(x,y)\|\le\left(\widehat{m}_0+\widehat{m}_1\|x\|+\widehat{m}_2\|y\|\right)Q_1+\left(\widehat{n}_0+\widehat{n}_1\|x\|+\widehat{n}_2\|y\|\right)Q_2+Q_3\|x\|+Q_4\|y\| \tag{19}
$$

Similarly, one can find that

$$
\|\mathcal{F}_2(x,y)\|\le\left(\widehat{m}_0+\widehat{m}_1\|x\|+\widehat{m}_2\|y\|\right)\widehat{Q}_1+\left(\widehat{n}_0+\widehat{n}_1\|x\|+\widehat{n}_2\|y\|\right)\widehat{Q}_2+\widehat{Q}_3\|x\|+\widehat{Q}_4\|y\|, \tag{20}
$$

where $\widehat{Q}_p,\ p=1,2,3,4$, are given in (15). From (19), (20) together with (17) and using the definition of norm, we have

$$
\begin{aligned}
\|\mathcal{F}(x,y)\| &= \|\mathcal{F}_1(x,y)\|+\|\mathcal{F}_2(x,y)\|\\
&\le\widehat{m}_0(Q_1+\widehat{Q}_1)+\widehat{n}_0(Q_2+\widehat{Q}_2)+\max(\mathcal{Z}_1,\mathcal{Z}_2)\tilde{\rho}\le\tilde{\rho},
\end{aligned}
$$

which implies that $\mathcal{F}(\Lambda_{\tilde{\rho}})$ is uniformly bounded.
To show that $\mathcal{F}(\Lambda_{\tilde{\rho}})$ is equicontinuous, we take $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $(x,y)\in\Lambda_{\tilde{\rho}}$. Then, we get

$$
\begin{aligned}
&|\mathcal{F}_1(x,y)(t_2)-\mathcal{F}_1(x,y)(t_1)|\\
\le\ &\frac{|t_2^{P-1}-t_1^{P-1}|}{\Delta}\Bigg[\mathcal{K}_1\Bigg(\frac{1}{|\Delta|}\Bigg[\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}+\frac{|A||v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta+2)}+\sum_{l=1}^{m}\frac{|A||b_l|\sigma_l^{\alpha_\zeta}}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Bigg)+\frac{\mathcal{K}_2}{|\Delta|}\Bigg[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu+2)}+\sum_{l=1}^{m}\frac{|a_l|\sigma_l^{\beta_\mu}}{|\rho_\mu|\Gamma(\beta_\mu+1)}+\frac{|A|}{|\rho_\mu|\Gamma(\beta_\mu+1)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg(\frac{1}{|\Delta|}\sum_{i=1,i\neq\zeta}^{p}\Big[\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}+\frac{|A||v_2||\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+2)}\Big]\\
&+\frac{|A|}{|\Delta|}\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|\sigma_l^{\alpha_\zeta-\alpha_i}}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}+\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}\bigg)\|x\|\\
&+\frac{1}{|\Delta|}\bigg[\sum_{j=1,j\neq\mu}^{q}\frac{|v_1||\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+2)}+\sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|\sigma_l^{\beta_\mu-\beta_j}}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\\
&+\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j||A|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j+1)}\bigg]\|y\|\\
&+\bigg|\int_0^{t_2}\Big[\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}(t_2-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}(t_2-s)^{\alpha_\zeta-\alpha_i-1}x(s)\Big]ds-\int_0^{t_1}\Big[\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}(t_1-s)^{\alpha_\zeta-1}f_1(s,x(s),y(s))ds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}(t_1-s)^{\alpha_\zeta-\alpha_i-1}x(s)\Big]ds\bigg|\\
&\le\frac{|t_2^{P-1}-t_1^{P-1}|}{\Delta}\Big[\mathcal{K}_1Q_1+\mathcal{K}_2Q_2+Q_3\|x\|+Q_4\|y\|\Big]\\
&+\frac{\mathcal{K}_1}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Big[2|t_2-t_1|^{\alpha_\zeta}+|t_2^{\alpha_\zeta}-t_1^{\alpha_\zeta}|\Big]\\
&+\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}\Big[2|t_2-t_1|^{\alpha_\zeta-\alpha_i}+|t_2^{\alpha_\zeta-\alpha_i}-t_1^{\alpha_\zeta-\alpha_i}|\Big]\|x\|\\
&\le\frac{|t_2^{P-1}-t_1^{P-1}|}{\Delta}\Big[(\widehat{m}_0Q_1+\widehat{n}_0Q_2)+\Big[(\widehat{m}_1+\widehat{m}_2)Q_1+(\widehat{n}_1+\widehat{n}_2)Q_2+Q_3+Q_4\Big]\tilde{\rho}\Big]\\
&+\frac{\Big[\widehat{m}_0+(\widehat{m}_1+\widehat{m}_2)\tilde{\rho}\Big]}{|\omega_\zeta|\Gamma(\alpha_\zeta+1)}\Big[2|t_2-t_1|^{\alpha_\zeta}+|t_2^{\alpha_\zeta}-t_1^{\alpha_\zeta}|\Big]\\
&+\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i+1)}\Big[2|t_2-t_1|^{\alpha_\zeta-\alpha_i}+|t_2^{\alpha_\zeta-\alpha_i}-t_1^{\alpha_\zeta-\alpha_i}|\Big]\tilde{\rho}\\
&\longrightarrow 0 \text{ as } (t_2-t_1)\to 0,
\end{aligned}
$$

independently of $(x,y)\in\Lambda_{\tilde{\rho}}$. Analogously, it can be shown that

$$|\mathcal{F}_2(x,y)(t_2)-\mathcal{F}_2(x,y)(t_1)|\longrightarrow 0 \text{ as } (t_2-t_1)\to 0 \text{ independently of } (x,y)\in\Lambda_{\tilde{\rho}}.$$

Thus, $\mathcal{F}_1(\Lambda_{\tilde{\rho}})$ and $\mathcal{F}_2(\Lambda_{\tilde{\rho}})$ are equicontinuous and hence $\mathcal{F}(\Lambda_{\tilde{\rho}})$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, the operator $\mathcal{F}(\Lambda_{\tilde{\rho}})$ is completely continuous.
Finally, we consider a set $\Psi=\{(x,y)\in\Omega\times\Omega:(x,y)=\eta\mathcal{F}(x,y),\ 0<\eta\le 1\}$ and verify its boundedness. For $(x,y)\in\Psi$, we have $(x,y)=\eta\mathcal{F}(x,y)$, which implies that $x(t)=\eta\mathcal{F}_1(x,y)(t)$ and $y(t)=\eta\mathcal{F}_2(x,y)(t)$ for $t\in[0,1]$. Then, by the assumption (H_1), we have

$$
\begin{aligned}
\|x\| &= \sup_{t\in[0,1]}|x(t)|\le\sup_{t\in[0,1]}|\mathcal{F}_1(x,y)(t)|\\
&\le\sup_{t\in[0,1]}\bigg\{\frac{t^{p-1}}{|\Delta|}\bigg[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}\Big(\widehat{n}_0+\widehat{n}_1|x|+\widehat{n}_2|y|\Big)duds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|v_1\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}|y(u)|duds\\
&+\sum_{l=1}^{m}\frac{|a_l|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}\Big(\widehat{n}_0+\widehat{n}_1|x|+\widehat{n}_2|y|\Big)ds\\
&+\sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}|y(s)|ds\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}\Big(\widehat{m}_0+\widehat{m}_1|x|+\widehat{m}_2|y|\Big)ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\\
&+|A|\Bigg(\frac{|v_2|}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}\Big(\widehat{m}_0+\widehat{m}_1|x|+\widehat{m}_2|y|\Big)duds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|v_2\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}|x(u)|duds\\
&+\sum_{l=1}^{m}\frac{|b_l|}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}\Big(\widehat{m}_0+\widehat{m}_1|x|+\widehat{m}_2|y|\Big)ds\\
&+\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\\
&+\frac{1}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}\Big(\widehat{n}_0+\widehat{n}_1|x|+\widehat{n}_2|y|\Big)ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}|y(s)|ds\Bigg)\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}\Big(\widehat{m}_0+\widehat{m}_1|x|+\widehat{m}_2|y|\Big)ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\Bigg\},
\end{aligned}
$$

which yields

$$\|x\|\leq(\widehat{m}_0Q_1+\widehat{n}_0Q_2)+(\widehat{m}_1Q_1+\widehat{n}_1Q_2+Q_3)\|x\|+(\widehat{m}_2Q_1+\widehat{n}_2Q_2+Q_4)\|y\|. \tag{21}$$

In a similar manner, we can find that

$$\|y\|\leq(\widehat{m}_0\widehat{Q}_1+\widehat{n}_0\widehat{Q}_2)+(\widehat{m}_1\widehat{Q}_1+\widehat{n}_1\widehat{Q}_2+\widehat{Q}_3)\|x\|+(\widehat{m}_2\widehat{Q}_1+\widehat{n}_2\widehat{Q}_2+\widehat{Q}_4)\|y\|. \tag{22}$$

From (21) and (22), we have

$$\|x\|+\|y\|\leq\widehat{m}_0(Q_1+\widehat{Q}_1)+\widehat{n}_0(Q_2+\widehat{Q}_2)+\max(\mathcal{Z}_1,\mathcal{Z}_2)\|(x,y)\|,$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are given in (16). The above inequality can alternatively be written as

$$\|(x,y)\|\leq\frac{1}{G_0}\Big[\widehat{m}_0(Q_1+\widehat{Q}_1)+\widehat{n}_0(Q_2+\widehat{Q}_2)\Big],$$

where

$$G_0 = 1 - \max(\mathcal{Z}_1, \mathcal{Z}_2).$$

Thus, the set Ψ is bounded. Consequently, by the Leray-Schauder alternative [18], there exists at least one fixed point for the operator $\mathcal{F}$. Therefore, the problem (1)-(2) admits at least one solution on $[0, 1]$. □

In the next result, we accomplish a uniqueness result for the problem (1)-(2) by applying Banach's fixed point theorem [18].

Theorem 3.2. *Let $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the condition (H_2) and that*

$$\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2) + \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4) < 1, \tag{23}$$

where Q_p and $\widehat{Q}_p$, $p = 1, 2, 3, 4$, are given in (15). Then, a unique solution exists to the problem (1)-(2) on $[0, 1]$.

Proof. We verify the hypothesis of Banach's fixed point theorem in two steps. In the first step, it will be shown that $\mathcal{F}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{F} : \mathcal{B}_r \longrightarrow \Omega \times \Omega$ is given in (12), $\mathcal{B}_r$ is a closed ball defined as $\mathcal{B}_r = \{(x, y) \in \Omega \times \Omega : \|(x, y)\| \leq r\}$ with

$$r \geq \frac{\mathcal{N}_1(Q_1 + \widehat{Q}_1) + \mathcal{N}_2(Q_2 + \widehat{Q}_2)}{1 - \left[\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2)\right] - \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4)}, \tag{24}$$

$\sup_{t\in[0,1]} |f_1(t, 0, 0)| = \mathcal{N}_1 < \infty$ and $\sup_{t\in[0,1]} |f_2(t, 0, 0)| = \mathcal{N}_2 < \infty$. By (H_2), we have

$$\begin{aligned} |f_1(t, x(t), y(t))| &= |f_1(t, x(t), y(t)) - f_1(t, 0, 0) + f_1(t, 0, 0)| \leq \mathcal{L}_1(\|x\| + \|y\|) + \mathcal{N}_1 \leq \mathcal{L}_1 r + \mathcal{N}_1, \\ |f_2(t, x(t), y(t)| &\leq \mathcal{L}_2(\|x\| + \|y\|) + \mathcal{N}_2 \leq \mathcal{L}_2 r + \mathcal{N}_2. \end{aligned} \tag{25}$$

For $(x, y) \in B_r$, it follows by using (25) that

$$\begin{aligned} \|\mathcal{F}_1(x, y)\| \leq \sup_{t\in[0,1]} \Bigg\{ & \frac{t^{p-1}}{|\Delta|} \Bigg[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - 1} \Big(\mathcal{L}_2 r + \mathcal{N}_2\Big) du ds \\ & + \sum_{j=1,\ j\neq\mu}^{q} \frac{|v_1 \rho_j|}{|\rho_\mu|\Gamma(\beta_\mu - \beta_j)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - \beta_j - 1} |y(u)| du ds \\ & + \sum_{l=1}^{m} \frac{|a_l|}{|\rho_\mu|\Gamma(\beta_\mu)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - 1} \Big(\mathcal{L}_2 r + \mathcal{N}_2\Big) ds \\ & + \sum_{l=1}^{m} |a_l| \sum_{j=1, j\neq\mu}^{q} \frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu - \beta_j)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - \beta_j - 1} |y(s)| ds \\ & + \frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)} \int_0^1 (1-s)^{\alpha_\zeta - 1} \Big(\mathcal{L}_1 r + \mathcal{N}_1\Big) ds \\ & + \sum_{i=1,\ i\neq\zeta}^{p} \frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta - \alpha_i)} \int_0^1 (1-s)^{\alpha_\zeta - \alpha_i - 1} |x(s)| ds \\ & + |A| \Bigg(\frac{|v_2|}{\omega_\zeta \Gamma(\alpha_\zeta)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta - 1} \Big(\mathcal{L}_1 r + \mathcal{N}_1\Big) du ds \\ & + \sum_{i=1,\ i\neq\zeta}^{p} \frac{|v_2 \omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta - \alpha_i)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta - \alpha_i - 1} |x(u)| du ds \\ & + \sum_{l=1}^{m} \frac{|b_l|}{|\omega_\zeta|\Gamma(\alpha_\zeta)} \int_0^{\sigma_l} (\sigma_l - s)^{\alpha_\zeta - 1} \Big(\mathcal{L}_1 r + \mathcal{N}_1\Big) ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\\
&+\frac{1}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}\Big(\mathcal{L}_2r+\mathcal{N}_2\Big)ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}|y(s)|ds\Big)\Big]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}\Big(\mathcal{L}_1r+\mathcal{N}_1\Big)ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}|x(s)|ds\Big\}\\
&\leq\Big(\mathcal{L}_1r+\mathcal{N}_1\Big)Q_1+Q_3\|x\|+\Big(\mathcal{L}_2r+\mathcal{N}_2\Big)Q_2+Q_4\|y\|. \qquad (26)
\end{aligned}
$$

Likewise, we can find that

$$
\|\mathcal{F}_2(x,y)\|\leq\Big(\mathcal{L}_1r+\mathcal{N}_1\Big)\widehat{Q}_1+\widehat{Q}_3\|x\|+\Big(\mathcal{L}_2r+\mathcal{N}_2\Big)\widehat{Q}_2+\widehat{Q}_4\|y\|. \qquad (27)
$$

From (26) and (27) together with (24), we obtain

$$
\begin{aligned}
\|\mathcal{F}(x,y)\| &= \|\mathcal{F}_1(x,y)\|+\|\mathcal{F}_2(x,y)\|\\
&\leq\Big[\mathcal{L}_1(Q_1+\widehat{Q}_1)+\mathcal{L}_2(Q_2+\widehat{Q}_2)\Big]r+\mathcal{N}_1(Q_1+\widehat{Q}_1)+\mathcal{N}_2(Q_2+\widehat{Q}_2)\\
&\quad+\max(Q_3+\widehat{Q}_3,Q_4+\widehat{Q}_4)r\leq r,
\end{aligned}
$$

which shows that $\mathcal{F}(x,y)\in\mathcal{B}_r$. Hence, $\mathcal{F}\mathcal{B}_r\subset\mathcal{B}_r$ since $(x,y)\in\mathcal{B}_r$ is an arbitrary element.

Next, we verify that the operator $\mathcal{F}$ given in (12) is a contraction. Using the condition (H_2), we obtain

$$
\begin{aligned}
&\|\mathcal{F}_1(x_2,y_2)-\mathcal{F}_1(x_1,y_1)\|\\
&\leq\sup_{t\in[0,1]}\Big\{\frac{t^{p-1}}{|\Delta|}\Big[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}|f_2(u,x_2(u),y_2(u))-f_2(u,x_1(u),y_1(u))|duds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|v_1\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}|y_2(u)-y_1(u)|duds\\
&+\sum_{l=1}^{m}\frac{|a_l|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|ds\\
&+\sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}|y_2(s)-y_1(s)|ds\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}|x_2(s)-x_1(s)|ds\\
&+|A|\Big(\frac{|v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}|f_1(u,x_2(u),y_2(u))-f_1(u,x_1(u),y_1(u))|duds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}|x_2(u)-x_1(u)|duds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|ds\\
&+\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}|x_2(s)-x_1(s)|ds\\
&+\frac{1}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}|y_2(s)-y_1(s)|ds\Bigg)\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}|x_2(s)-x_1(s)|ds\Bigg]\Bigg\}\\
&\leq\Big(\mathcal{L}_1Q_1+\mathcal{L}_2Q_2\Big)\Big(\|x_2-x_1\|+\|y_2-y_1\|\Big)+Q_3\|x_2-x_1\|+Q_4\|y_2-y_1\|.
\end{aligned}
$$

Similarly, we can find that

$$
\|\mathcal{F}_2(x_2,y_2)-\mathcal{F}_2(x_1,y_1)\|\leq\Big(\mathcal{L}_1\widehat{Q}_1+\mathcal{L}_2\widehat{Q}_2\Big)\Big(\|x_2-x_1\|+\|y_2-y_1\|\Big)+\widehat{Q}_3\|x_2-x_1\|+\widehat{Q}_4\|y_2-y_1\|.
$$

From the foregoing inequalities, we get

$$
\begin{aligned}
&\|\mathcal{F}(x_2,y_2)-\mathcal{F}(x_1,y_1)\|\\
&\leq\Big[\mathcal{L}_1(Q_1+\widehat{Q}_1)+\mathcal{L}_2(Q_2+\widehat{Q}_2)+\max\big(Q_3+\widehat{Q}_3,Q_4+\widehat{Q}_4\big)\Big]\Big(\|x_2-x_1\|+\|y_2-y_1\|\Big),
\end{aligned}
$$

which, by (23), implies that $\mathcal{F}$ is contraction. Thus, the hypothesis of Banach's fixed point theorem is verified and hence its conclusion implies that the operator $\mathcal{F}$ has a unique fixed point. Therefore, there exists a unique solution to problem (1)-(2) on $[0,1]$. □

4. Examples

Here, we present examples illustrating the results obtained in the last section.

Example 4.1. *Consider a coupled system of nonlinear multi-term fractional differential equations*

$$
\begin{cases}
\displaystyle\sum_{i=1}^{4}\omega_i{}^cD^{\alpha_i}x(t)=f_1(t,x(t),y(t)),\ t\in[0,1],\\
\displaystyle\sum_{j=1}^{5}\rho_j{}^cD^{\beta_j}y(t)=f_2(t,x(t),y(t)),\ t\in[0,1],
\end{cases}
\tag{28}
$$

complemented with nonlocal integral-multipoint boundary conditions

$$
\begin{cases}
x(0)=0,\ x'(0)=0,\ x''(0)=0,\ y(0)=0,\ y'(0)=0,\ y''(0)=0,\ y'''(0)=0,\\
\displaystyle x(1)=4\int_0^1y(s)ds-\sum_{l=1}^{3}a_ly(\sigma_l),\quad v_1,\ a_l\ \in\mathbb{R},\ \sigma_l\in(0,1),\\
\displaystyle y(1)=3\int_0^1x(s)ds-\sum_{l=1}^{3}b_lx(\sigma_l),\ v_2,\ b_l\ \in\mathbb{R},\ \sigma_l\in(0,1),
\end{cases}
\tag{29}
$$

where $\alpha_1 = 0.9$, $\alpha_2 = 0.75$, $\alpha_\zeta = \alpha_3 = 7/2$, $\alpha_4 = 0.5$, $\beta_1 = 0.8$, $\beta_2 = 0.6$, $\beta_\mu = \beta_3 = 9/2$, $\beta_4 = 0.4$, $\beta_5 = 0.2$, $v_1 = 4$, $v_2 = 3$, $\omega_1 = 1/20$, $\omega_2 = -1/10$, $\omega_\zeta = \omega_3 = 3$, $\omega_4 = 7/15$, $\rho_1 = 1/2$, $\rho_2 = 1/5$, $\rho_\mu = \rho_3 = 4$, $\rho_4 = 5/19$, $\rho_5 = 5/8$, $\sigma_1 = 1/3$, $\sigma_2 = 2/3$, $\sigma_3 = 3/8$, $m = 3$, $p = 4$, $q = 5$, $a_1 = -3$, $a_2 = 2$, $a_3 = 5$, $b_1 = 4$, $b_2 = 5$, $b_3 = 1$.

Using the given values, we find that $Q_1 \approx 0.06522380$, $Q_2 \approx 0.00533702$, $Q_3 \approx 0.09174221$, $Q_4 \approx 0.02207270$, $\widehat{Q}_1 \approx 0.06353920$, $\widehat{Q}_2 \approx 0.01221331$, $\widehat{Q}_3 \approx 0.09980610$, *and* $\widehat{Q}_4 \approx 0.04630676$. $\big(Q_p$ and $\widehat{Q}_p$, $p = 1, 2, 3, 4$, are given in (15)$\big)$.

(a) For illustrating Theorem 3.1, we take

$$\begin{aligned} f_1(t, x(t), y(t)) &= \frac{1}{12} + \frac{|x(t)|}{210(1 + |x(t)|)} + \frac{|y(t)|}{(t+2)^3(1 + |y(t)|)}, \\ f_2(t, x(t), y(t)) &= \frac{1}{6} + \frac{|x(t)||y(t)|}{2(1 + |y(t)|)} + \frac{|\sin y(t)|}{(t+3)^3(1 + |\sin y(t)|)}. \end{aligned} \tag{30}$$

From (30), it follows that the assumption (H_1) holds with $\widehat{m}_0 = \frac{1}{12}$, $\widehat{m}_1 = \frac{1}{210}$, $\widehat{m}_2 = \frac{1}{8}$, $\widehat{n}_0 = \frac{1}{6}$, $\widehat{n}_1 = \frac{1}{2}$, $\widehat{n}_2 = \frac{1}{27}$. Moreover, we find that $\mathcal{Z}_1 \approx 0.20093663 < 1$ and $\mathcal{Z}_2 \approx 0.08512486 < 1$. Thus, the hypothesis of Theorem 3.1 is verified. Therefore, by the conclusion of Theorem 3.1, the problem (28)-(29) with the nonlinear functions f_1 and f_2 given in (30) has at least one solution on $[0, 1]$.

(b) We demonstrate the application of Theorem 3.2 by considering

$$\begin{aligned} f_1(t, x(t), y(t)) &= \frac{1}{100} tan^{-1} x(t) + \frac{1}{4(t+5)^2(1 + |y(t)|)} + 2, \\ f_2(t, x(t), y(t)) &= \frac{1}{3\sqrt{t^3 + 144}} \Big(\cos x(t) + \frac{|y(t)|}{1 + |y(t)|} \Big) + 5e^t. \end{aligned} \tag{31}$$

It is easy to verify that the condition (H_2) holds true with $\mathcal{L}_1 = \frac{1}{100}$ and $\mathcal{L}_2 = \frac{1}{36}$. Further, we have

$$\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2) + \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4) \approx 0.19332345 < 1.$$

Clearly, the assumptions of Theorem 3.2 are satisfied and hence its conclusion implies that there exists a unique solution to the problem (28)-(29) with the nonlinear functions f_1 and f_2 defined in (31) on $[0, 1]$.

5. Stability Analysis

This section is devoted to the Ulam–Hyers stability for the problem at hand. We begin this section with some basic definitions [28].

Definition 5.1. *For* $\epsilon = (\epsilon_1, \epsilon_2) > 0$, *consider a system of inequalities*

$$\begin{cases} \Big| \sum_{i=1}^{p} \omega_i {}^c D^{\alpha_i} \widehat{x}(t) - f_1(t, \widehat{x}(t), \widehat{y}(t)) \Big| \le \epsilon_1, \quad t \in [0, 1], \\ \Big| \sum_{j=1}^{q} \rho_j {}^c D^{\beta_j} \widehat{y}(t) - f_2(t, \widehat{x}(t), \widehat{y}(t)) \Big| \le \epsilon_2, \quad t \in [0, 1], \end{cases} \tag{32}$$

with boundary conditions (2). Then, the system (1)–(2) is called Ulam–Hyers stable if we can find $c = (c_{f_1}, c_{f_2}) > 0$, *such that, for each solution* $(\widehat{x}, \widehat{y}) \in \Omega \times \Omega$ *of* (32) *with boundary conditions (2), there exists a unique solution* $(x, y) \in \Omega \times \Omega$ *of the system (1)–(2) satisfying*

$$\|(\widehat{x}, \widehat{y}) - (x, y)\| \le c\epsilon^T, \quad t \in [0, 1].$$

Definition 5.2. *If there exists $\Phi = (\Phi_1, \Phi_2) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi(0) = 0$ $\big(\Phi_1(0) = \Phi_2(0) = 0\big)$, such that, for each solution $(\widehat{x}, \widehat{y}) \in \Omega \times \Omega$ of (32) with boundary conditions (2), there exists a unique solution $(x, y) \in \Omega \times \Omega$ of the system (1)–(2) satisfying*

$$\|(\widehat{x}, \widehat{y}) - (x, y)\| \le \Phi(\epsilon), \quad t \in [0, 1].$$

Then, the system (1)–(2) is generalized Ulam–Hyers stable.

Remark 5.3. *$(\widehat{x}, \widehat{y}) \in \Omega \times \Omega$ is a solution of the system of inequalities (32) with boundary conditions (2) if and only if there exist functions $\varpi_1, \varpi_2 \in C(\mathcal{J}, \mathbb{R})$ such that $|\varpi_1(t)| \le \epsilon_1$ and $|\varpi_2(t)| \le \epsilon_2$, $t \in [0, 1]$ and*

$$\begin{cases} \displaystyle\sum_{i=1}^{p} \omega_i {}^cD^{\alpha_i}\widehat{x}(t) = f_1(t, \widehat{x}(t), \widehat{y}(t)) + \varpi_1(t), \quad t \in [0, 1], \\ \displaystyle\sum_{j=1}^{q} \rho_j {}^cD^{\beta_j}\widehat{y}(t) = f_2(t, \widehat{x}(t), \widehat{y}(t)) + \varpi_2(t), \quad t \in [0, 1]. \end{cases}$$

Theorem 5.4. *If the assumption (H_2) and the condition (23) are satisfied, then the system (1)–(2) is Ulam–Hyers stable and hence generalized Ulam–Hyers stable in $\Omega \times \Omega$.*

Proof. Let (x, y) be a unique solution of the system (1)–(2) and $(\widehat{x}, \widehat{y})$ be any solution of the system of inequalities (32) with boundary conditions (2). Then, by Remark 5.3, there exist functions $\varpi_1, \varpi_2 \in C([0, 1], \mathbb{R})$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{p} \omega_i {}^cD^{\alpha_i}\widehat{x}(t) = f_1(t, \widehat{x}(t), \widehat{y}(t)) + \varpi_1, \ \alpha_\zeta \in (p-1, p], \ p \ge 2, \ 0 < \alpha_i \le 1, \ i \ne \zeta, \ \omega_\zeta \ne 0, \ \omega_i \in \mathbb{R}, \ t \in [0, 1], \\ \displaystyle\sum_{j=1}^{q} \rho_j {}^cD^{\beta_j}\widehat{y}(t) = f_2(t, \widehat{x}(t), \widehat{y}(t)) + \varpi_2, \ \beta_\mu \in (q-1, q], q \ge 2, \ 0 < \beta_j \le 1, \ j \ne \mu, \ \rho_\mu \ne 0, \ \rho_j \in \mathbb{R}, \ t \in [0, 1], \\ \widehat{x}(0) = 0, \ \widehat{x}^{(\kappa_1)} = 0, \ \kappa_1 = 1, 2, 3, \ldots, p-2, \ \widehat{y}(0) = 0, \ \widehat{y}^{(\kappa_2)} = 0, \ \kappa_2 = 1, 2, 3, \ldots, q-2, \\ \widehat{x}(1) = v_1 \displaystyle\int_0^1 \widehat{y}(s)ds - \sum_{l=1}^{m} a_l \widehat{y}(\sigma_l), \ \widehat{y}(1) = v_2 \int_0^1 \widehat{x}(s)ds - \sum_{l=1}^{m} b_l \widehat{x}(\sigma_l), \ v_1, \ v_2, \ a_l, \ b_l \in \mathbb{R}, \ \sigma_l \in (0, 1). \end{cases} \tag{33}$$

By Lemma 2.6, the solution of (33) can be written as

$$\begin{aligned} \widehat{x}(t) \ = \ & \frac{t^{p-1}}{\Delta}\Bigg[\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - 1}\Big(f_2(u, \widehat{x}(u), \widehat{y}(u)) + \varpi_2(u)\Big)duds \\ & - \sum_{j=1,\, j\ne\mu}^{q} \frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu - \beta_j)} \int_0^1 \int_0^s (s-u)^{\beta_\mu - \beta_j - 1}\widehat{y}(u)duds \\ & - \sum_{l=1}^{m} \frac{a_l}{\rho_\mu\Gamma(\beta_\mu)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - 1}\Big(f_2(s, \widehat{x}(s), \widehat{y}(s)) + \varpi_2(s)\Big)ds \\ & + \sum_{l=1}^{m} a_l \sum_{j=1, j\ne\mu}^{q} \frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu - \beta_j)} \int_0^{\sigma_l} (\sigma_l - s)^{\beta_\mu - \beta_j - 1}\widehat{y}(s)ds \\ & - \frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)} \int_0^1 (1-s)^{\alpha_\zeta - 1}\Big(f_1(s, \widehat{x}(s), \widehat{y}(s)) + \varpi_1(s)\Big)ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds\\
&+A\bigg(\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}\Big(f_1(u,\widehat{x}(u),\widehat{y}(u))+\varpi_1(u)\Big)duds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(u)duds\\
&-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}\Big(f_1(s,\widehat{x}(s),\widehat{y}(s))+\varpi_1(s)\Big)ds\\
&+\sum_{l=1}^{m}b_l\sum_{i=1,i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds\\
&-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}\Big(f_2(s,\widehat{x}(s),\widehat{y}(s))+\varpi_2(s)\Big)ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\bigg)\bigg]\\
&+\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}\Big(f_1(s,\widehat{x}(s),\widehat{y}(s))+\varpi_1(s)\Big)ds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds
\end{aligned}\tag{34}
$$

and

$$
\begin{aligned}
\widehat{y}(t)=\frac{t^{q-1}}{\Delta}\bigg[&-B\bigg(\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}\Big(f_2(u,\widehat{x}(u),\widehat{y}(u))+\varpi_2(u)\Big)duds\\
&-\sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}\widehat{y}(u)duds\\
&-\sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}\Big(f_2(s,\widehat{x}(s),\widehat{y}(s))+\varpi_2(s)\Big)ds\\
&+\sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\\
&-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}\Big(f_1(s,\widehat{x}(s),\widehat{y}(s))+\varpi_1(s)\Big)ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds\bigg)\\
&+\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}\Big(f_1(u,\widehat{x}(u),\widehat{y}(u))+\varpi_1(u)\Big)duds\\
&-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(u)duds\\
&-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}\Big(f_1(s,\widehat{x}(s),\widehat{y}(s))+\varpi_1(s)\Big)ds
\end{aligned}
$$

$$+\sum_{l=1}^{m} b_l \sum_{i=1,i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)} \int_0^{\sigma_l} (\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds$$
$$-\frac{1}{\rho_\mu\Gamma(\beta_\mu)} \int_0^1 (1-s)^{\beta_\mu-1}\Big(f_2(s,\widehat{x}(s),\widehat{y}(s))+\varpi_2(s)\Big)ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)} \int_0^1 (1-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\Bigg]$$
$$+\frac{1}{\rho_\mu\Gamma(\beta_\mu)} \int_0^t (t-s)^{\beta_\mu-1}\Big(f_2(s,\widehat{x}(s),\widehat{y}(s))+\varpi_2(s)\Big)ds$$
$$-\sum_{j=1,\ j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)} \int_0^t (t-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds. \tag{35}$$

In view of $|\varpi_1| < \epsilon_1$, $|\varpi_2| < \epsilon_2$ and (15), we we have

$$\sup_{t\in[0,1]}\Bigg|\widehat{x}(t)-\frac{t^{p-1}}{\Delta}\Bigg[\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)} \int_0^1 \int_0^s (s-u)^{\beta_\mu-1}f_2(u,\widehat{x}(u),\widehat{y}(u))duds$$
$$-\sum_{j=1,\ j\neq\mu}^{q} \frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)} \int_0^1 \int_0^s (s-u)^{\beta_\mu-\beta_j-1}\widehat{y}(u)duds$$
$$-\sum_{l=1}^{m} \frac{a_l}{\rho_\mu\Gamma(\beta_\mu)} \int_0^{\sigma_l} (\sigma_l-s)^{\beta_\mu-1}f_2(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{l=1}^{m} a_l \sum_{j=1,j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)} \int_0^{\sigma_l} (\sigma_l-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds$$
$$-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)} \int_0^1 (1-s)^{\alpha_\zeta-1}f_1(s,\widehat{x}(s),\widehat{y}(s)))ds$$
$$+\sum_{i=1,\ i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)} \int_0^1 (1-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds$$
$$+A\Bigg(\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta-1}f_1(u,\widehat{x}(u),\widehat{y}(u))duds$$
$$-\sum_{i=1,\ i\neq\zeta}^{p} \frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)} \int_0^1 \int_0^s (s-u)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(u)duds$$
$$-\sum_{l=1}^{m} \frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)} \int_0^{\sigma_l} (\sigma_l-s)^{\alpha_\zeta-1}(f_1(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{l=1}^{m} b_l \sum_{i=1,i\neq\zeta}^{p} \frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)} \int_0^{\sigma_l} (\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds$$
$$-\frac{1}{\rho_\mu\Gamma(\beta_\mu)} \int_0^1 (1-s)^{\beta_\mu-1}f_2(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q} \frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)} \int_0^1 (1-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\Bigg)\Bigg]$$

$$-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^t (t-s)^{\alpha_\zeta-1} f_1(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t (t-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds\Bigg| \le Q_1\epsilon_1+Q_2\epsilon_2$$

and

$$\sup_{t\in[0,1]}\Bigg|\widehat{y}(t)-\frac{t^{q-1}}{\Delta}\Bigg[-B\Bigg(\frac{v_1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1\int_0^s (s-u)^{\beta_\mu-1} f_2(u,\widehat{x}(u),\widehat{y}(u)))duds$$
$$-\sum_{j=1,\ j\neq\mu}^{q}\frac{v_1\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s (s-u)^{\beta_\mu-\beta_j-1}\widehat{y}(u)duds$$
$$-\sum_{l=1}^{m}\frac{a_l}{\rho_\mu\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1} f_2(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{l=1}^{m}a_l\sum_{j=1,j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds$$
$$-\frac{1}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1 (1-s)^{\alpha_\zeta-1} f_1(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{i=1,\ i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1 (1-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds\Bigg)$$
$$+\frac{v_2}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^1\int_0^s (s-u)^{\alpha_\zeta-1} f_1(u,\widehat{x}(u),\widehat{y}(u))duds$$
$$-\sum_{i=1,\ i\neq\zeta}^{p}\frac{v_2\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s (s-u)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(u)duds$$
$$-\sum_{l=1}^{m}\frac{b_l}{\omega_\zeta\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1} f_1(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{l=1}^{m}b_l\sum_{i=1,i\neq\zeta}^{p}\frac{\omega_i}{\omega_\zeta\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}\widehat{x}(s)ds$$
$$-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^1 (1-s)^{\beta_\mu-1} f_2(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^1 (1-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\Bigg]$$
$$-\frac{1}{\rho_\mu\Gamma(\beta_\mu)}\int_0^t (t-s)^{\beta_\mu-1} f_2(s,\widehat{x}(s),\widehat{y}(s))ds$$
$$+\sum_{j=1,\ j\neq\mu}^{q}\frac{\rho_j}{\rho_\mu\Gamma(\beta_\mu-\beta_j)}\int_0^t (t-s)^{\beta_\mu-\beta_j-1}\widehat{y}(s)ds\Bigg| \le \widehat{Q}_1\epsilon_1+\widehat{Q}_2\epsilon_2.$$

Next, it follows by the assumption (H_2) that

$$\|\widehat{x}-x\| = \sup_{t\in[0,1]}|\widehat{x}(t)-x(t)| \le Q_1\epsilon_1+Q_2\epsilon_2$$

$$
\begin{aligned}
&+\sup_{t\in[0,1]}\Bigg\{\frac{t^{p-1}}{|\Delta|}\Bigg[\frac{|v_1|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1\int_0^s(s-u)^{\beta_\mu-1}|f_2(u,\widehat{x}(u),\widehat{y}(u))-f_2(u,x(u),y(u))|duds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|v_1\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1\int_0^s(s-u)^{\beta_\mu-\beta_j-1}|\widehat{y}(u)-y(u)|duds\\
&+\sum_{l=1}^{m}\frac{|a_l|}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-1}|f_2(s,\widehat{x}(s),\widehat{y}(s))-f_2(s,x(s),y(s))|ds\\
&+\sum_{l=1}^{m}|a_l|\sum_{j=1,j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^{\sigma_l}(\sigma_l-s)^{\beta_\mu-\beta_j-1}|\widehat{y}(s)-y(s)|ds\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^1(1-s)^{\alpha_\zeta-1}|f_1(s,\widehat{x}(s),\widehat{y}(s))-f_1(s,x(s),y(s))|ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1(1-s)^{\alpha_\zeta-\alpha_i-1}|\widehat{x}(s)-x(s)|ds\\
&+|A|\Bigg(\frac{|v_2|}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-1}|f_1(u,\widehat{x}(u),\widehat{y}(u))-f_1(u,x(u),y(u))|duds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|v_2\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^1\int_0^s(s-u)^{\alpha_\zeta-\alpha_i-1}|\widehat{x}(u)-x(u)|duds\\
&+\sum_{l=1}^{m}\frac{|b_l|}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-1}|f_1(s,\widehat{x}(s),\widehat{y}(s))-f_1(s,x(s),y(s))|ds\\
&+\sum_{l=1}^{m}|b_l|\sum_{i=1,i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^{\sigma_l}(\sigma_l-s)^{\alpha_\zeta-\alpha_i-1}|\widehat{x}(s)-x(s)|ds\\
&+\frac{1}{|\rho_\mu|\Gamma(\beta_\mu)}\int_0^1(1-s)^{\beta_\mu-1}|f_2(s,\widehat{x}(s),\widehat{y}(s))-f_2(s,x(s),y(s))|ds\\
&+\sum_{j=1,\ j\neq\mu}^{q}\frac{|\rho_j|}{|\rho_\mu|\Gamma(\beta_\mu-\beta_j)}\int_0^1(1-s)^{\beta_\mu-\beta_j-1}|\widehat{y}(s)-y(s)|ds\Bigg)\Bigg]\\
&+\frac{1}{|\omega_\zeta|\Gamma(\alpha_\zeta)}\int_0^t(t-s)^{\alpha_\zeta-1}|f_1(s,\widehat{x}(s),\widehat{y}(s))-f_1(s,x(s),y(s))|ds\\
&+\sum_{i=1,\ i\neq\zeta}^{p}\frac{|\omega_i|}{|\omega_\zeta|\Gamma(\alpha_\zeta-\alpha_i)}\int_0^t(t-s)^{\alpha_\zeta-\alpha_i-1}|\widehat{x}(s)-x(s)|ds\Bigg\}\\
&\leq Q_1\epsilon_1+Q_2\epsilon_2+\Big(\mathcal{L}_1Q_1+\mathcal{L}_2Q_2\Big)\Big(\|\widehat{x}-x\|+\|\widehat{y}-y\|\Big)+Q_3\|\widehat{x}-x\|+Q_4\|\widehat{y}-y\|,
\end{aligned}
$$

which implies that

$$
\|\widehat{x}-x\|\leq Q_1\epsilon_1+Q_2\epsilon_2+\Big(\mathcal{L}_1Q_1+\mathcal{L}_2Q_2\Big)\Big(\|\widehat{x}-x\|+\|\widehat{y}-y\|\Big)+Q_3\|\widehat{x}-x\|+Q_4\|\widehat{y}-y\|. \tag{36}
$$

In a similar manner, one can find that

$$
\|\widehat{y}-y\|\leq \widehat{Q}_1\epsilon_1+\widehat{Q}_2\epsilon_2+\Big(\mathcal{L}_1\widehat{Q}_1+\mathcal{L}_2\widehat{Q}_2\Big)\Big(\|\widehat{x}-x\|+\|\widehat{y}-y\|\Big)+\widehat{Q}_3\|\widehat{x}-x\|+\widehat{Q}_4\|\widehat{y}-y\|. \tag{37}
$$

From (36) and (37), we have

$$
\|(\widehat{x},\widehat{y})-(x,y)\|\leq\|\widehat{x}-x\|+\|\widehat{y}-y\|
$$

$$\leq (Q_1 + \widehat{Q}_1)\epsilon_1 + (Q_2 + \widehat{Q}_2)\epsilon_2 + \Big(\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2)\Big)\Big(\|\widehat{x} - x\| + \|\widehat{y} - y\|\Big)$$
$$+ \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4)(\|\widehat{x} - x\| + \|\widehat{y} - y\|)$$
$$\leq \frac{(Q_1 + \widehat{Q}_1)\epsilon_1 + (Q_2 + \widehat{Q}_2)\epsilon_2}{1 - \Big[\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2)\Big] - \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4)}, \tag{38}$$

where we have used (23). Letting $c = (c_{f_1}, c_{f_2}) = \Big(\frac{Q_1 + \widehat{Q}_1}{1 - \mathbb{G}}, \frac{Q_2 + \widehat{Q}_2}{1 - \mathbb{G}}\Big)$, we get

$$\|(\widehat{x}, \widehat{y}) - (x, y)\| \leq c\epsilon^T,$$

where $\mathbb{G} = \Big[\mathcal{L}_1(Q_1 + \widehat{Q}_1) + \mathcal{L}_2(Q_2 + \widehat{Q}_2)\Big] + \max(Q_3 + \widehat{Q}_3, Q_4 + \widehat{Q}_4) < 1$ by the condition (23). Hence, the system (1)–(2) is Ulam–Hyers stable. Moreover, it is generalized Ulam–Hyers stable as $\|(\widehat{x}, \widehat{y}) - (x, y)\| \leq \Phi(\epsilon)$ with $\Phi(\epsilon) = c\epsilon^T$, $\Phi(0) = 0$. This completes the proof.

Example 5.5. *The problem (28)-(29) with f_1 and f_2 in (31) is Ulam–Hyers stable, and generalized Ulam–Hyers stable since $\mathbb{G} \approx 0.19332345 < 1$.*

6. Conclusions

In this study, we applied the tools of the fixed point theory to prove the existence and uniqueness of solutions to a new fully coupled integral-multipoint boundary value problem involving a system of nonlinear multi-term fractional differential equations. We also discussed the Ulam-Hyers, and generalized Ulam-Hyers stability for the given problem. Our results enrich the literature on integral-multipoint boundary value problems as it is the first paper dealing with a coupled system of multi-term nonlinear fractional differential equations. Our results are not only new in the given setting but also specialize to some new results for specific values of the parameters involved in the boundary data. For instance, our results correspond to the ones for coupled integral boundary conditions by fixing $a_l = 0 = b_l, \forall l = 1, 2, \ldots, m$ in the present results. We obtain the results associated with nonlocal coupled multipoint boundary conditions when we take $v_1 = 0 = v_2$ in the results of this paper. In our future work, we plan to extend the present work to a coupled system of multi-term fractional differential inclusions with coupled integral-multipoint boundary conditions.

Conflict of interest. The authors declare that they have no conflict of interest.

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