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The pseudospectra of linear combinations of two orthogonal projections in the Hilbert space

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Abstract. Let P and Q be two orthogonal projections in Hilbert space $\mathcal{H}$. For $\alpha, \beta \in \mathbb{C}\backslash\{0\}$, the lower bound and the upper bound of the pseudospectra of $\alpha P + \beta Q$ are obtained. The bounds are represented by the product PQ which are independent of the choice of scalars α, β. For $\alpha, \beta \in \mathbb{C}\backslash\{0\}, \alpha + \beta \neq \xi, \xi \in \mathbb{C}$, the bounds of the pseudospectra of $\alpha P + \beta Q - \xi PQ$ are also obtained in the same way. Finally, two examples are constructed to show the effectiveness of the results.

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be an idempotent if $P^2 = P$ and an orthogonal projection if $P^2 = P = P^*$. The spectrum of $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not invertible}\}.$$

The ε-pseudospectrum of T with $\varepsilon > 0$ is defined as

$$\sigma_\varepsilon(T) = \sigma(T) \cup \sigma_{\rho,\varepsilon}(T),$$

where $\sigma_{\rho,\varepsilon}(T) = \{\lambda \in \rho(T) : \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\}$ and $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

For the sake of simplicity, write $m_{|\lambda_1|,|\lambda_2|} := \min\{|\lambda_1|, |\lambda_2|\}$ and $M_{|\lambda_1|,|\lambda_2|} := \max\{|\lambda_1|, |\lambda_2|\}$ for $\lambda_1, \lambda_2 \in \mathbb{C}$. Also, $B(a, \varepsilon) := \{z \in \mathbb{C} : |z - a| < \varepsilon\}$ with $\varepsilon > 0$. The identity operator in Hilbert space is denoted by I, $\forall a \in \mathbb{C}$, aI is denoted by a.

There have been many studies on orthogonal projections, see [1–5] and the references cited therein. In addition, the properties for linear combinations of idempotents or orthogonal projections also have many applications. The spectral properties of the linear combination of idempotents or orthogonal projections are proved in [6–8]. Then, the Fredholmness, equivalence form, and the necessary and sufficient conditions

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for the invertibility of the linear combination of idempotents are obtained in [9–12]. Last but not least, the literatures [13, 14] show the equivalence of two spectral sets, which is a prerequisite for studying their pseudospectral bounds. But the pseudospectra of linear combinations of idempotents or orthogonal projections have not yet been fully investigated.

The studies of pseudospectra of matrices and linear operators are increasingly improved, see [15–20]. Cui [21] summarized the pseudospectra of some special operator classes like normal operators, nontrivial projections, nilpotents and so on. Then Jia and Feng [22] provided a characterization of those operators T satisfying $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$ for all $\varepsilon > 0$. There are also detailed classifications of pseudospectra, such as essential pseudospectra. The note [23] studied the essential and the structured essential pseudospectra of closed densely defined linear operators acting on a Banach space. The pseudospectra are shown to be so useful in various mathematical applications. For instance, applications to Markov chains, cutoff phenomenon and lasers can be found in [24].

To enrich the properties for linear combinations of orthogonal projections, the bounds of the resolvent norms of $\alpha P + \beta Q$ (resp. $\alpha P + \beta Q - \xi PQ$) are considered, and then the bounds of pseudospectra of $\alpha P + \beta Q$ (resp. $\alpha P + \beta Q - \xi PQ$) are established by the product PQ in the present paper. Finally, two examples show that the pseudospectra of PQ (resp. $1 - PQ$) are easier to calculate than $\alpha P + \beta Q$ (resp. $\alpha P + \beta Q - \xi PQ$).

2. Main results and proofs

From [25, Theorem 2.7] and [13, Lemma 3.1], the following lemma is given.

Lemma 2.1. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. If $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}$, then*

$$\lambda \in \sigma(\alpha P + \beta Q) \Longleftrightarrow (1 - \alpha^{-1}\lambda)(1 - \beta^{-1}\lambda) \in \sigma(PQ). \tag{1}$$

Lemma 2.1 indicates that the spectra of linear combinations $\alpha P + \beta Q$ equal to the spectra of the product PQ. If T is normal, it is well-known that $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$. When P, Q are orthogonal projections and commutative, one can check that $\alpha P + \beta Q$ and PQ are normal operators. Therefore, the following proposition is obvious.

Proposition 2.2. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. If P, Q are commutative and $\lambda \in \mathbb{C}\backslash\{0\}$, then*

$$\lambda \in \sigma_\varepsilon(\alpha P + \beta Q) \Longleftrightarrow (1 - \alpha^{-1}\lambda)(1 - \beta^{-1}\lambda) \in \sigma_\varepsilon(PQ).$$

Next we consider the bounds of $\sigma_\varepsilon(\alpha P + \beta Q)$ without the commutability through the information of the product PQ based on Lemma 2.1.

Theorem 2.3. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. If $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}$ and set $\mu := (1 - \alpha^{-1}\lambda)(1 - \beta^{-1}\lambda)$, then*

$$\mu \in \sigma(PQ) \cup L_1 \Rightarrow \lambda \in \sigma_\varepsilon(\alpha P + \beta Q) \Rightarrow \mu \in \sigma(PQ) \cup U_1,$$

where

$$L_1 = \{\mu \in \rho(PQ) : |\lambda^{-1}| m_{|\alpha^{-1}\lambda|, |1-\alpha^{-1}\lambda|} \cdot m_{|\beta^{-1}\lambda|, |1-\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| > \frac{1}{\varepsilon}\},$$

$$U_1 = \{\mu \in \rho(PQ) : |\lambda^{-1}| M_{|\alpha^{-1}\lambda|, |1-\alpha^{-1}\lambda|} \cdot M_{|\beta^{-1}\lambda|, |1-\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| > \frac{1}{\varepsilon}\}.$$

Proof. From [13, Equation (3.2)], it can be seen that

$$(\lambda - (\alpha P + \beta Q))^{-1} = \lambda^{-1}(1 - \beta^{-1}\lambda - Q)(\mu - PQ))^{-1}(1 - \alpha^{-1}\lambda - P).$$

Moreover, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
&\|(\lambda-(\alpha P+\beta Q))^{-1})x\|^2\\
=&|\lambda^{-1}|^2\|(1-\beta^{-1}\lambda-Q)(\mu-PQ)^{-1}(1-\alpha^{-1}\lambda-P)x\|^2\\
=&|\lambda^{-1}|^2\langle(c_1Q+c_2)(\mu-PQ)^{-1}(1-\alpha^{-1}\lambda-P)x,(\mu-PQ)^{-1}(1-\alpha^{-1}\lambda-P)x\rangle,
\end{aligned}$$

where

$$c_1=\beta^{-1}\lambda+\overline{\beta^{-1}\lambda}-1,\ c_2=|1-\beta^{-1}\lambda|^2.$$

Since c_1Q+c_2 is a positive operator, there exists a positive square root $S=\xi Q+\eta$ such that $S^2=c_1Q+c_2$, in which $\eta=\sqrt{c_2}, \xi=-\sqrt{c_2}+|\beta^{-1}\lambda|$. And then

$$\|((\lambda-(\alpha P+\beta Q))^{-1})x\|^2=|\lambda^{-1}|^2\ \|\ S(\mu-PQ)^{-1}(1-\alpha^{-1}\lambda-P)x\ \|^2\ . \tag{2}$$

Because $(1-\alpha^{-1}\lambda-P)$ is invertible and combining Equation (2), we have

$$\begin{aligned}
\|S(\mu-PQ)^{-1}\|&=\sup_{x\neq0}\frac{\|S(\mu-PQ)^{-1}(1-\alpha^{-1}\lambda-P)x\|}{\|(1-\alpha^{-1}\lambda-P)x\|}\\
&=|\lambda|\sup_{x\neq0}\frac{\|((\lambda-(\alpha P+\beta Q))^{-1})x\|}{\|(1-\alpha^{-1}\lambda-P)x\|}.
\end{aligned}\tag{3}$$

First, for $x\in\mathcal{H}=R(P)\oplus R(P)^{\perp}, x=x_1+x_2$, where $x_1\in R(P), x_2\in R(P)^{\perp}$,

$$\|(1-\alpha^{-1}\lambda-P)x\|^2=\|-Px+(1-\alpha^{-1}\lambda)x\|^2=|\alpha^{-1}\lambda|^2\|x_1\|^2+|1-\alpha^{-1}\lambda|^2\|x_2\|^2,$$

this concludes that

$$m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|}\|x\|\leq\|(1-\alpha^{-1}\lambda-P)x\|\leq M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|}\|x\|. \tag{4}$$

Next we estimate the bounds of $\|S\|$ and $\|S^{-1}\|$, here $S^{-1}=(|\beta\lambda^{-1}|-\frac{1}{\sqrt{c_2}})Q+\frac{1}{\sqrt{c_2}}$. For $x\in\mathcal{H}=R(Q)\oplus R(Q)^{\perp}$, $x=x_3+x_4$, where $x_3\in R(Q), x_4\in R(Q)^{\perp}$, it is similar to get

$$\|Sx\|^2=\|((|\beta^{-1}\lambda|-\sqrt{c_2})Qx+\sqrt{c_2})x\|^2=|\beta^{-1}\lambda|^2\|x_3\|^2+c_2\|x_4\|^2.$$

Hence

$$m_{\sqrt{c_2},|\beta^{-1}\lambda|}\|x\|\leq\|Sx\|\leq M_{\sqrt{c_2},|\beta^{-1}\lambda|}\|x\|$$

and

$$\frac{1}{M_{\sqrt{c_2},|\beta^{-1}\lambda|}}\|x\|\leq\|S^{-1}x\|\leq\frac{1}{m_{\sqrt{c_2},|\beta^{-1}\lambda|}}\|x\|.$$

Noting that $\|S\|=\sup\limits_{x\neq0}\frac{\|Sx\|}{\|x\|}, \|S^{-1}\|=\sup\limits_{x\neq0}\frac{\|S^{-1}x\|}{\|x\|}$, then the following estimates are obtained

$$m_{\sqrt{c_2},|\beta^{-1}\lambda|}\leq\|S\|\leq M_{\sqrt{c_2},|\beta^{-1}\lambda|}, \tag{5}$$

$$\frac{1}{M_{\sqrt{c_2},|\beta^{-1}\lambda|}}\leq\|S^{-1}\|\leq\frac{1}{m_{\sqrt{c_2},|\beta^{-1}\lambda|}}.$$

Setting $T=S(\mu-PQ)^{-1}$ and by (2)-(5), we have

$$\frac{|\lambda|\|(\lambda-(\alpha P+\beta Q))^{-1}\|}{M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|}}\leq\|T\|\leq\frac{|\lambda|\|(\lambda-(\alpha P+\beta Q))^{-1}\|}{m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|}}. \tag{6}$$

The inequalities (5) and (6) show that

$$\|(\mu - PQ)^{-1}\| \geq \frac{\|T\|}{\|S\|} \geq \frac{|\lambda|\|(\lambda - (\alpha P + \beta Q))^{-1}\|}{M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot M_{\sqrt{c_2},|\beta^{-1}\lambda|}},$$

$(\mu - PQ)^{-1} = S^{-1}T$ indicates that

$$\|(\mu - PQ)^{-1}\| \leq \|S^{-1}\| \cdot \|T\| \leq \frac{|\lambda|\|(\lambda - (\alpha P + \beta Q))^{-1}\|}{m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot m_{\sqrt{c_2},|\beta^{-1}\lambda|}}.$$

Afterwards,

$$\begin{aligned}\frac{1}{\varepsilon} &< |\lambda^{-1}|m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot m_{\sqrt{c_2},|\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| \leq \|(\lambda - (\alpha P + \beta Q))^{-1}\| \\ &\leq |\lambda^{-1}|M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot M_{\sqrt{c_2},|\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\|\end{aligned}$$

implies that

$$\begin{aligned}&\{\mu \in \rho(PQ) : |\lambda^{-1}|m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot m_{\sqrt{c_2}|\beta^{-1}\lambda|} \cdot \|\mu - PQ)^{-1}\| > \frac{1}{\varepsilon}\} \\ \Rightarrow&\{\lambda \in \rho(\alpha P + \beta Q) : \|(\lambda - (\alpha P + \beta Q))^{-1}\| > \frac{1}{\varepsilon}\} \\ \Rightarrow&\{\mu \in \rho(PQ) : |\lambda^{-1}|M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot M_{\sqrt{c_2}|\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| > \frac{1}{\varepsilon})\}.\end{aligned}$$

Then the lower bound and upper bound of $\sigma_{\rho,\varepsilon}(\alpha P + \beta Q)$ are obtained and written as

$$L_1 := \{\mu \in \rho(PQ) : |\lambda^{-1}|m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot m_{|\beta^{-1}\lambda|,|1-\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| > \frac{1}{\varepsilon}\},$$

and

$$U_1 := \{\mu \in \rho(PQ) : |\lambda^{-1}|M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot M_{|\beta^{-1}\lambda|,|1-\beta^{-1}\lambda|} \cdot \|(\mu - PQ)^{-1}\| > \frac{1}{\varepsilon}\},$$

where $\mu = (1 - \alpha^{-1}\lambda)(1 - \beta^{-1}\lambda)$ in L_1 and U_1.
Combining the relation (1), it can be seen that

$$\mu \in \sigma(PQ) \cup L_1 \Rightarrow \lambda \in \sigma(\alpha P + \beta Q) \cup \sigma_{\rho,\varepsilon}(\alpha P + \beta Q) \Rightarrow \mu \in \sigma(PQ) \cup U_1.$$

□

Example 2.4. *Define operators P and Q in $\mathcal{B}(\mathcal{H})$ by*

$$P = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

One can check that P, Q are orthogonal projections and $\sigma(2P+Q) = \{\frac{3}{2} \pm \frac{\sqrt{5}}{2}\}$. Let $\alpha = 2, \beta = 1$, $\lambda \in \mathbb{C}\backslash\{0, 1, 2, \frac{3}{2} \pm \frac{\sqrt{5}}{2}\}$ and $\mu = (1 - \frac{1}{2}\lambda)(1 - \lambda)$. Then

$$(\lambda - (2P + Q))^{-1} = \frac{1}{(\lambda - 2)(\lambda - 1) - 1} \begin{pmatrix} (\lambda - 1)I & I \\ I & (\lambda - 2)I \end{pmatrix},$$

$$(\mu - PQ)^{-1} = \frac{2}{(2\mu - 1)\mu} \begin{pmatrix} \mu I & 0 \\ \frac{1}{2}I & (\mu - \frac{1}{2})I \end{pmatrix}.$$

For different values of λ, $M_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}$ *and* $M_{|\lambda|,|1-\lambda|}$ *can be divided into three cases. First suppose that* $M_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|} = |\frac{1}{2}\lambda|$ *and* $M_{|\lambda|,|1-\lambda|} = |\lambda|$. *Then,*

$$|1-\frac{1}{2}\lambda|\cdot|1-\lambda|\cdot\|(\mu-PQ)^{-1}\| = \max\{|\frac{(\lambda-1)(\lambda-2)}{2\mu-1}|, |\frac{1}{2\mu-1}|, 1\},$$

$$|\lambda|\cdot\|(\lambda-(2P+Q))^{-1}\| = \max\{|\frac{\lambda(\lambda-1)}{2\mu-1}|, |\frac{\lambda}{2\mu-1}|, |\frac{\lambda(\lambda-2)}{2\mu-1}|\},$$

$$|\frac{1}{2}\lambda|\cdot|\lambda|\cdot\|(\mu-PQ)^{-1}\| = \max\{|\frac{\lambda^2}{2\mu-1}|, |\frac{\lambda^2}{2\mu(2\mu-1)}|, |\frac{\lambda^2}{2\mu}|\}.$$

From the assumption, we get

$$|\frac{1}{2\mu-1}| \le |\frac{\lambda}{2\mu-1}| \le |\frac{\lambda^2}{2\mu-1}|, \quad |\frac{\lambda(\lambda-2)}{2\mu-1}| \le |\frac{\lambda^2}{2\mu-1}|,$$

$$1 \le |\frac{(\lambda-1)(\lambda-2)}{2\mu-1}| \le |\frac{\lambda(\lambda-1)}{2\mu-1}| \le |\frac{\lambda^2}{2\mu-1}|.$$

For all $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{\varepsilon} &< |\lambda^{-1}|\cdot|1-\frac{1}{2}\lambda|\cdot|1-\lambda|\cdot\|(\mu-PQ)^{-1}\| \le \|(\lambda-(2P+Q))^{-1}\|\\ &\le |\lambda^{-1}|\cdot|\frac{1}{2}\lambda|\cdot|\lambda|\cdot\|(\mu-PQ)^{-1}\|.\end{aligned}$$

In the other two cases, the verification method is similar and omitted here.
Hence,

$$\begin{aligned}&\{\mu\in\rho(PQ) : |\lambda^{-1}|m_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}\cdot m_{|\lambda|,|1-\lambda|}\cdot\|\mu-PQ)^{-1}\| > \frac{1}{\varepsilon}\}\\ \Rightarrow&\{\lambda\in\rho(2P+Q) : \|(\lambda-(2P+Q))^{-1}\| > \frac{1}{\varepsilon}\}\\ \Rightarrow&\{\mu\in\rho(PQ) : |\lambda^{-1}|M_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}\cdot M_{|\lambda|,|1-\lambda|}\cdot\|(\mu-PQ)^{-1}\| > \frac{1}{\varepsilon})\}.\end{aligned}$$

Combining the relation (1), it can be seen that

$$\begin{aligned}&\mu\in\sigma(PQ)\cup\{\mu\in\rho(PQ) : |\lambda^{-1}|m_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}\cdot m_{|\lambda|,|1-\lambda|}\cdot\|\mu-PQ)^{-1}\| > \frac{1}{\varepsilon}\}\\ \Rightarrow&\lambda\in\sigma_\varepsilon(2P+Q) = \sigma(2P+Q)\cup\sigma_{\rho,\varepsilon}(2P+Q)\\ \Rightarrow&\mu\in\sigma(PQ)\cup\{\mu\in\rho(PQ) : |\lambda^{-1}|M_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}\cdot M_{|\lambda|,|1-\lambda|}\cdot\|(\mu-PQ)^{-1}\| > \frac{1}{\varepsilon})\}.\end{aligned}$$

This justifies Theorem 2.3.

Corollary 2.5. *Let* P, Q *be orthogonal projections in* $\mathcal{B}(\mathcal{H})$ *and* $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. *If* $\lambda \in \mathbb{C}\backslash\{0\}$ *and* $Re(\alpha^{-1}\lambda) = Re(\beta^{-1}\lambda) = \frac{1}{2}, |\lambda| = |\alpha\beta|$, *then*

$$\lambda\in\sigma_\varepsilon(\alpha P+\beta Q) \Longleftrightarrow \mu\in\sigma_\varepsilon(PQ),$$

where $\mu = (1-\alpha^{-1}\lambda)(1-\beta^{-1}\lambda)$.

Proof. Under the conditions and combining the inequality (6), it is clearly that

$$|\lambda^{-1}|m_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot m_{\sqrt{c_2},|\beta^{-1}\lambda|} = |\lambda^{-1}|M_{|\alpha^{-1}\lambda|,|1-\alpha^{-1}\lambda|} \cdot M_{\sqrt{c_2},|\beta^{-1}\lambda|} = 1,$$

thus

$$\|(\lambda - (\alpha P + \beta Q))^{-1}\| = \|(\mu - PQ)^{-1}\|,$$

and

$$\lambda \in \sigma_{\rho,\varepsilon}(\alpha P + \beta Q) \Longleftrightarrow \mu \in \sigma_{\rho,\varepsilon}(PQ).$$

Combining the relation (1), it is easy to get

$$\lambda \in \sigma_{\varepsilon}(\alpha P + \beta Q) \Longleftrightarrow \mu \in \sigma_{\varepsilon}(PQ).$$

□

By analogous argument, we obtain the results about another linear combination $\alpha P + \beta Q - \xi PQ$.

Lemma 2.6. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}, \alpha + \beta \neq \xi, \xi \in \mathbb{C}$. If $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}$ and $\xi\lambda \neq \alpha\beta$, then*

$$\lambda \in \sigma(\alpha P + \beta Q - \xi PQ) \Longleftrightarrow \frac{\lambda(\lambda + \xi - \alpha - \beta)}{\xi\lambda - \alpha\beta} \in \sigma(1 - PQ). \tag{7}$$

Proof. Let $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}, \xi\lambda \neq \alpha\beta$. If $0 < |\lambda| < \min\{|\alpha|, |\beta|, \frac{|\alpha\beta|}{|\xi|+1}\}$, following [14, Theorem 2.2], we note that

$$\begin{aligned} &\lambda^{-1}(\lambda - \alpha(1-P))(\lambda - (\alpha P + \beta Q - \xi PQ))(\lambda - \beta(1-Q)) \\ =&(\xi\lambda - \alpha\beta)\left(\frac{\lambda(\lambda + \xi - \alpha - \beta)}{\xi\lambda - \alpha\beta} - (1 - PQ)\right), \end{aligned} \tag{8}$$

where $\lambda - \alpha(1 - P)$ and $\lambda - \beta(1 - Q)$ are invertible, this implies the relation (7). □

Besides, it is easy to know that $\alpha P + \beta Q - \xi PQ$ and $1 - PQ$ are normal operators when P, Q are orthogonal projections and commutative. So the following conclusion is evident.

Proposition 2.7. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}, \alpha + \beta \neq \xi, \xi \in \mathbb{C}$. If P, Q are commutative and $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}$, $\xi\lambda \neq \alpha\beta$, then*

$$\lambda \in \sigma_{\varepsilon}(\alpha P + \beta Q - \xi PQ) \Longleftrightarrow \frac{\lambda(\lambda + \xi - \alpha - \beta)}{\xi\lambda - \alpha\beta} \in \sigma_{\varepsilon}(1 - PQ).$$

Next we investigate the general properties without the commutability of $\sigma_{\varepsilon}(\alpha P + \beta Q - \xi PQ)$.

Theorem 2.8. *Let P, Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}, \alpha+\beta \neq \xi, \xi \in \mathbb{C}$. If $\lambda \in \mathbb{C}\backslash\{0, \alpha, \beta\}, \xi\lambda \neq \alpha\beta$ and set $\nu := \frac{\lambda(\lambda+\xi-\alpha-\beta)}{\xi\lambda-\alpha\beta}$, then*

$$\nu \in \sigma(1 - PQ) \cup L_2 \Rightarrow \lambda \in \sigma_{\varepsilon}(\alpha P + \beta Q - \xi PQ) \Rightarrow \nu \in \sigma(1 - PQ) \cup U_2,$$

where

$$L_2 = \{\nu \in \rho(1 - PQ) : |\lambda|^{-1} \cdot |\xi\lambda - \alpha\beta|^{-1} m_{|\lambda|,|\lambda-\alpha|} \cdot m_{|\lambda|,|\lambda-\beta|} \cdot \|(\nu - (1 - PQ))^{-1}\| > \frac{1}{\varepsilon}\},$$

$$U_2 = \{\nu \in \rho(1 - PQ) : |\lambda|^{-1} \cdot |\xi\lambda - \alpha\beta|^{-1} M_{|\lambda|,|\lambda-\alpha|} \cdot M_{|\lambda|,|\lambda-\beta|} \cdot \|(\nu - (1 - PQ))^{-1}\| > \frac{1}{\varepsilon}\}.$$

Proof. From the Equation (8), it is observed that

$$
\begin{aligned}
&(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}\\
=&\lambda^{-1}(\xi\lambda-\alpha\beta)^{-1}(\lambda-\beta(1-Q))(\nu-(1-PQ))^{-1}(\lambda-\alpha(1-P)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}x\|^2\\
=&|\lambda^{-1}|^2|(\xi\lambda-\alpha\beta)^{-1}|^2\langle(c_3Q+c_4)(\nu-(1-PQ))^{-1}(\lambda-\alpha(1-P))x,(\nu-(1-PQ))^{-1}(\lambda-\alpha(1-P))x\rangle
\end{aligned}
\tag{9}
$$

where $c_3=\overline{\lambda}\beta+\lambda\overline{\beta}-|\beta|^2, c_4=|\lambda|^2-c_3$. Since c_3Q+c_4 is a positive operator, there exists a positive square root $B=\xi Q+\eta$ such that $B^2=c_3Q+c_4$, in which $\eta=\sqrt{c_4}, \xi=|\lambda|-\sqrt{c_4}$. And then

$$
\begin{aligned}
&\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}x\|^2\\
=&|\lambda^{-1}|^2|(\xi\lambda-\alpha\beta)^{-1}|^2\|B(\nu-(1-PQ))^{-1}(\lambda-\alpha(1-P))x\|^2.
\end{aligned}
$$

Because $(\lambda-\alpha(1-P)$ is invertible and combining Equation (9), we have

$$
\begin{aligned}
\|B(\nu-(1-PQ))^{-1}\|&=\sup_{x\neq 0}\frac{\|B(\nu-(1-PQ))^{-1}(\lambda-\alpha(1-P))x\|}{\|(\lambda-\alpha(1-P))x\|}\\
&=\sup_{x\neq 0}\frac{|\lambda|\cdot|\xi\lambda-\alpha\beta|\cdot\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}x\|}{\|(\lambda-\alpha(1-P))x\|}.
\end{aligned}
$$

First, for $x\in\mathcal{H}=R(P)\oplus R(P)^{\perp}$, it is similar to conclude that

$$m_{|\lambda|,|\lambda-\alpha|}\|x\|\leq\|(\lambda-\alpha(1-P))x\|\leq M_{|\lambda|,|\lambda-\alpha|}\|x\|.$$

Then we gauge the bounds of $\|B\|$ and $\|B^{-1}\|$, here $B^{-1}=(\frac{1}{|\lambda|}-\frac{1}{\sqrt{c_4}})Q+\frac{1}{\sqrt{c_4}}$. For $x\in\mathcal{H}=R(Q)\oplus R(Q)^{\perp}$, it is similar to get

$$m_{\sqrt{c_4},|\lambda|}\|x\|\leq\|Bx\|\leq M_{\sqrt{c_4},|\lambda|}\|x\|$$

and

$$\frac{1}{M_{\sqrt{c_4},|\lambda|}}\|x\|\leq\|B^{-1}x\|\leq\frac{1}{m_{\sqrt{c_4},|\lambda|}}\|x\|.$$

Setting $A=B(\nu-(1-PQ))^{-1}$ and combining with the proof above, we obtain that

$$\frac{|\lambda|\cdot|\xi\lambda-\alpha\beta|\cdot\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}\|}{M_{|\lambda|,|\lambda-\alpha|}}\leq\|A\|\leq\frac{|\lambda|\cdot|\xi\lambda-\alpha\beta|\cdot\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}\|}{m_{|\lambda|,|\lambda-\alpha|}}.$$

$A=B(\nu-(1-PQ))^{-1}$ shows that

$$\|(\nu-(1-PQ))^{-1}\|\geq\frac{\|A\|}{\|B\|}\geq\frac{|\lambda|\cdot|\xi\lambda-\alpha\beta|\cdot\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}\|}{M_{|\lambda|,|\lambda-\alpha|}\cdot M_{\sqrt{c_4},|\lambda|}},$$

$(\nu-(1-PQ))^{-1}=B^{-1}A$ indicates that

$$\|(\nu-(1-PQ))^{-1}\|\leq\|B^{-1}\|\cdot\|A\|\leq\frac{|\lambda|\cdot|\xi\lambda-\alpha\beta|\cdot\|(\lambda-(\alpha P+\beta Q-\xi PQ))^{-1}\|}{m_{|\lambda|,|\lambda-\alpha|}\cdot m_{\sqrt{c_4},|\lambda|}}.$$

$$\{\nu \in \rho(1-PQ) : |\lambda^{-1}| \cdot |(\xi\lambda - \alpha\beta)^{-1}| \cdot m_{|\lambda|,|\lambda-\alpha|} \cdot m_{\sqrt{c_4},|\lambda|} \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\}$$
$$\Rightarrow\{\lambda \in \rho(\alpha P + \beta Q - \xi PQ) : \|(\lambda - (\alpha P + \beta Q - \xi PQ))^{-1}\| > \frac{1}{\varepsilon}\}$$
$$\Rightarrow\{\nu \in \rho(1-PQ) : |\lambda^{-1}| \cdot |(\xi\lambda - \alpha\beta)^{-1}| \cdot M_{|\lambda|,|\lambda-\alpha|} \cdot M_{\sqrt{c_4},|\lambda|} \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\}.$$

Then the lower bound and upper bound of $\sigma_{\rho,\varepsilon}(\alpha P + \beta Q - \xi PQ)$ are obtained and written as

$$\{\nu \in \rho(1-PQ) : |\lambda^{-1}| \cdot |(\xi\lambda - \alpha\beta)^{-1}| \cdot m_{|\lambda|,|\lambda-\alpha|} \cdot m_{\sqrt{c_4},|\lambda|} \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\} = L_2$$

and

$$\{\nu \in \rho(1-PQ) : |\lambda^{-1}| \cdot |(\xi\lambda - \alpha\beta)^{-1}| \cdot M_{|\lambda|,|\lambda-\alpha|} \cdot M_{\sqrt{c_4},|\lambda|} \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\} = U_2,$$

where $\nu =: \frac{\lambda(\lambda+\xi-\alpha-\beta)}{\xi\lambda-\alpha\beta}$ in L_2 and U_2.
Combining the relation (7), it is known that

$$\nu \in \sigma(1-PQ) \cup L_2 \Rightarrow \lambda \in \sigma_\varepsilon(\alpha P + \beta Q - \xi PQ) \Rightarrow \nu \in \sigma(1-PQ) \cup U_2.$$

□

Example 2.9. *Consider P, Q are the orthogonal projections given in the Example 2.4. Let $\alpha = 2, \beta = 1, \xi = 2$, one can check that $\sigma(2P + Q - 2PQ) = \{1\}$. If $\lambda \in \mathbb{C}\backslash\{0, 1, 2\}$, then $\xi\lambda \neq \alpha\beta$ and $\nu = \frac{\lambda(\lambda+\xi-\alpha-\beta)}{\xi\lambda-\alpha\beta} = \frac{\lambda}{2}$. Moreover*

$$(\lambda - (2P + Q - 2PQ))^{-1} = \begin{pmatrix} \frac{1}{(\lambda-1)}I & \frac{1}{(\lambda-1)^2}I \\ 0 & \frac{1}{(\lambda-1)}I \end{pmatrix},$$
$$(\nu - (1-PQ))^{-1} = \begin{pmatrix} \frac{2}{(2\nu-1)}I & 0 \\ \frac{-1}{(2\nu-1)(\mu-1)}I & \frac{1}{(\mu-1)}I \end{pmatrix}.$$

For different values of λ, $M_{|\frac{1}{2}\lambda|,|1-\frac{1}{2}\lambda|}$ and $M_{|\lambda|,|1-\lambda|}$ also can be divided into three cases. First suppose that $M_{|\lambda|,|\lambda-1|} = |\lambda|$ and $M_{|\lambda|,|\lambda-2|} = |\lambda|$. Then,

$$|\lambda - 1| \cdot |\lambda - 2| \cdot \|(\nu - (1-PQ))^{-1}\| = \max\{2|\lambda - 2|, |\lambda - 1|, 1\},$$
$$|\lambda(\xi\lambda - \alpha\beta)| \cdot \|(\lambda - (2P + Q - 2PQ))^{-1}\| = \max\{2|\lambda|, \frac{2|\lambda|}{|\lambda - 1|}\},$$
$$|\lambda| \cdot |\lambda| \cdot \|(\nu - (1-PQ))^{-1}\| = \max\{|\frac{2|\lambda|^2}{|\lambda - 1|}|, |\frac{|\lambda|^2}{|\lambda - 2|}|, |\frac{|\lambda|^2}{|(\lambda - 1)(\lambda - 2)}|\}.$$

From the assumption and for all $\varepsilon > 0$,

$$\frac{1}{\varepsilon} < |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| \cdot |\lambda - 1| \cdot |\lambda - 2| \cdot \|(\nu - (1-PQ))^{-1}\|$$
$$\leq \|(\lambda - (2P + Q - 2PQ))^{-1}\| \leq |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| \cdot |\lambda| \cdot |\lambda| \cdot \|(\nu - (1-PQ))^{-1}\|.$$

In the other two cases, the verification method is similar and omitted here.
Hence,

$$\{\nu \in \rho(1-PQ) : |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| m_{|\lambda|,|\lambda-2|} \cdot m_{|\lambda|,|\lambda-1|} \cdot \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\}$$
$$\Rightarrow\{\lambda \in \rho(2P + Q - 2PQ) : \|(\lambda - (2P + Q - 2PQ))^{-1}\| > \frac{1}{\varepsilon}\}$$
$$\Rightarrow\{\nu \in \rho(1-PQ) : |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| M_{|\lambda|,|\lambda-2|} \cdot M_{|\lambda|,|\lambda-1|} \cdot \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\}.$$

Combining the relation (7), it can be seen that

$$\begin{aligned} &\nu \in \sigma(1-PQ) \cup \{\nu \in \rho(1-PQ) : |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| \cdot m_{|\lambda|,|\lambda-2|} \cdot m_{|\lambda|,|\lambda-1|} \cdot \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\} \\ \Rightarrow &\lambda \in \sigma_\varepsilon(2P+Q-2PQ) \\ \Rightarrow &\nu \in \sigma(1-PQ) \cup \{\nu \in \rho(1-PQ) : |\lambda^{-1}(\xi\lambda - \alpha\beta)^{-1}| \cdot M_{|\lambda|,|\lambda-2|} \cdot M_{|\lambda|,|\lambda-1|} \cdot \|(\nu - (1-PQ))^{-1}\| > \frac{1}{\varepsilon}\}. \end{aligned}$$

This verifies Theorem 2.8.

3. Summary and discussion

In Hilbert space, we recall the equivalence relation of the spectrum between linear combinations of two orthogonal projections and the product of that. What is more, the bounds of the pseudospectra of $\alpha P + \beta Q$ and $\alpha P + \beta Q - \xi PQ$ are established. Therefore, the results and the examples of this paper are the abundance of the linear combinations of projections. They can be used as a reference for future research.

The pseudospectra of bounded operators have many good properties, how to study the pseudospectra of linear combinations of other bounded operators is worth considering in the future.

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