



Some new bounds for the sum of powers of the normalized Laplacian eigenvalues of graphs

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Abstract. Let $G = (V, E)$ be a simple connected graph of order $n \geq 2$, size m with normalized Laplacian eigenvalues $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > \gamma_n = 0$. Denote with $s_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha$, where α is an arbitrary real number, the sum of powers of normalized Laplacian eigenvalues of graphs. In this paper several inequalities involving invariants of the form $s_\alpha(G)$, for various real α are proved. Our results not only generalize and improve some previous results on $s_\alpha(G)$, Kemeny constant and Laplacian incidence energy, but also present new bounds for these graph invariants.

1. Introduction

Let $G = (V, E)$, $V(G) = \{v_1, v_2, \dots, v_n\}$ be a connected graph with $n \geq 2$ vertices and m edges, with the sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta > 0$, $d_i = d(v_i)$ given in a nonincreasing order. If vertices v_i and v_j are adjacent in G we write $i \sim j$.

Let $A = A(G)$ be the adjacency matrix of G , and $D = D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ the diagonal matrix of its vertex degrees. Then $L = L(G) = D(G) - A(G)$ and $\mathcal{L} = \mathcal{L}(G) = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2} = I - R$ are, respectively, the Laplacian matrix and the normalized Laplacian matrix of G , where $R = R(G) = D^{-1/2}AD^{-1/2}$ is the Randić matrix.

Denote with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$ the eigenvalues of Randić matrix. The following identities are valid for ρ_i [3]:

$$\sum_{i=1}^n \rho_i = \text{tr}(R) = 0 \quad \text{and} \quad \sum_{i=1}^n \rho_i^2 = \text{tr}(R^2) = 2R_{-1}(G), \quad (1)$$

where $R_{-1}(G)$ is the general Randić index R_{-1} (see [5]).

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The eigenvalues of matrix $\mathcal{L}$, $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > \gamma_n = 0$, form the normalized Laplacian spectrum of graph G . These eigenvalues have the following properties [7, 30]:

$$\sum_{i=1}^{n-1} \gamma_i = \text{tr}(\mathcal{L}) = n \quad \text{and} \quad \sum_{i=1}^{n-1} \gamma_i^2 = \text{tr}(\mathcal{L}^2) = n + 2R_{-1}(G). \quad (2)$$

Spectral-based topological indices, or graph invariants, are molecular descriptors defined in terms of the eigenvalues of the corresponding graph matrix. There are a number of such indices defined in the literature. Here we are interested in the one defined in [4] as

$$s_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha, \quad s_0(G) = n - 1,$$

where α is an arbitrary real number. More on mathematical properties of this invariant one can find in, for example, [1, 4, 8, 17]. Some special cases of this invariant are:

- the Laplacian incidence energy of graph, $LIE(G)$, defined as [28]

$$LIE = LIE(G) = s_{1/2}(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i},$$

and

- Kemeny constant, $K(G)$, introduced in [14] (see also [15, 16]), as

$$K(G) = s_{-1}(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i}. \quad (3)$$

Let us note that Kemeny's constant [14] (see also [15]) represents the expected number of steps needed for a random walker to reach a random vertex from some random starting vertex, with the starting and ending vertices being selected according to the equilibrium distribution of the Markov chain. The nature of random walks on a graph is closely tied to the structure of the graph. In [16] it was shown that this constant can be studied through the use of the characteristic polynomial of the normalized Laplacian matrix and in that case it is defined as (3).

The degree Kirchhoff index is graph invariant defined in terms of normalized Laplacian eigenvalues as [6]

$$Kf^*(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\gamma_i}.$$

It is obvious that $Kf^*(G) = 2mK(G)$, so we will not consider $Kf^*(G)$ as a distinct invariant.

In this paper several inequalities involving invariants of the form $s_\alpha(G)$, for various real α , are proved. Our results not only generalize and improve some previous results on $s_\alpha(G)$, Kemeny constant and Laplacian incidence energy, but also present new bounds for these graph invariants.

2. Preliminaries

In this section we recall a couple of analytical inequalities for real number sequences that will be used in proofs of theorems later in the paper.

Lemma 2.1. [23] *Let $p = (p_i)$, $i = 1, 2, \dots, n-1$, be a sequence of non-negative real numbers and $a = (a_i)$, $i = 1, 2, \dots, n-1$, sequence of positive real numbers. Then, for any r , $r \leq 0$ or $r \geq 1$, holds*

$$\left(\sum_{i=1}^{n-1} p_i \right)^{r-1} \sum_{i=1}^{n-1} p_i a_i^r \geq \left(\sum_{i=1}^{n-1} p_i a_i \right)^r. \quad (4)$$

When $0 \leq r \leq 1$ the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = 1$, or $a_1 = a_2 = \dots = a_{n-1}$, or $p_1 = p_2 = \dots = p_t = 0$ and $a_{t+1} = \dots = a_{n-1}$, or $p_{t+1} = \dots = p_{n-1} = 0$ and $a_1 = a_2 = \dots = a_{n-1}$, for some t , $1 \leq t \leq n-2$.

Lemma 2.2. [27] Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \dots, n-1$, are two positive real number sequences. Then for any $r \geq 0$ holds

$$\sum_{i=1}^{n-1} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n-1} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n-1} a_i\right)^r}. \quad (5)$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_{n-1}}{a_{n-1}}$.

Remark 2.3. The inequality (4) is known as Jensen's inequality. However, (4) is quite different from the original Jensen's inequality proved in [13]. For the historical background on Jensen's inequality one can refer to the monograph [24] and paper [25]. The inequality (4) is very powerful since large number of classical analytical inequalities can be obtained as a special case of (4). Some of the inequalities that can be obtained from (4) include Cauchy–Bunyakovsky–Schwarz inequality, Hölder's inequality, arithmetic–geometric mean inequality, etc. We will show that (5) is a corollary of (4). Moreover, we will extend the conditions under which (5) is valid as well as when equality is achieved.

For $r := r+1$, $p_i := a_i$, $a_i := \frac{x_i}{a_i}$, $i = 1, 2, \dots, n-1$, $r \leq -1$ or $r \geq 0$, the inequality (4) becomes

$$\left(\sum_{i=1}^{n-1} a_i\right)^r \sum_{i=1}^{n-1} a_i \frac{x_i^{r+1}}{a_i^{r+1}} \geq \left(\sum_{i=1}^{n-1} x_i\right)^{r+1},$$

that is

$$\left(\sum_{i=1}^{n-1} a_i\right)^r \sum_{i=1}^{n-1} \frac{x_i^{r+1}}{a_i^r} \geq \left(\sum_{i=1}^{n-1} x_i\right)^{r+1},$$

from which inequality (5) is obtained. This implies that inequality (5) holds if and only if $r \leq -1$, or $r \geq 0$. When $-1 \leq r \leq 0$ the opposite inequality in (5) holds. Equality in (5) holds if and only if $r = -1$, or $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_{n-1}}{a_{n-1}}$.

In monograph [7] the following result was proven.

Lemma 2.4. [7] Let G be a connected graph of order n . Then $0 < \gamma_i \leq 2$, for $i = 1, 2, \dots, n-1$. Moreover, $\gamma_1 = 2$ if and only if G is bipartite and $\gamma_1 \geq \frac{n}{n-1}$ with equality if and only if $G \cong K_n$.

Lemma 2.5. [9] Let G be a connected graph of order $n \geq 2$. Then $\gamma_1 = \gamma_2 = \dots = \gamma_{n-1}$ if and only if $G \cong K_n$ and $\gamma_2 = \gamma_3 = \dots = \gamma_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$, $p+q = n$.

Lemma 2.6. [19] Let G be a connected graph of order n . Then

$$\gamma_i = 1 - \rho_{n-i+1}, \quad (6)$$

for $i = 1, 2, \dots, n$.

Lemma 2.7. [3] Let G be a connected graph with n vertices and Randić eigenvalues $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$. Then

$$\sum_{i=1}^n \rho_i^3 = \text{tr}(R^3) = 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left(\sum_{k \sim i, k \sim j} \frac{1}{d_k} \right).$$

By Lemma 2.7 and (6) we have the following result.

Lemma 2.8. Let G be a connected graph with $n \geq 2$ vertices. Then

$$\sum_{i=1}^{n-1} \gamma_i^3 = \text{tr}(\mathcal{L}^3) = 6R_{-1}(G) + n - 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left(\sum_{k \sim i, k \sim j} \frac{1}{d_k} \right).$$

3. Main results

In this section, we present main results of this paper. At first, we determine a lower bound for the expression $s_{\alpha+1}(G) - 2as_{\alpha}(G) + a^2s_{\alpha-1}(G)$, where a is an arbitrary real number with the property $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$.

Theorem 3.1. *Let G be a connected graph with $n \geq 2$ vertices and a be an arbitrary real number such that $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$. Then for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$s_{\alpha+1}(G) - 2as_{\alpha}(G) + a^2s_{\alpha-1}(G) \geq \frac{(2R_{-1}(G) - n(2a-1) + a^2(n-1))^{\alpha}}{(a^2K(G) + n - 2a(n-1))^{\alpha-1}}. \quad (7)$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = a$, for some t , $1 \leq t \leq n-2$.

Proof. For any real α and a holds

$$\begin{aligned} s_{\alpha+1}(G) - 2as_{\alpha}(G) + a^2s_{\alpha-1}(G) &= \sum_{i=1}^{n-1} \gamma_i^{\alpha+1} - 2a \sum_{i=1}^{n-1} \gamma_i^{\alpha} + a^2 \sum_{i=1}^{n-1} \gamma_i^{\alpha-1} \\ &= \sum_{i=1}^{n-1} \gamma_i^{\alpha-1} (\gamma_i - a)^2. \end{aligned} \quad (8)$$

On the other hand, for $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $p_i = \frac{(\gamma_i - a)^2}{\gamma_i}$, $a_i = \gamma_i$, $i = 1, 2, \dots, n-1$, the inequality (4) becomes

$$\left(\sum_{i=1}^{n-1} \frac{(\gamma_i - a)^2}{\gamma_i} \right)^{\alpha-1} \sum_{i=1}^{n-1} \gamma_i^{\alpha-1} (\gamma_i - a)^2 \geq \left(\sum_{i=1}^{n-1} (\gamma_i - a)^2 \right)^{\alpha}. \quad (9)$$

From (2) we obtain that

$$\sum_{i=1}^{n-1} \frac{(\gamma_i - a)^2}{\gamma_i} = \sum_{i=1}^{n-1} \left(\gamma_i - 2a + \frac{a^2}{\gamma_i} \right) = a^2K(G) + n - 2a(n-1)$$

and

$$\sum_{i=1}^{n-1} (\gamma_i - a)^2 = \sum_{i=1}^{n-1} (\gamma_i^2 - 2a\gamma_i + a^2) = n + 2R_{-1}(G) - 2an + a^2(n-1).$$

From the above identities and (8) and (9), we obtain

$$\begin{aligned} (a^2K(G) + n - 2a(n-1))^{\alpha-1} (s_{\alpha+1}(G) - 2as_{\alpha}(G) + a^2s_{\alpha-1}(G)) &\geq \\ &\geq (2R_{-1}(G) - n(2a-1) + a^2(n-1))^{\alpha}. \end{aligned} \quad (10)$$

Since $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$, it follows that $a^2K(G) + n - 2a(n-1) \neq 0$, so from (10) immediately follows (7). The case when $0 \leq \alpha \leq 1$ can be proved analogously.

Equality in (9) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $\gamma_1 = \dots = \gamma_{n-1}$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = a$, for some t , $1 \leq t \leq n-2$. By Lemma 2.5, this implies that equality in (7) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = a$, for some t , $1 \leq t \leq n-2$. $\square$

For $a = 0$ we obtain the following corollaries of Theorem 3.1.

Corollary 3.2. Let G be a connected graph with $n \geq 2$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+1}(G) \geq \frac{(n + 2R_{-1}(G))^\alpha}{n^{\alpha-1}}. \quad (11)$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$.

The inequality (11) was proven in [17].

Corollary 3.3. Let G be a connected graph with $n \geq 2$ vertices. Then

$$K(G) \geq \frac{n^3}{(n + 2R_{-1}(G))^2}. \quad (12)$$

Equality holds if and only if $G \cong K_n$.

The inequality (12) was proven in [17].

Corollary 3.4. Let G be a connected graph with $n \geq 2$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_\alpha(G) \geq \frac{n^\alpha}{(n-1)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if $G \cong K_n$.

Corollary 3.5. Let G be a connected graph with $n \geq 2$ vertices. Then

$$K(G) \geq \frac{(n-1)^2}{n}. \quad (13)$$

Equality holds if and only if $G \cong K_n$.

The inequality (13) was proven in [26] (see also [2, 10, 12, 21]).

For $a = 1$ we have the following corollary of Theorem 3.1.

Corollary 3.6. Let G be a connected graph with $n \geq 2$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+1}(G) - 2s_\alpha(G) + s_{\alpha-1}(G) \geq \frac{(2R_{-1}(G) - 1)^\alpha}{(K(G) - n + 2)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = 1$, for some t , $1 \leq t \leq n-2$.

Lemma 2.8 gives the expression for calculating the exact value for $s_3(G)$ which is complex and time consuming to compute. Therefore it is purposeful to determine its boundaries.

Corollary 3.7. Let G be a connected graph with $n \geq 2$ vertices. Then

$$s_3(G) \geq n + 4R_{-1}(G) + \frac{(2R_{-1}(G) - 1)^2}{K(G) - n + 2}.$$

Equality holds if and only if $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = 1$, for some t , $1 \leq t \leq n-2$.

Corollary 3.8. Let G be a connected graph with $n \geq 2$ vertices. Then

$$LIE(G) \geq \sqrt{\frac{n^3}{n + 2R_{-1}(G)}}. \quad (14)$$

Equality holds if and only if $G \cong K_n$.

Inequality (14) was proven in [28].

Corollary 3.9. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$LIE(G) \geq n \sqrt{\frac{\delta}{\delta + 1}}. \quad (15)$$

Equality holds if and only if $G \cong K_n$.

Inequality (15) was proven in [28].

Using Lemmas 2.1, 2.4 and 2.5, the proof of the next theorem is analogous to that of Theorem 3.1, thus omitted.

Theorem 3.10. *Let G be a connected bipartite graph with $n \geq 3$ vertices and a an arbitrary real number such that $a \neq \gamma_i$, for at least one i , $2 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$s_{\alpha+1}(G) - 2as_{\alpha}(G) + a^2s_{\alpha-1}(G) \geq 2^{\alpha-1}(2-a)^2 + \frac{(2R_{-1}(G) - n(2a-1) + a^2(n-1) - (2-a)^2)^{\alpha}}{\left(a^2K(G) - n(2a-1) + 2a - \frac{(2-a)^2}{2}\right)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q = n$, or $a = \gamma_2 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, or $\gamma_2 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = a$, for some t , $2 \leq t \leq n-2$.

For $a = 0$ we have the following corollary of Theorem 3.10.

Corollary 3.11. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$s_{\alpha+1}(G) \geq 2^{\alpha+1} + \frac{(2R_{-1}(G) + n - 4)^{\alpha}}{(n-2)^{\alpha-1}}. \quad (16)$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q = n$.

The inequality (16) was proven in [17].

Corollary 3.12. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$K(G) \geq \frac{1}{2} + \frac{(n-2)^3}{(2R_{-1}(G) + n - 4)^2}. \quad (17)$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q = n$.

The inequality (17) was also proven in [17].

Corollary 3.13. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$s_3(G) \geq 8 + \frac{(2R_{-1}(G) + n - 4)^2}{n-2}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q = n$.

Corollary 3.14. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$LIE(G) \geq \sqrt{2} + \sqrt{\frac{(n-2)^3}{2R_{-1}(G) + n - 4}}. \quad (18)$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q = n$.

The inequality (18) is a special case of more general result obtained in [17].

Theorem 3.15. Let G be a connected graph with $n \geq 3$ vertices, β be an arbitrary real number and b real number such that $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq b > 0$ and $b \neq \gamma_i$ for at least one i . Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+\beta}(G) \geq bs_{\alpha+\beta-1}(G) + \frac{(s_{\beta+1}(G) - bs_{\beta}(G))^{\alpha}}{(s_{\beta}(G) - bs_{\beta-1}(G))^{\alpha-1}}. \quad (19)$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = b$, for some t , $1 \leq t \leq n-2$.

Proof. For any α, β, b hold

$$s_{\alpha+\beta}(G) - bs_{\alpha+\beta-1}(G) = \sum_{i=1}^{n-1} (\gamma_i - b) \gamma_i^{\alpha+\beta-1}. \quad (20)$$

On the other hand, for $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $p_i = (\gamma_i - b) \gamma_i^{\beta-1}$, $a_i = \gamma_i$, $i = 1, 2, \dots, n-1$, the inequality (4) becomes

$$\left(\sum_{i=1}^{n-1} (\gamma_i - b) \gamma_i^{\beta-1} \right)^{\alpha-1} \sum_{i=1}^{n-1} (\gamma_i - b) \gamma_i^{\alpha+\beta-1} \geq \left(\sum_{i=1}^{n-1} (\gamma_i - b) \gamma_i^{\beta} \right)^{\alpha}, \quad (21)$$

that is

$$(s_{\beta}(G) - bs_{\beta-1}(G))^{\alpha-1} (s_{\alpha+\beta}(G) - bs_{\alpha+\beta-1}(G)) \geq (s_{\beta+1}(G) - bs_{\beta}(G))^{\alpha}.$$

If $b \neq \gamma_i$, for at least one i , $1 \leq i \leq n-1$, then $s_{\beta}(G) - bs_{\beta-1}(G) \neq 0$, therefore, from the above inequality we obtain (19). Similarly, it can be proved that when $0 \leq \alpha \leq 1$, opposite inequality is valid in (19).

Equality in (21) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $\gamma_1 = \gamma_2 = \dots = \gamma_{n-1}$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = b$, for some t , $1 \leq t \leq n-2$. By Lemma 2.5, this implies that equality in (19) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = b$, for some t , $1 \leq t \leq n-2$.

□

For $\beta = 0$ we have the following corollary of Theorem 3.15.

Corollary 3.16. Let G be a connected graph with $n \geq 2$ vertices, and b real number such that $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq b > 0$ and $b \neq \gamma_i$ for at least one i . Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \geq bs_{\alpha-1}(G) + \frac{(n - b(n-1))^{\alpha}}{(n-1 - bK(G))^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = b$, for some t , $1 \leq t \leq n-2$.

Corollary 3.17. Let G be a connected graph with $n \geq 2$ vertices and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq 1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \geq s_{\alpha-1}(G) + (n-1 - K(G))^{1-\alpha}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = 1$, for some t , $1 \leq t \leq n-2$.

Corollary 3.18. Let G be a connected graph with $n \geq 2$ vertices and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq 1$. Then

$$K(G) \leq n-1 - \frac{1}{2R_{-1}(G)}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = 1$, for some t , $1 \leq t \leq n-2$.

Corollary 3.19. Let G be a connected graph with $n \geq 2$ vertices and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq 1$. Then

$$s_3(G) \geq n + 2R_{-1}(G) + \frac{1}{(n-1-K(G))^2}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if $G \cong K_n$, or $\gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = 1$, for some t , $1 \leq t \leq n-2$.

Considering Lemmas 2.1, 2.4 and 2.5 with the similar procedure in Theorem 3.15, the proof of the next theorem can be obtained.

Theorem 3.20. Let G be a connected bipartite graph with $n \geq 3$ vertices and b be an arbitrary real number such that $\gamma_2 \geq \dots \geq \gamma_{n-1} \geq b > 0$ where $b \neq \gamma_i$ for at least one i , $2 \leq i \leq n-1$, and β an arbitrary real number. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+\beta}(G) \geq bs_{\alpha+\beta-1}(G) + (2-b)2^{\alpha+\beta-1} + \frac{(s_{\beta+1}(G) - bs_{\beta}(G) - (2-b)2^{\beta})^{\alpha}}{(s_{\beta}(G) - bs_{\beta-1}(G) - (2-b)2^{\beta-1})^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q=n$, or $\gamma_2 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1} = b$, for some t , $1 \leq t \leq n-2$.

Corollary 3.21. Let G be a connected bipartite graph with $n \geq 3$ vertices and b be an arbitrary real number such that $\gamma_2 \geq \gamma_3 \geq \dots \geq \gamma_{n-1} \geq b > 0$ and $b \neq \gamma_i$ for at least one i , $2 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \geq bs_{\alpha-1}(G) + (2-b)2^{\alpha-1} + \frac{(n-2)^{\alpha}(1-b)^{\alpha}}{\left(n-2-b\left(K(G)-\frac{1}{2}\right)\right)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q=n$.

Corollary 3.22. Let G be a connected bipartite graph with $n \geq 2$ vertices and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} \geq 1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \geq s_{\alpha-1}(G) + 2^{\alpha-1}.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q=n$.

Corollary 3.23. Let G be a connected bipartite graph with $n \geq 2$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \geq n - 2 + 2^{\alpha}. \quad (22)$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q=n$.

The inequality (22) was proven in [4] (see also [1]).

Corollary 3.24. Let G be a connected bipartite graph with $n \geq 2$ vertices. Then

$$LIE(G) \leq \sqrt{2} + n - 2. \quad (23)$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

The inequality (23) was proven in [11] (see also [20]).

Remark 3.25. Every tree is a bipartite graph. Thus, for any tree T according to (23) we have that

$$LIE(T) \leq \sqrt{2} + n - 2,$$

with equality holding if and only if $T \cong K_{1,n-1}$.

The above result was proven in [11].

Corollary 3.26. Let G be a connected bipartite graph with $n \geq 2$ vertices. Then

$$s_3(G) \geq n + 6.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Corollary 3.27. Let G be a connected bipartite graph with $n \geq 2$ vertices. Then

$$K(G) \geq \frac{2n-3}{2}. \quad (24)$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

The inequality (24) was proven in [29].

In the next theorem we establish a relationship between $s_{\alpha+\beta}(G)$ and $s_{\alpha+\beta-1}(G)$, where a , α and β are real numbers.

Theorem 3.28. Let G be a connected graph with n vertices, β an arbitrary real number, and a real number a such that $a \geq \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > 0$ and $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+\beta}(G) \leq a \cdot s_{\alpha+\beta-1}(G) - \frac{(as_{\beta}(G) - s_{\beta+1}(G))^{\alpha}}{(as_{\beta-1}(G) - s_{\beta}(G))^{\alpha-1}}. \quad (25)$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

Proof. For any real a , α and β , the following identities are valid

$$\begin{aligned} as_{\alpha+\beta-1}(G) - s_{\alpha+\beta}(G) &= a \sum_{i=1}^{n-1} \gamma_i^{\alpha+\beta-1} - \sum_{i=1}^{n-1} \gamma_i^{\alpha+\beta} = \\ &= \sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^{\alpha+\beta-1}. \end{aligned} \quad (26)$$

On the other hand, for $r = \alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, $p_i = (a - \gamma_i) \gamma_i^{\beta-1}$, $a_i = \gamma_i$, $i = 1, 2, \dots, n-1$, where β is an arbitrary real number, the inequality (4) becomes

$$\left(\sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^{\beta-1} \right)^{\alpha-1} \sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^{\alpha+\beta-1} \geq \left(\sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^{\beta} \right)^{\alpha}. \quad (27)$$

Also, the following identities are valid

$$\sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^{\beta-1} = as_{\beta-1}(G) - s_{\beta}(G),$$

and

$$\sum_{i=1}^{n-1} (a - \gamma_i) \gamma_i^\beta = as_\beta(G) - s_{\beta+1}(G).$$

From the above and (26) and (27) we obtain

$$\left(as_{\beta-1}(G) - s_\beta(G)\right)^{\alpha-1} \left(as_{\alpha+\beta-1}(G) - s_{\alpha+\beta}(G)\right) \geq \left(as_\beta(G) - s_{\beta+1}(G)\right)^\alpha.$$

Since $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$, it follows that $as_{\beta-1}(G) - s_\beta(G) \neq 0$. Thus, we obtain that

$$as_{\alpha+\beta-1}(G) - s_{\alpha+\beta}(G) \geq \frac{(as_\beta(G) - s_{\beta+1}(G))^\alpha}{(as_{\beta-1}(G) - s_\beta(G))^{\alpha-1}},$$

from which (25) is obtained. The case when $0 \leq \alpha \leq 1$ is proved analogously.

By Lemmas 2.1 and 2.5, equality in (27), and consequently in (25), holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

□

For $\beta = 0$ we obtain the following corollary of Theorem 3.28.

Corollary 3.29. *Let G be a connected graph with $n \geq 2$ vertices and a arbitrary real number such that $a \geq \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > 0$ and $a \neq \gamma_i$ for at least one i , $1 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$s_\alpha(G) \leq as_{\alpha-1}(G) - \frac{(a(n-1) - n)^\alpha}{(aK(G) - n + 1)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $a = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

Corollary 3.30. *Let G be a connected graph with $n \geq 2$ vertices. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$s_\alpha(G) \leq 2s_{\alpha-1}(G) - \frac{(n-2)^\alpha}{(2K(G) - n + 1)^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$, or $2 = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

Corollary 3.31. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$K(G) \geq \frac{1}{2} \left(n - 1 + \frac{(n-2)^2}{n - 2R_{-1}(G)} \right). \quad (28)$$

Equality holds if and only if $G \cong K_n$, or $2 = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

Corollary 3.32. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$s_3(G) \leq 2(n + 2R_{-1}(G)) - \frac{(n-2)^3}{(2K(G) - n + 1)^2},$$

and

$$s_3(G) \leq 2(n + 2R_{-1}(G)) - \frac{(n - 2R_{-1}(G))^2}{n - 2}.$$

Equality holds if and only if $G \cong K_n$, or $2 = \gamma_1 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $1 \leq t \leq n-2$.

Using Lemmas 2.1, 2.4 and 2.5, the proof of the next theorem is obtained analogous to that of Theorem 3.28, hence omitted.

Theorem 3.33. Let G be a connected bipartite graph with $n \geq 3$ vertices, β an arbitrary real number, and a real number such that $a \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > 0$ and $a \neq \gamma_i$ for at least one i , $2 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha+\beta}(G) \leq as_{\alpha+\beta-1}(G) - (a-2)2^{\alpha+\beta-1} - \frac{(as_{\beta}(G) - s_{\beta+1}(G) - (a-2)2^{\beta})^{\alpha}}{(as_{\beta-1}(G) - s_{\beta}(G) - (a-2)2^{\beta-1})^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q = n$, or $a = \gamma_2 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $2 \leq t \leq n-2$.

Corollary 3.34. Let G be a connected bipartite graph with n vertices and a be a real number such that $a \geq \gamma_2 \geq \dots \geq \gamma_{n-1} > 0$ and $a \neq \gamma_i$ for at least one i , $2 \leq i \leq n-1$. Then, for any real α , $\alpha \leq 0$ or $\alpha \geq 1$, holds

$$s_{\alpha}(G) \leq as_{\alpha-1}(G) - (a-2)2^{\alpha-1} - \frac{(a-1)^{\alpha}(n-2)^{\alpha}}{(aK(G) - n + 1 - \frac{a-2}{2})^{\alpha-1}}.$$

When $0 \leq \alpha \leq 1$ the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$, $p+q = n$, or $2 = \gamma_2 = \dots = \gamma_t > \gamma_{t+1} = \dots = \gamma_{n-1}$, for some t , $2 \leq t \leq n-2$.

Theorem 3.35. Let G be a connected graph with $n \geq 2$ vertices and α, β arbitrary real numbers. Then, for any real r , $r \leq -1$ or $r \geq 0$, holds

$$s_{\alpha}(G)^{r+1} \leq s_{\beta}(G)^r s_{(r+1)\alpha-r\beta}(G). \quad (29)$$

When $-1 \leq r \leq 0$, the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = -1$, or $\alpha = \beta$, or $G \cong K_n$.

Proof. The following identity is valid

$$s_{(r+1)\alpha-r\beta}(G) = \sum_{i=1}^{n-1} \gamma_i^{(r+1)\alpha-r\beta} = \sum_{i=1}^{n-1} \frac{(\gamma_i^{\alpha})^{r+1}}{(\gamma_i^{\beta})^r}. \quad (30)$$

On the other hand, when $r \leq -1$ or $r \geq 0$, for $x_i = \gamma_i^{\alpha}$, $a_i = \gamma_i^{\beta}$, $i = 1, 2, \dots, n-1$, the inequality (5) becomes

$$\sum_{i=1}^{n-1} \frac{(\gamma_i^{\alpha})^{r+1}}{(\gamma_i^{\beta})^r} \geq \frac{(\sum_{i=1}^{n-1} \gamma_i^{\alpha})^{r+1}}{(\sum_{i=1}^{n-1} \gamma_i^{\beta})^r} = \frac{s_{\alpha}(G)^{r+1}}{s_{\beta}(G)^r}. \quad (31)$$

Now, from the above and Eq. (30) we obtain (29).

The case when $-1 \leq r \leq 0$ is proved analogously.

By Lemmas 2.2 and 2.5, equality in (31), and thus in (29), holds if and only if either $r = -1$, or $r = 0$, or $\alpha = \beta$, or $G \cong K_n$. $\square$

For real parameters α and β we have the following corollaries of Theorem 3.35.

Corollary 3.36. Let G be a connected graph with $n \geq 2$ vertices. Then

$$R_{-1}(G) \geq \frac{n}{2(n-1)}, \quad (32)$$

with equality holding if and only if $G \cong K_n$.

The inequality (32) was proven in [18].

Corollary 3.37. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$\begin{aligned}s_3(G) &\geq \frac{n^2}{K(G)}, \\s_3(G) &\geq \frac{(n + 2R_{-1}(G))^2}{n}, \\s_3(G) &\geq \frac{n^3}{(n-1)^2}.\end{aligned}$$

Equalities hold if and only if $G \cong K_n$.

Corollary 3.38. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$LIE(G) \leq \sqrt{n(n-1)}. \quad (33)$$

Equality holds if and only if $G \cong K_n$.

The inequality (33) was proven in [28].

Corollary 3.39. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$K(G) \geq \frac{LIE(G)^2}{n + 2R_{-1}(G)}. \quad (34)$$

Equality holds if and only if $G \cong K_n$.

The inequality (34) was proven in [22].

Corollary 3.40. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$LIE(G) \leq \sqrt[4]{(n-1)^3(n + 2R_{-1}(G))}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 3.41. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$K(G) \geq \frac{LIE(G)^4}{n^3}$$

Equality holds if and only if $G \cong K_n$.

The above inequality is stronger than (34).

Considering Lemmas 2.2, 2.4 and 2.5 and similar arguments as in the case of Theorem 3.35, the following result can be proven.

Theorem 3.42. *Let G be a connected bipartite graph with $n \geq 3$ vertices and α, β arbitrary real numbers. Then for any real r , $r \leq -1$ or $r \geq 0$, holds*

$$s_{(r+1)\alpha-r\beta}(G) \geq 2^{(r+1)\alpha-r\beta} + \frac{(s_\alpha(G) - 2^\alpha)^{r+1}}{(s_\beta(G) - 2^\beta)^r}.$$

When $-1 \leq r \leq 0$, the opposite inequality is valid. Equality holds if and only if either $r = -1$, or $r = 0$, or $\alpha = \beta$, or $G \cong K_{p,q}$, $p + q = n$.

Now, we have the following corollaries of Theorem 3.42.

Corollary 3.43. *Let G be a connected bipartite graph with $n \geq 2$ vertices. Then*

$$R_{-1}(G) \geq 1. \quad (35)$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Corollary 3.44. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$\begin{aligned} (s_3(G) - 8) \left(K(G) - \frac{1}{2} \right) &\geq (n - 2)^2, \\ s_3(G) &\geq 8 + \frac{(n + 2R_{-1}(G) - 4)^2}{n - 2}. \end{aligned}$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Corollary 3.45. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$K(G) \geq \frac{1}{2} + \frac{(LIE(G) - \sqrt{2})^2}{n + 2R_{-1}(G) - 4}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Corollary 3.46. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$K(G) \geq \frac{1}{2} + \frac{(LIE(G) - \sqrt{2})^4}{(n - 2)^3}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

The above inequality is stronger than the one given in Corollary 3.45.

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